Maths HUBS Resources

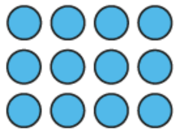
National Centre for Excellence in the Teaching of Mathematics

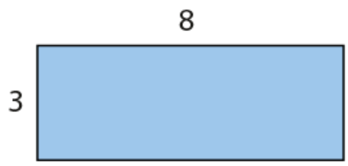
# Mastery Professional Development

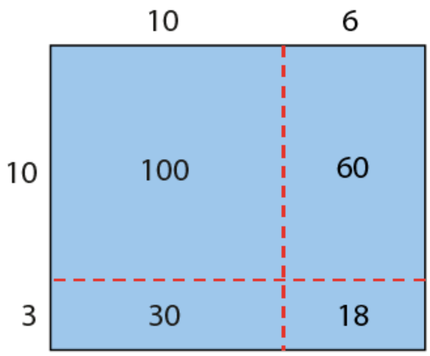
*Mathematical representations*

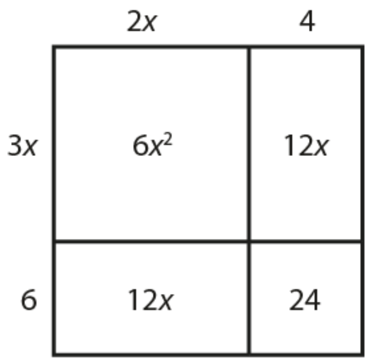
## Arrays and area models

Guidance document | Key Stage 3





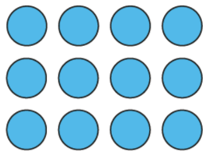




*Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.*

## What they are

An array is a rectangular arrangement of dots in rows and columns, where each column has the same number of dots. The number of dots in each row (or column) multiplied by the number of dots in each column (or row) gives the total number of dots.

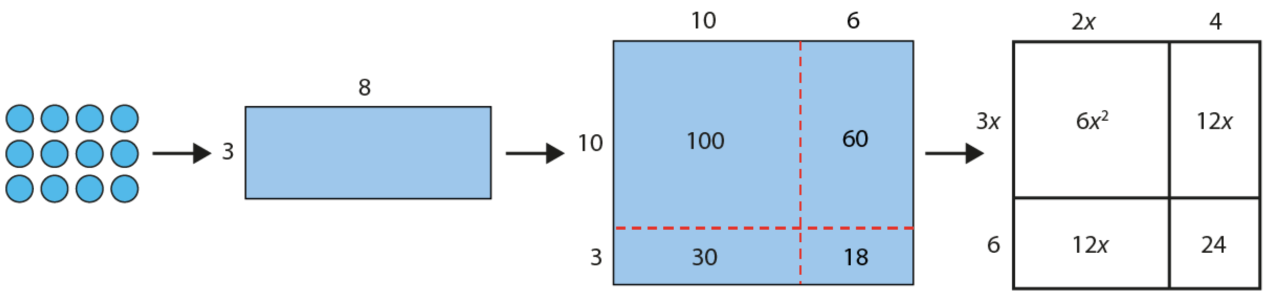


The above array can be used to model a number of key aspects of the multiplicative relationship between 4, 3 and 12, namely:

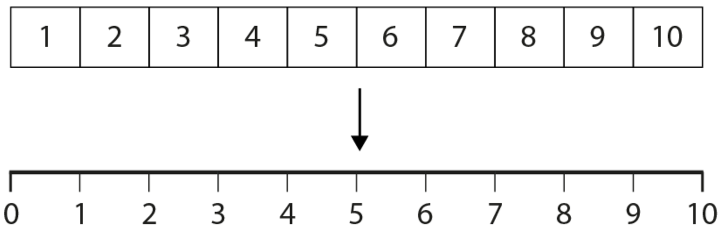
- repeated addition of 4s, i.e. $4 + 4 + 4$, or 4 multiplied 3 times is 12
- repeated addition of 3s, i.e. $3 + 3 + 3 + 3$, or 3 multiplied 4 times is 12
- the structural equivalence of $4 \times 3$ and $3 \times 4$ (the commutative law of multiplication)
- the corresponding division facts, i.e. that $12 \div 3 = 4$ and $12 \div 4 = 3$.

Despite the apparent transparency of the link between the number of dots and multiplication, some research suggests that many secondary school students struggle to make this connection, even if they can find the area of a rectangle.

Arrays are essentially a way of organising a set of countable objects. This means arrays can only be used to represent the product of two integers, but they provide a starting point for the development of a more generalised area model for multiplication. In this model, the dimensions, which can be any real numbers, represent the factors, and the area represents the product of those factors. The development of these representations ultimately provides students with a model of multiplication that helps them to think about the structures involved in multiplying larger numbers as well as algebraic expressions.




At the most basic level, the area model is a simple rectangle with no partitioning. However, as the factors involved become larger and more complex, partitioning of the rectangle helps to break the calculation down into more manageable steps, as the area model mirrors the standard algorithm. When working with algebraic expressions, such as the product of $(3x + 6)$ and $(2x + 4)$, shown above, the area model becomes a more concise image of an algebra tiles representation of multiplying two binomials. This process of progressing from counting dots to continuous measurement can be likened to the move from number tracks to the continuous number line, already made by students at primary school.

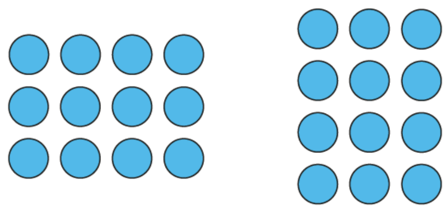



Arrays and area models provide an effective way of linking multiplication to area. It is important for these representations to be introduced in a progressive way, so that students understand how they are linked. The use of counters as a precursor to a picture of an array of dots may also be useful for some students.

## Why they are important

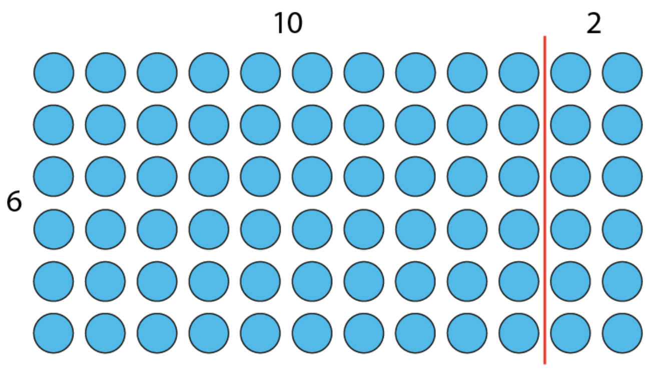
Arrays and area models support students in developing a robust understanding of both the 'how' and 'why' of multiplication, and the relationship between multiplication and division.

Arrays can help students to understand the commutativity of multiplication, i.e. $a \times b = b \times a$. When an array is rotated or viewed in a different orientation, 3 multiplied 4 times and 4 multiplied 3 times can be seen to be the same.



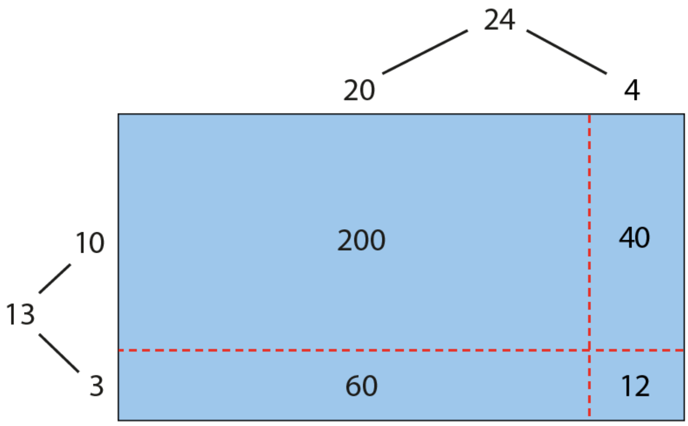
The concept of related number facts ('number families') can also be shown through the use of arrays. Students are often told that if you know $3 \times 4 = 12$, you also know the associated facts of $4 \times 3 = 12$, $12 \div 3 = 4$ and $12 \div 4 = 3$. Many students accept this as being the case, without necessarily seeing why it is so. The array is a powerful visual model to show why these other facts automatically follow.

Arrays and area models can provide a meaningful illustration of how partitioning can be used when multiplying larger numbers, while maintaining a clear visual understanding of the magnitude of the numbers involved.




$6 \times 12 = 6 \times 10 + 6 \times 2$

A visual model of the distributive property can also be demonstrated, by emphasising the link between multiplication and area.

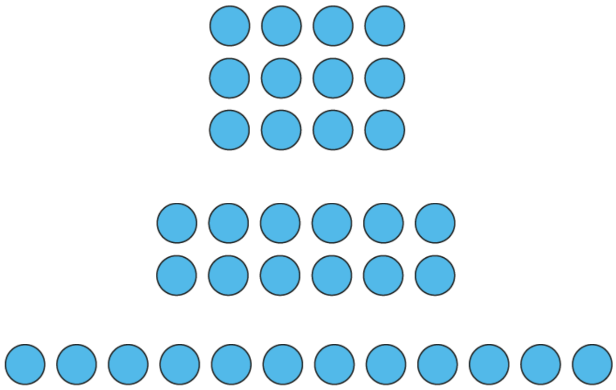



Arrays and area models support the shift from an additive view of multiplication (a 'groups of' model) to a 'factor × factor = product' structure, as well as helping students to extend the properties of number and laws of arithmetic to algebraic terms and expressions.

## How they might be used

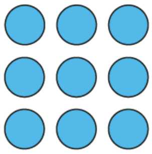
### Identifying factors

A way of investigating the factors of a number is to make as many different rectangular arrays for that number as possible. For example, the possible rectangular arrays for the number 12 are 3 by 4 (or 4 by 3), 2 by 6 (or 6 by 2) and 1 by 12 (or 12 by 1).



The corresponding factors of 12 can be identified by the number of dots in each row and column, i.e. 3, 4, 2, 6, 1 and 12. Exploring factors in this way highlights the fact that some numbers (composite numbers) can be represented with more than one array, whereas others (prime numbers) can only be made with one-row (or one-column) arrays, i.e. a single line of dots; for example, the number 3.

Arrays can also provide images for square numbers, by considering the number of dots that can be arranged to make square arrays.

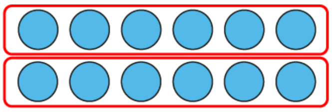


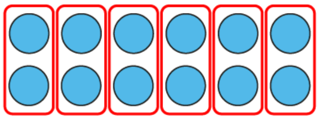
Here, the square number 9 is represented by a 3 by 3 array, highlighting the factors of 9 as being 1, 3 and 9 (as a positive integer can always be represented by a single line of dots).

### Commutative property of multiplication

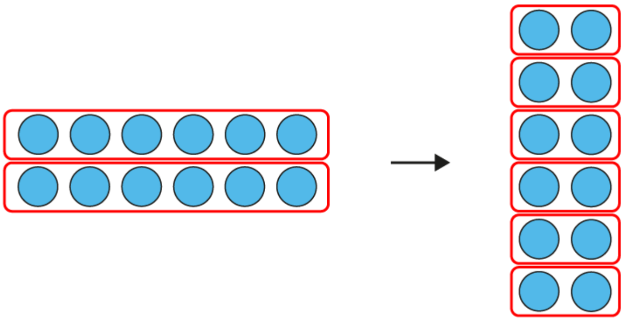
The commutative property of multiplication can be illustrated using an array.

This array can be read as 2 rows of 6, or as 6 columns of 2.





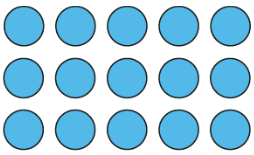
The array can also be rotated to show that 2 rows of 6 has the same number of dots as 6 rows of 2.



Regardless of how the array is viewed, it contains 12 dots and illustrates that $2 \times 6 = 6 \times 2$, a visual image of the commutative property of multiplication.

### Inverse relationships

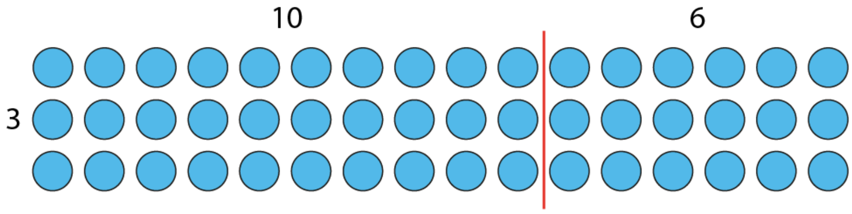
The inverse relationship between division and multiplication can be illustrated using arrays. For example, $3 \times 5 = 15$ can be represented by the following array.



Looking at the array as 15 arranged into 3 rows, making 5 columns (or 5 in each row) reveals the two inverse relationships, i.e. that $15 \div 3 = 5$ and $15 \div 5 = 3$.

## Distributive property of multiplication

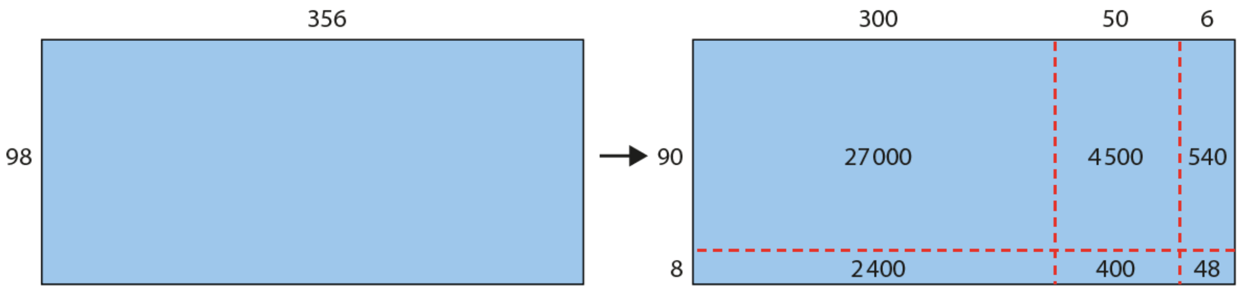
The distributive property of multiplication tells us that multiplying a number by the sum of two (or more) numbers is the same as performing two (or more) multiplications separately and then finding their sum, i.e. $a(b + c) = ab + ac$. For example, $3 \times 16$ can be represented with an array containing 3 rows, each containing 16 dots.




$$3 \times 16 = 3 \times (10 + 6) = 3 \times 10 + 3 \times 6$$

The array can be partitioned, producing two smaller arrays involving number facts that are easier for students to recall. Dividing the 16 into 10 and 6 may be an obvious choice. However, it is important for students to recognise that an alternative partitioning may result in multiplications involving numbers that are equally as accessible to them. They should understand that the result will be the same, regardless of how the numbers are partitioned.

As the factors involved in a multiplication become larger, the transition from arrays of dots to area models becomes appropriate. For $356 \times 98$, for example, an area model can be drawn, and both the height and length of the rectangle partitioned, in order to give more manageable numbers to work with.




The standard algorithm for multiplication is based on the distributive law, and the use of the area model helps to support students' full understanding of why this procedure works.

| | | | | | |
|---|---|---|---|---|---|
| | | | 3 | 5 | 6 |
| × | | | | 9 | 8 |
| | | | | 4 | 8 |
| | | | 4 | 0 | 0 |
| | | 2 | 4 | 0 | 0 |
| | | | 5 | 4 | 0 |
| | | 4 | 5 | 0 | 0 |
| | 2 | 7 | 0 | 0 | 0 |
| | 3 | 4 | 8 | 8 | 8 |
| | 1 | 1 | | | |

| | | | | | |
|---|---|---|---|---|---|
| | | ~~5~~ | ~~5~~ | | |
| | | | ~~4~~ | ~~4~~ | |
| | | | 3 | 5 | 6 |
| × | | | | 9 | 8 |
| | | 2 | 8 | 4 | 8 |
| | 3 | 2 | 0 | 4 | 0 |
| | 3 | 4 | 8 | 8 | 8 |

Each of the six rectangles can be seen in the expanded written method. Using the area model representation alongside the symbolic representation can help to strengthen students' understanding of where the 'extra zeros' come from when using the standard algorithm. Students may find the compact method a more efficient way of recording the multiplication and may transition to that once they fully understand how the standard algorithm works. As with any representation, the use of symbols alongside the area model plays an important part in the development of students' understanding, fluency and efficiency.

## Products of binomials

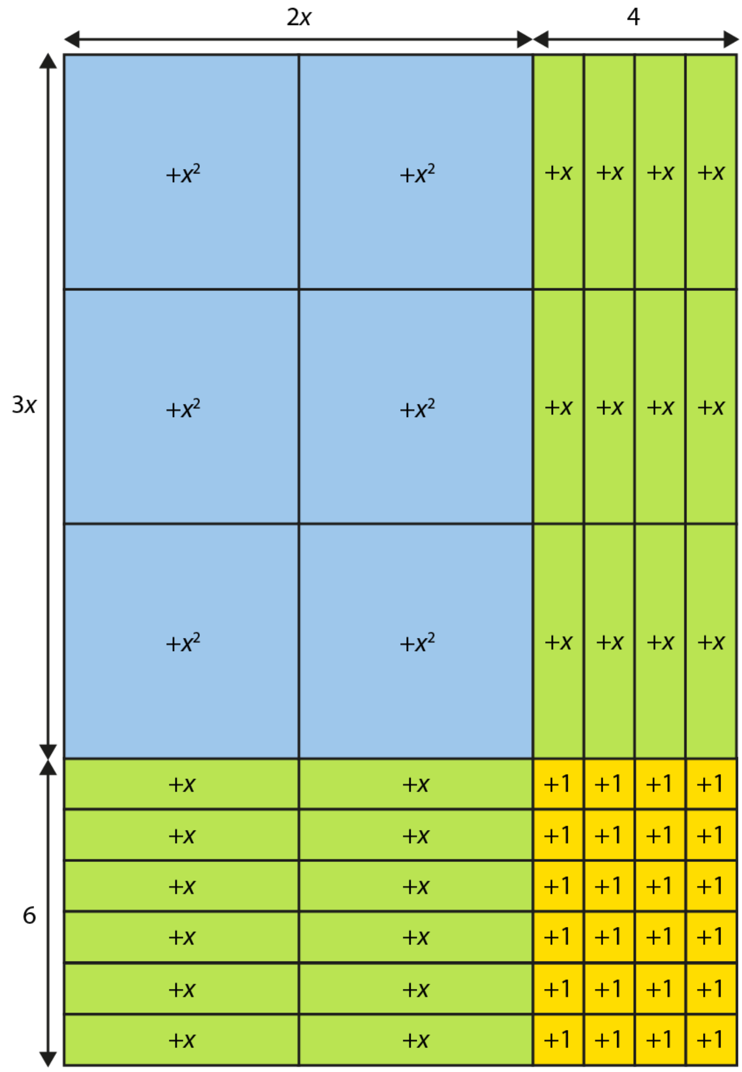
Area models can be used to support students in relating the structure of multiplication with number, to the multiplication of algebraic expressions. For example, for the multiplication $36 \times 24$, we know that we can partition 36 as $30 + 6$, and 24 as $20 + 4$, and represent this with an area model.

| | 20 | 4 |
|---|---|---|
| 30 | 600 | 120 |
| 6 | 120 | 24 |

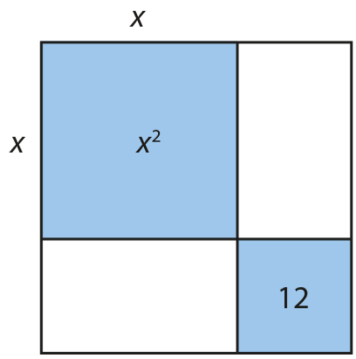
This partitioning of $36 \times 24$ as $(30 + 6)(20 + 4)$ can be generalised to $(3x + 6)(2x + 4)$ (where $x = 10$ in the above case) and represented in the same way.

| | $2x$ | 4 |
|---|---|---|
| $3x$ | $6x^2$ | $12x$ |
| 6 | $12x$ | 24 |

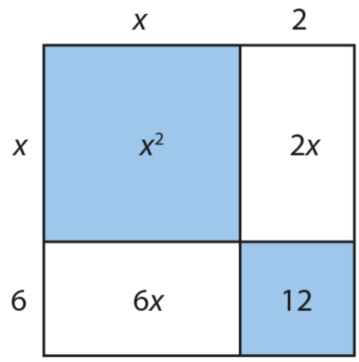
The use of algebra tiles to represent this may also help to make the connection with the area model of multiplication more explicit.




The area model can be used equally well as a model to support the factorisation of quadratic expressions when students encounter this at Key Stage 4. For example, for the quadratic expression $x^2 + 8x + 12$, a grid with four sections can be set up and the factors identified.



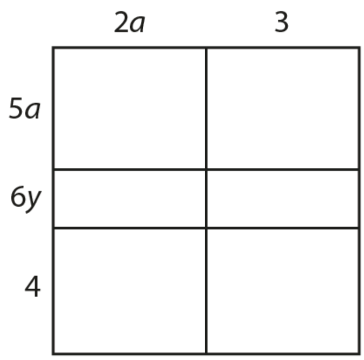

$x^2$ must be in the top left, with $x$ and $x$ on each side, and the 12 must be in the bottom right.




The factors of 12 can then be explored to find two numbers with a product of 12 that would give a total of $8x$ when adding the other two areas (i.e. multiply to give 12 and add to give 8).

The use of a representation at Key Stage 3, that also has usefulness at Key Stage 4, means that students are already familiar with the representation and will have explored why it is an appropriate model. This can then be built on, and, in this case, clearly demonstrates the relationship between expanding and factorising quadratic expressions.

The area model can also be used to help students understand and justify that the product of an expression with, for example, two terms in the first expression and three terms in the second expression will have six (i.e. $2 \times 3$) terms before simplifying. For example, $(2a + 3)(5a + 6y + 4)$ can be represented with the following area model:




Students can be encouraged to generalise further, to situations where there are more than two binomials, and to realise that the product of more than two binomials can be reduced to two polynomials by successive multiplication of pairs.

## Further resources

When physical counters are not available, or when wanting to display an array of counters to the whole class, using an online resource that can be projected on a screen can provide a helpful means of demonstrating arrays and area models.

Consider the usability of a resource as a tool for independent student use and its appropriateness in terms of enabling students to focus on the way in which the representation supports the mathematics, without them being hindered by the process of producing the representation.

**ICT games**

https://www.ictgames.com/mobilePage/arrayDisplay/index.html

This array generator produces arrays of dots corresponding to multiplications up to 12 × 12. The multiplication can be entered via the numerical keypad at the bottom left of the screen. When the equals button is pressed, a labelled array is displayed. The rotation button rotates the array 90 degrees and changes the order of the multiplicand and multiplier. There is also the option to divide the dots in the array. When the division symbol is pressed, the product of the multiplicand and multiplier appears. The multiplication is rewritten as a division, with the dots grouped corresponding to the solution to the division.

**Conceptua® Math**

https://teach.conceptuamath.com/client/html5/MatEngine.html?tool=placeValueMultiplicationMat&direct=true

This tool provides a labelled area model partitioned into four sections (as default), with an expanded version of the standard multiplication algorithm alongside, which is completed as the multiplication details are input. Entering the multiplicand and multiplier in the boxes at the bottom of the screen labels the sides of the area model and partitions the factors (into tens and ones), and the symbolic representation is set up. The areas of the four sections can then be found and entered into the area model. As this is done, the symbolic representation is completed simultaneously. Summing the four areas and inputting this as the product in the box at the bottom of the screen completes the solution in the standard algorithm workings. There is, however, the option to unlink the two representations by deselecting the 'chain link' option and the corresponding details can be completed manually.

For calculations where the four sections of the area model are not required, the type of multiplication problem can be changed via the rotation button on the right-hand side of the screen, with the option of one-digit × one-, two-, three- or four-digit problems. There is the option to hide the area model and just have the symbolic representation, or just the area model can be viewed. The calculation at the bottom of the screen can also be hidden. The way in which the area model mirrors the standard algorithm can be demonstrated by hovering over either elements of the symbolic representation or components of the area model. The corresponding counterpart is highlighted to show the relationship between the two representations.

This tool is well suited to whole-class demonstrations, with the option to add a multiplication problem via the text prompt box. It may also be appropriate to allow students to use the tool for themselves, to explore the link between multiplication and area.

A demonstration of the tool is available via this online video:
https://teach.conceptuamath.com/sites/default/files/ccvideos/open-array-multiplication.mp4

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