MathsHUBS Resources

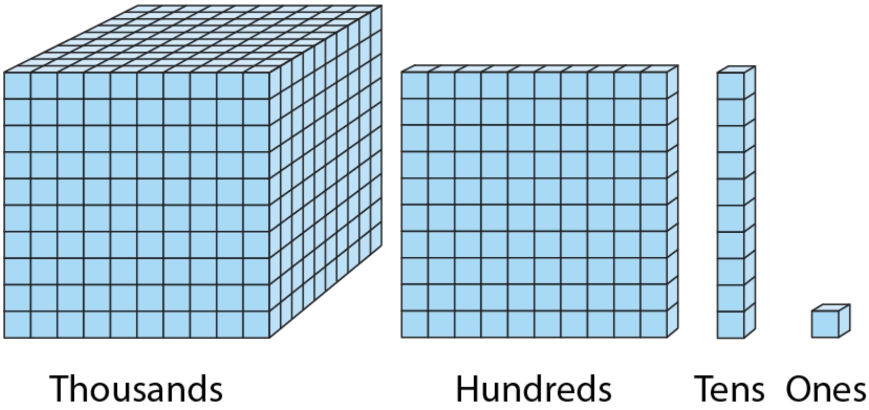
National Centre for Excellence in the Teaching of Mathematics

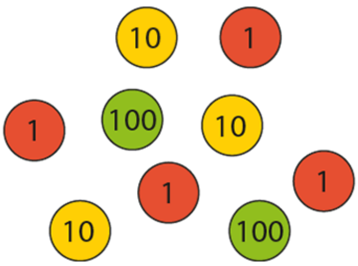
Mastery Professional Development

Mathematical representations

Dienes (and place-value counters)

Guidance document | Key Stage 3

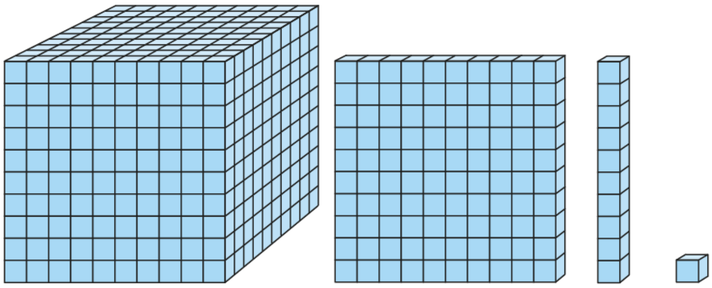




Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

What they are

Dienes blocks (referred to as 'Dienes' from hereon in) consist of ones (known as 'cubes'), tens (known as 'rods'*), hundreds (10 by 10 arrays, sometimes known as 'squares'*) and thousands (10 by 10 by 10 'blocks') and are traditionally plastic or wooden blocks. (*N.B. We refer to tens 'rods' and hundreds 'squares', but tens are also commonly called 'longs' and hundreds 'flats'.)



Thousands Hundreds Tens Ones

Please note: For ease of representation, in the following document, hundreds, tens and ones have been shown in 2D.

Why they are important

Dienes expose the base-ten structure of the decimal number system. They can be used to reveal the 'multiply by ten' and 'divide by ten' relationships that exist between adjacent column headings in the place-value system. Dienes are also useful for exploring the concept of 'trading', by regrouping and exchanging within the base-ten structure, providing support for mastery of the standard columnar methods for addition and subtraction.

Dienes are directly related to thousands, hundreds, tens and ones. However, their use can be extended to decimals by using:

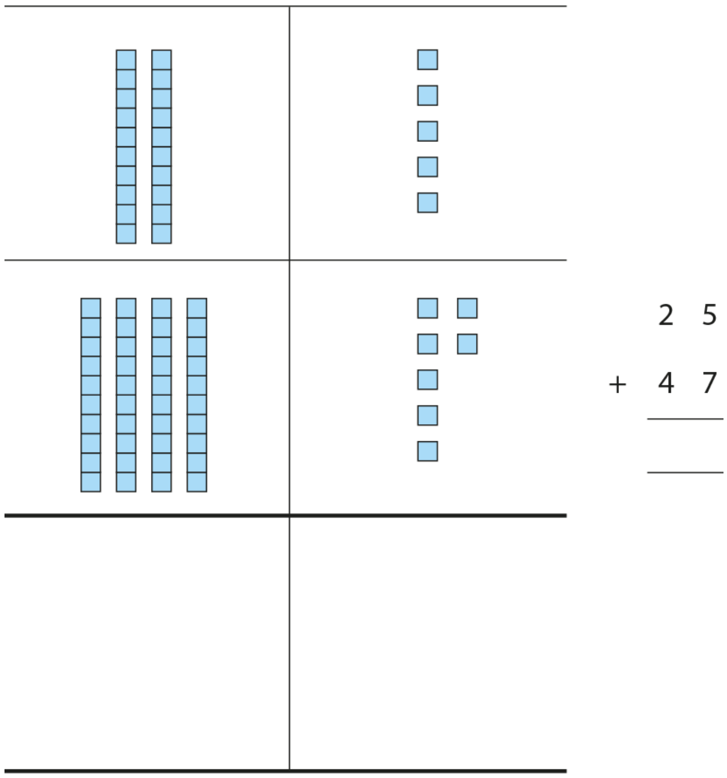
- the hundreds square to represent 1, the tens rod to represent 0.1 and the ones cube to represent 0.01, or
- the thousands block to represent 1, the hundreds square to represent 0.1, the tens rod to represent 0.01 and the ones cube to represent 0.001.

Students may have used one or the other of the above to support their work on decimals at primary school. It is important to spend some time ensuring that students understand what is being represented, before using Dienes to support understanding of structure and calculation strategies.

How they might be used

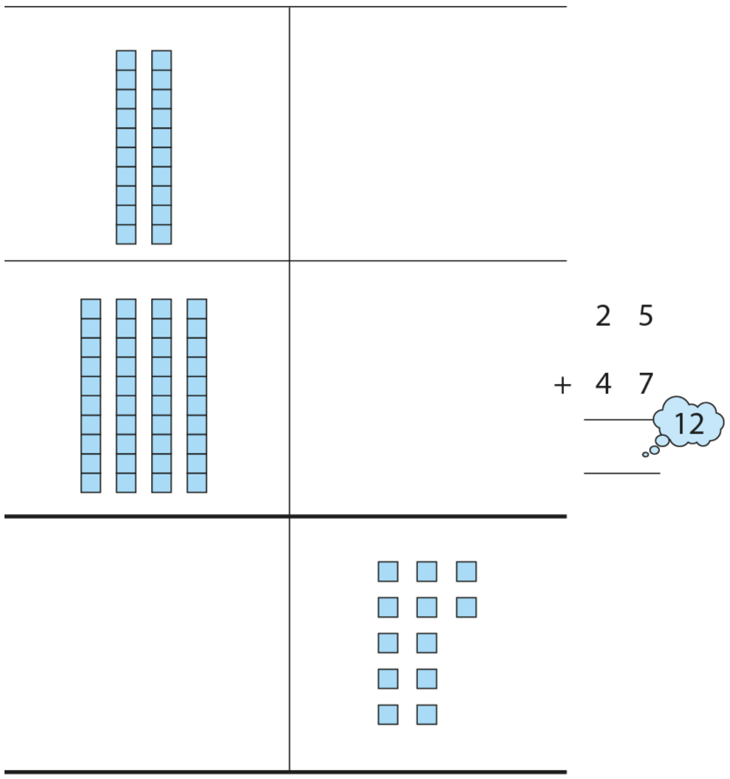
Addition

At primary school, when students first met problems involving the addition of 2 two-digit numbers, it's likely they will have been introduced to the column addition method, where digits are written in columns that represent place value. Dienes can help to make the base-ten structure visible to students and support an understanding of the process of 'regrouping' and 'exchange', i.e. 10 ones for a ten (or 10 tens for a hundred). Using Dienes alongside the number symbols can help students to appreciate what each digit represents. For example, the addition 25 + 47 can be represented using Dienes, with the addends arranged into tens and ones columns.

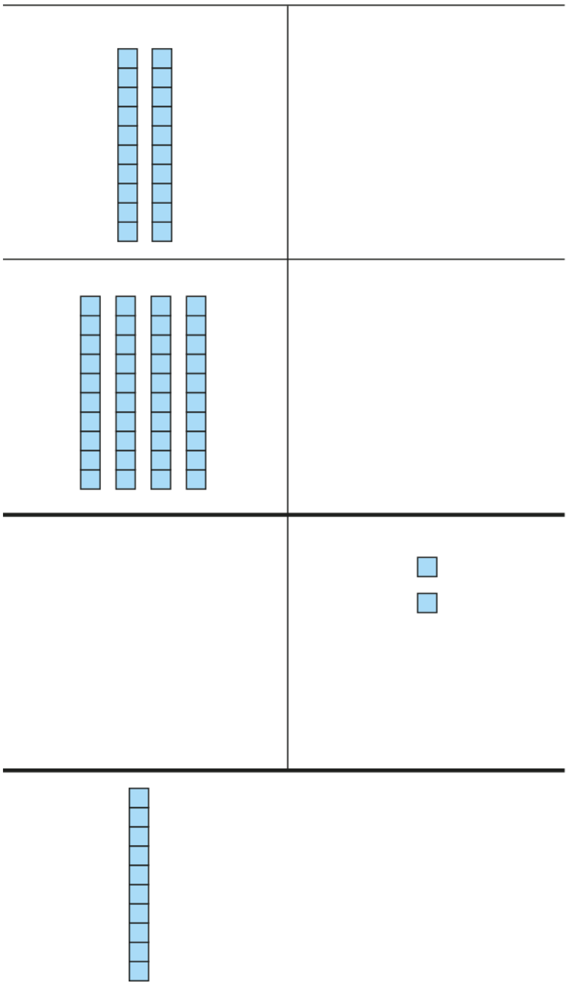


It is important not to label the columns with place-value headings. This results in incorrect representation of value (for example, four tens rods placed in a column labelled '10s' would represent a value of 40 tens, i.e. 400).

Regrouping the Dienes so that all the ones are together (i.e. adding the 5 and the 7) gives a total of 12 ones.

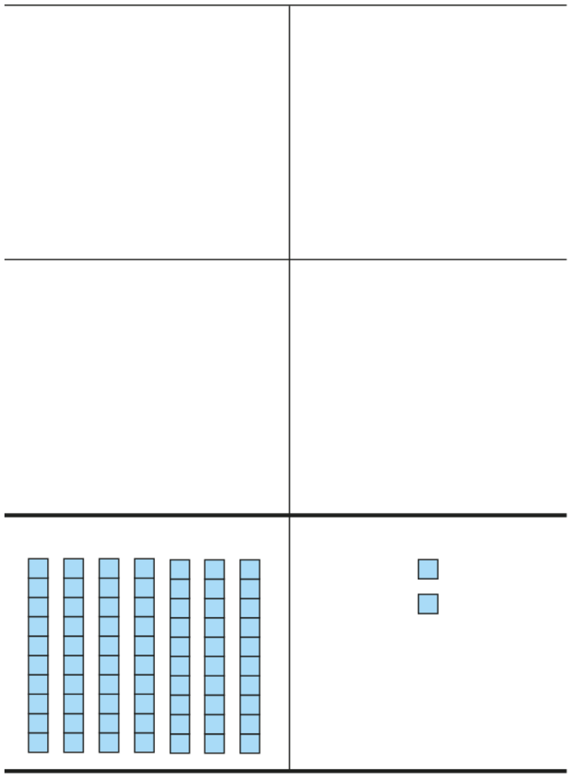


Ten of the 12 ones can be exchanged for a rod, to leave just two ones. Encourage students to reason *why* a total of 10 ones should be exchanged for a rod (ten).



$$\begin{array}{rr} & 25 \\ + & 47 \\ \hline & 2 \\ \hline & {\scriptstyle 1} \end{array}$$

Regrouping the Dienes so that all the tens are together (i.e. adding the 20 and the 40 and the 10 carried over) gives a total of 7 tens (70), and the solution of 72 can be obtained.



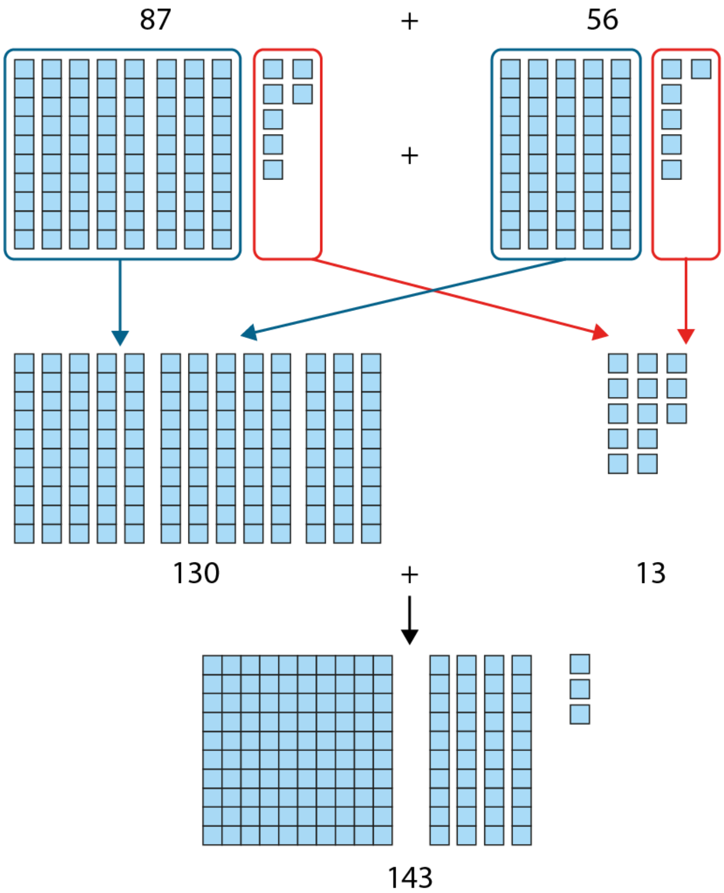
$$\begin{array}{r} 2\ 5 \\ +\ 4\ 7 \\ \hline 7\ 2 \\ \hline {\scriptstyle 1} \end{array}$$

This process of using Dienes alongside the symbolic representation enables students to make sense of the addition algorithm and so use it with fluency and understanding. Once students have a full appreciation of what each digit represents, however, they should be confident to work with just the symbolic representation, relying less and less on Dienes to model the column addition process.

It is important for students to have experienced regrouping of both ones and tens, and to have explored a variety of calculations, to be able to understand the structure of the addition algorithm for different scenarios.

From their work at primary school, students should be familiar, for example, with calculations where:

- both the tens and ones columns require regrouping (both due to the original number values, e.g. 426 + 397, or where regrouping of the tens is 'caused' by regrouping of the ones, e.g. 148 + 253)
- there are several addends which add up to a number greater than 20 in a given column (e.g. 18 + 36 + 29) so students don't begin to believe that you can only 'carry a one'
- the addends are two-digit numbers that sum to greater than 100; here we begin with only two columns in the algorithm, but end up with three in the sum, e.g. 87 + 56 = 143:



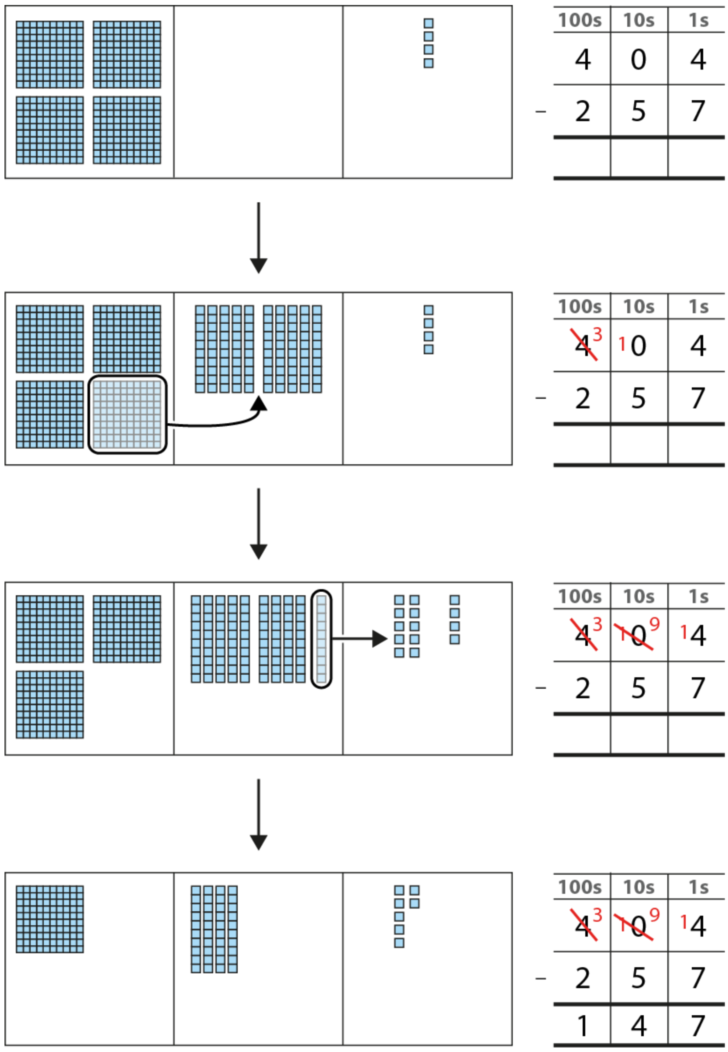
$$\begin{array}{r} 8\ 7 \\ +\ \ 5\ 6 \\ \hline 1\ 4\ 3 \\ \hline \end{array}$$

Once students have grasped the process of regrouping across one boundary (for example, the hundreds boundary, as above), extend to the thousands boundary and beyond. Regrouping across boundaries should become a generalisation of the process of addition, with students able to apply their understanding to a variety of scenarios and be confident in the constituent steps.

Subtraction

When Dienes are used to model problems involving subtraction, the process of 'exchanging' is a fundamental part of the calculation process itself, as tens (or hundreds) often need to be 'ungrouped' to units (or tens) to enable the subtraction to take place. It is important for students to recognise that if there is an insufficient number of any unit to subtract from in any given column, we must exchange from the column to the left.

There are a number of different scenarios, and students may have experience of some or all of these at primary level. For example, it is likely that students will be familiar with and have mastered exchanging tens for ones. They may have progressed to calculations with three-digit numbers, exchanging 1 hundred for 10 tens. It is important for students to identify what is the same and what is different, compared with exchanging 1 ten with 10 ones. Using Dienes to explore more complex calculations, where a variety of exchanges are required, enables students to understand the structure of the calculation, including, for example, scenarios that require 'working through a zero', as in the case of 404 – 257, shown below.



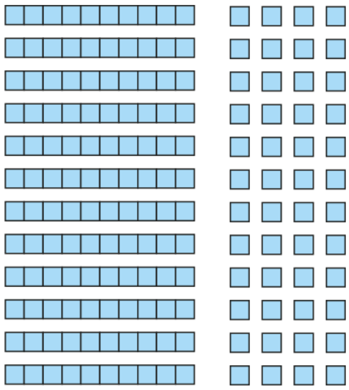
In this example, the 7 cannot be subtracted from the 4 in the 1s column and neither can the 5 from the 0 in the 10s column. This adds to the complexity of the exchange process. It means that the 100 exchanged for 10 tens is not sufficient for the calculation to proceed, and so a ten has to be exchanged for 10 ones. The use of Dienes supports students in making sense of the vertical column layout of subtraction and in understanding that the standard column method involves rewriting 404 as 300 plus 90 plus 14. The awareness that this method of redistributing the minuend can be used to ensure that the minuend in each column will always be larger than the corresponding subtrahend, is an important one.

When using Dienes to support students in solving problems involving subtraction, it is important that the equipment is used to support students' understanding of the *structure* of the algorithm and not just as a tool for finding the answer. Encourage students to use known facts to perform the calculation in each column.

Multiplication

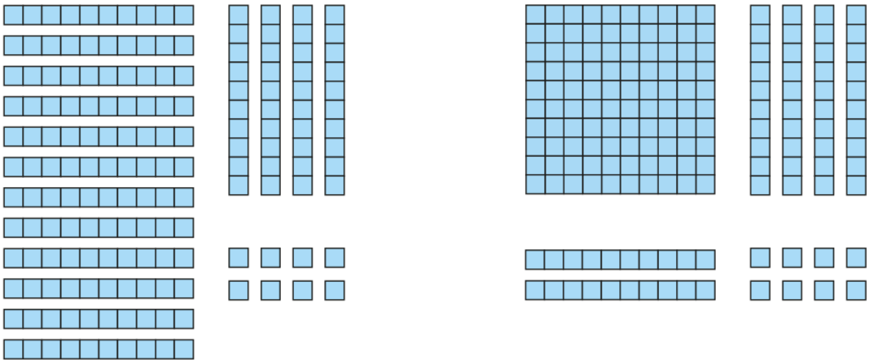
At primary level, students may have used Dienes to represent multiplication problems and to help make sense of the standard algorithm for multiplying numbers by a one- or two-digit number. For example, when multiplying 14 by 12, 14 can be represented with 1 rod and 4 ones.

Students are likely to be familiar with using Dienes to represent multiplications involving a two-digit number multiplied by a one-digit number as repeated addition of the two-digit number, repeated 'one-digit number' times. This can be applied in the same way to multiplying 2 two-digit numbers.



		1	4
×		1	2

When using Dienes, it is important for students to recognise when an exchange can take place. Exchanging some of the ones for tens and 10 tens for a hundred results in a representation using the least possible number of blocks.



		1	4
×		1	2
		2	8
	1	4	0
	1	6	8

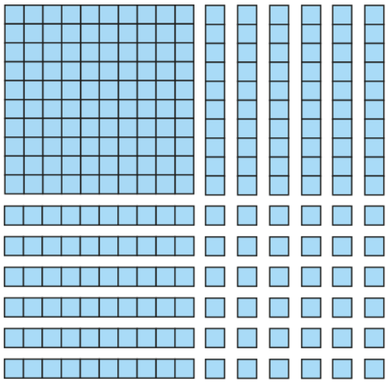
It is likely that students will not have seen vertical and horizontal rods mixed like this in the same representation.

It is again helpful to consider the symbolic representation of the calculation, together with Dienes, to help students see the link between the manipulation of the materials and the symbolic manipulation. The first stage in multiplying 14 by 12 is to multiply all of the digits in one of the numbers by the number of ones in the other number. Carrying out this calculation gives 28, and this amount can be seen in the bottom two rows of the Dienes representation. Exchanging 10 tens for a hundred may help students to see the other amount in the calculation when multiplying the 14 by the digit in the 10s column (remembering to insert the zero in the 1s column, because the 1 is in the 10s column so represents 10) as shown by the blocks in the top section of the representation. The total of the two 'sections' gives the overall solution of 1 square, 6 rods and 8 cubes (168).

As well as thinking about the Dienes representation as the two sections described, the array contains four 'same-block' rectangles, one made from a square alone, two made from rods alone and one made from cubes alone. Recognising that the sum of the blocks in the four regions is equivalent to the two numbers multiplied together emphasises the distributive law of multiplication:
$14 \times 12 = (10 \times 10) + (4 \times 10) + (10 \times 2) + (4 \times 2) = 168$.

Squares as a special case

Students may notice that when squaring a number, the Dienes array takes the form of a square. They may also notice that the two rectangles that are made up of rods only are always the same size but are oriented differently. For example, 16 multiplied by 16 can be represented using Dienes as follows:

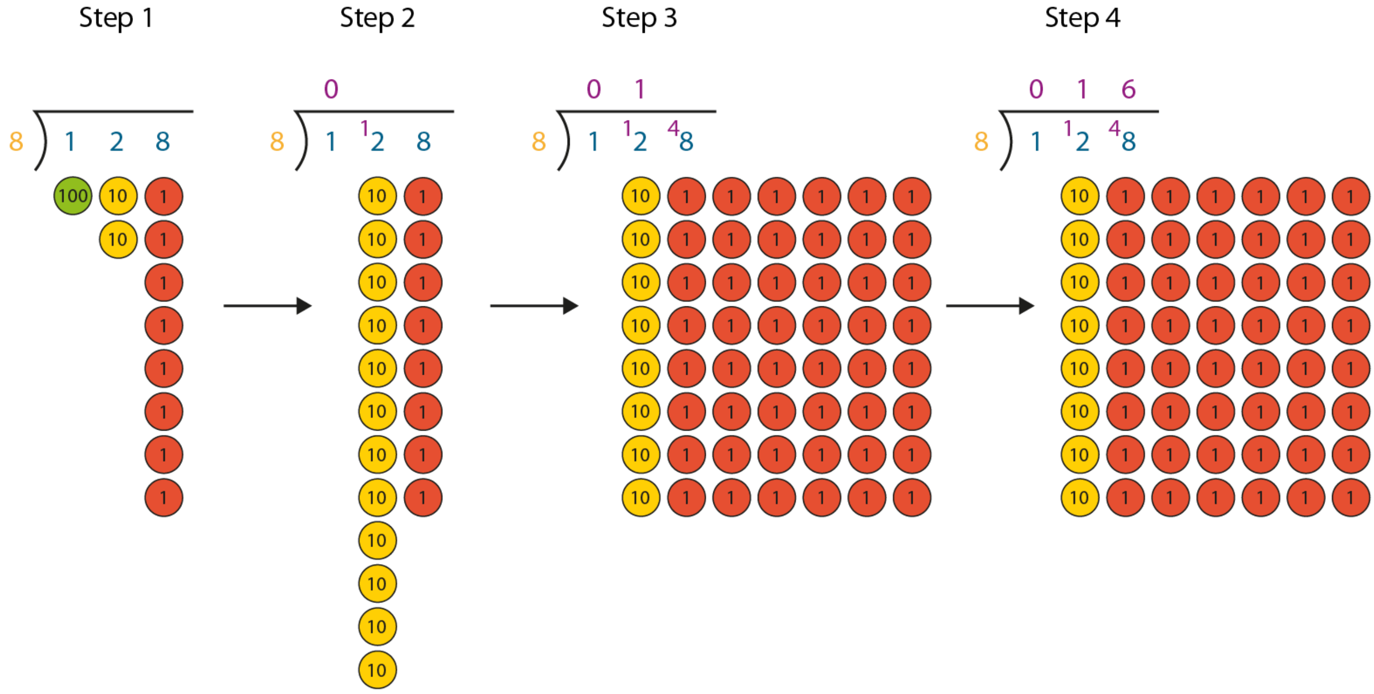


As students begin to work with larger numbers, using Dienes to model multiplication can become unwieldy, and it may no longer be appropriate for students to represent multiplications in this way. It is important that Dienes are used to support students' understanding of the structure of the algorithm and not just as a tool for finding the answer. So, once students are confident in their grasp of the base-ten structure of the number system, encourage them to perform multiplication calculations without the need to represent them using Dienes.

Division using place-value counters

The progression from the use of Dienes to place-value counters is not insignificant, and it's important to understand the different ways in which Dienes and counters model how numbers are represented. Dienes are an effective resource for developing a sense of number size. This is because there is a direct and accurate relationship between the size of each block and its value (for example, the hundreds square is ten times bigger than the tens rod, which in turn is made up of 10 ones). Once students are confident in the sense of relative size that Dienes provide, place-value counters are a useful representation and can be helpful when modelling division.

For example, the steps for dividing 128 by 8 show how the counters can initially be used to represent the dividend as $(1 \times 100) + (2 \times 10) + (8 \times 1)$. However, then the need to exchange the 100 counters for ten 10 counters arises as a result of not being able to divide the one in the hundreds column by the divisor of eight. Similarly, because dividing the 12 (in the tens column) by eight gives a remainder of four (tens), which cannot be partitioned into eight groups as needed, the four 10 counters have to be exchanged for forty 1 counters to enable the division to be completed.

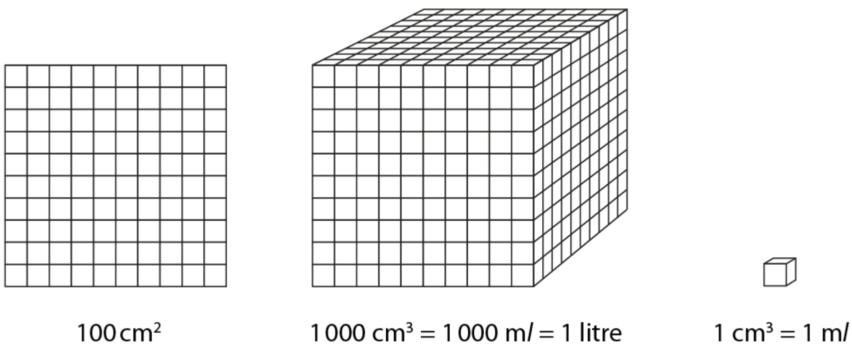


The step 4 diagram shows eight identical rows of counters, each containing one 10 counter and six 1 counters, representing 128 partitioned into eight groups of 16, and so providing a visual representation of the division. It is important that when students are using the place-value counters, they recognise when and why an exchange needs to take place, and are able to directly relate this to the process of dividing one number by another.

Perimeter, area and volume

The squares along each face make Dienes excellent materials for visualising the concepts of perimeter and surface area. They can also form the basis for understanding volume.

Assigning the lengths of the sides of the small cubes (ones) as 1 cm, the cubes can be used for finding area in square centimetres, since the face of the cube has an area of 1 cm^2. The face of the hundreds square, therefore, has a surface area of 100 cm^2. Similarly, since the volume of a small cube with assigned side lengths 1 cm would be 1 cm^3, the large cube could be filled with 1 000 of the smaller cubes, exemplifying the volume of the larger cube as 1 000 cm^3. Students may also develop a sense of litres and millilitres as the capacity of a container that holds exactly 1 000 cubes and one small cube respectively.



Dienes can also provide a useful tool for supporting students when converting between different units of measure. For example, finding how many square centimetres there are in a square metre, or cubic centimetres in a cubic metre, etc.

Further resources

Whilst having hands-on equipment that students can use for themselves can be beneficial and enables student autonomy, Dienes may not always be practical for use with larger classes and cannot always be easily used for whole-class demonstrations. With advances in technology and interactive whiteboard software in classrooms, teachers have access to ways of manipulating images of hundreds, tens and ones onscreen, via free online resources with varying degrees of usability and appropriateness.

See, for example:

Coolmath4kids

https://www.coolmath4kids.com/manipulatives/base-ten-blocks

Clicking on the 3-dimensional ones cube, tens rod and hundreds square at the top of the screen generates the blocks in the work area beneath. There is a menu option that allows students to ungroup a tens rod into ones or a hundreds square into tens rods, and there is also an option to select and group 10 ones into a rod or 10 rods into a hundreds square. As well as generating blocks at the top of the screen, there is also the option to duplicate a selected object in the work area. Tens rods can be selected and rotated 90 degrees. Additional features include the ability to change the colour of the cubes, an onscreen extendable ruler, keypad and drawing tools.

MathsBot.com

https://mathsbot.com/manipulatives/blocks

3-dimensional (or 2-dimensional by deselecting the 3D tick box) Dienes (thousands, hundreds, tens and ones) can be dragged into the work area from the bottom left-hand corner of the screen. Double-clicking on a block breaks it down into ten (i.e. a thousands block into 10 hundreds, a hundreds square into 10 tens and a tens rod into 10 ones). The 'Exchange' button at the top of the screen does the same action. A block on the screen can be copied and positioned above, below or to the right or left of the existing block, using the arrow keys at the top of the screen. A square grid can be added to the work area, as well as horizontal and vertical lines. 'Tidy' clears the screen. In the control panel there is also the option to change from the default base ten to any number between 2 and 20. The control panel at the top of the screen can be hidden to give a full-screen work area.

CPM Tiles

https://technology.cpm.org/general/tiles/

Selecting 'Base Ten Blocks' from the menu on the left-hand side of the screen gives access to 2-dimensional ones, tens and hundreds blocks, which can be dragged and dropped into the work area. Double-clicking the tens rod rotates it 90 degrees. Blocks can be removed from the work area by dragging them back to the left-hand side of the screen.

ABCya.com

http://www.abcya.com/base_ten.htm

A basic drag-and-drop block generator, with onscreen display of the number of blocks generated (Thousands, Hundreds, Tens and Ones). The resource includes a sort feature with limited functionality. Blocks cannot be manipulated in any way, other than being moved around the screen.

The importance of students being able to handle and manipulate blocks of this type for themselves, to support them in solving number problems, should not be underestimated and so the somewhat cumbersome nature of sets of physical blocks versus the online virtual Dienes has to be weighed up. The function that some of the online resources described above have, of being able to ungroup a block (i.e. a tens rod can be separated into 10 ones cubes) rather than exchanging it, is an interesting one and a discussion of the difference between these two strategies (and the way in which the physical Dienes don't allow for this) can be encouraged. Linking the use of Dienes to the symbolic representation of a calculation should always feature when using both the familiar manipulatives and the online versions of the Dienes and, in so doing, students' understanding of the place-value structure can be supported and developed.

When students are no longer reliant on the relative size of Dienes and transition to place-value counters as a means of modelling division, for example, an online resource can be used to generate counters that can be arranged onscreen.

See, for example:

MathsBot.com
https://mathsbot.com/manipulatives/placeValueCounters
Place-value counters can be dragged into the work area from the top left-hand corner of the screen. Default counters are set to 1, 10, 100 and 1 000, but the range of counters can be selected from powers of ten between 0.000001 and 1 000 000 inclusive. Double-clicking on a counter that has already been generated, duplicates it. Counters can also be duplicated using the copy arrows in the control panel at the top of the screen. The newly generated counter is positioned according to the selected arrow, either above, below or to either side of the original counter. The 'Exchange Down' feature replaces a counter with ten equivalent counters, i.e. a 10 counter is replaced with ten 1 counters. In a similar way, ten of the same counters can be exchanged up using the 'Exchange Up' feature, i.e. ten 1 counters are replaced with a 10 counter. There is the ability to change the base of the exchange from ten to any whole number between 2 and 20, e.g. with a base of seven selected, a 10 counter will be exchanged for seven 1 counters. However, as the counters are always powers of ten, this feature has no real sensible use. There is the option to add a square grid and a table with column headings (i.e. 1 000s, 100s, 10s and 1s, etc.) that resembles a place-value chart. The column headings of the table can be represented as words rather than figures by ticking the 'Worded Values' box. It is important to use only 1 counters in this table, otherwise an incorrect representation will occur (for example, a 10 counter in the 10s column would represent a value of 10 tens, i.e. 100).