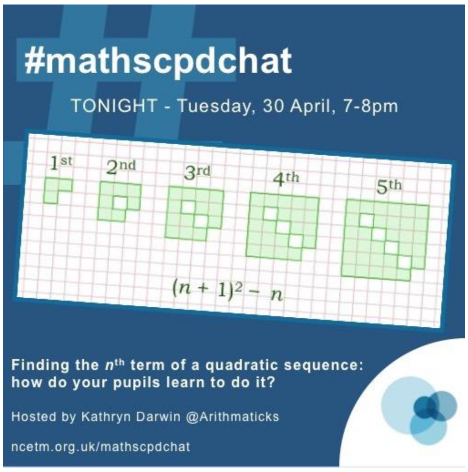
National Centre for Excellence in the Teaching of Mathematics

#mathscpdchat 30 April 2019

Finding the *n*th term of a quadratic sequence: how do your pupils learn to do it?
Hosted by Kathryn Darwin

*This is a brief summary of the discussion – to see all the tweets, follow the hashtag **#mathscpdchat** in Twitter*



Some of the areas where discussion focussed were:

- expectations that pupils' understanding and use of **quadratic sequences will 'flow on naturally from'** their understanding of, and ability to use, **linear sequences**;
- requiring pupils to be **able to solve simultaneous linear equations** before trying to find the general term of a quadratic sequence … the method used being 'write the general term as $an^2 + bn + c$, then create three simultaneous equations by substituting in that equation, in turn, $n = 1$, $n = 2$ and $n = 3$' and equating the resulting expressions to the appropriate (1st, 2nd, 3rd) terms of the given sequence;
- that there are at least **five different methods** for (ways of) finding the *n*th term of a quadratic sequence;

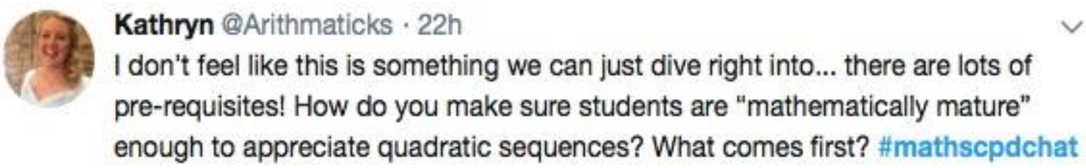
- that studying **quadratics is a bridge between linear and higher polynomials** … which is why they are studied in KS4 … therefore methods used with quadratic sequences should be **generalisable to higher polynomials**;
- pupils appreciating and **using the general form of a quadratic expression: an^2 + bn + c** … representing the coefficients by letters of the alphabet used **in 'reverse-alphabetic' order** so that when comparing linear/quadratic/cubic expressions **'a' is always the constant term**;
- writing the general term of 'the k times table sequence' as 'kn' … seeing that the **coefficient of *n* and the 'gap' between terms of a linear sequence are the same** … adapting/extending this understanding when looking at 'second differences' of quadratic sequences;
- **building up gradually to using algebraic 'methods'** for finding the n^{th} term of a quadratic sequence … with the aim of **avoiding** such algebraic 'methods' becoming, for pupils, **memorised routines that they don't understand** … eg looking at the second differences in sequences with general terms n^2, then $2n^2$, then $3n^2$, then $4n^2$, … providing opportunities for pupils to **see (find out) for themselves** how the coefficient of n^2 and the second difference are related;
- seeing the general term of a quadratic sequence as composed of a **simple quadratic part (an^2) together with a linear part (bn + c)** … given some terms of a quadratic sequence an^2 + bn + c, subtracting terms of the (guessed at) related simple quadratic sequence an^2 to get the related linear sequence bn + c … for example, given the (quadratic) sequence 2, 10, 24, 44, 70, 102, …, subtracting from it the (simpler quadratic) sequence 0, 3, 12, 27, 48, 75, …, to get the (linear) sequence 2, 7, 12, 17, 22, 27, 32, …;
- looking first at sequences with general term **n^2 + c**, then at some with general term **n^2 + bn**, then at some with general term **an^2**;
- **finding the n^{th} term of a given quadratic sequence by comparing it with the general case** … that is, comparing numbers with general expressions in tables showing (in separate rows) terms, first differences and second differences; then, by equating numbers with corresponding algebraic expressions, forming and solving linear equations;
- using visual-image-sequences to enable pupils **literally to see the quadratic, linear and constant components**;
- ways of 'just seeing' an expression for the n^{th} term when you've got (or a pupil has created) a **visual sequence that gives-rise to a quadratic sequence**;
- sequences of figurative patterns that allow pupils to **find the n^{th} term of a quadratic sequence without recourse to an 'automated rule'** … pupils seeing in various

different ways the ‘shapes’ (eg ‘shapes’ composed of tiny circles or unit-squares) in a **sequence of ‘shapes’** as sharing a common structure … and thereby arriving at **equivalent quadratic expressions for the *n*th term of the sequence formed by the number of ‘units’ (eg tiny circles or unit-squares) in each ‘shape’** …pupils creating their own such sequences of ‘shapes’;

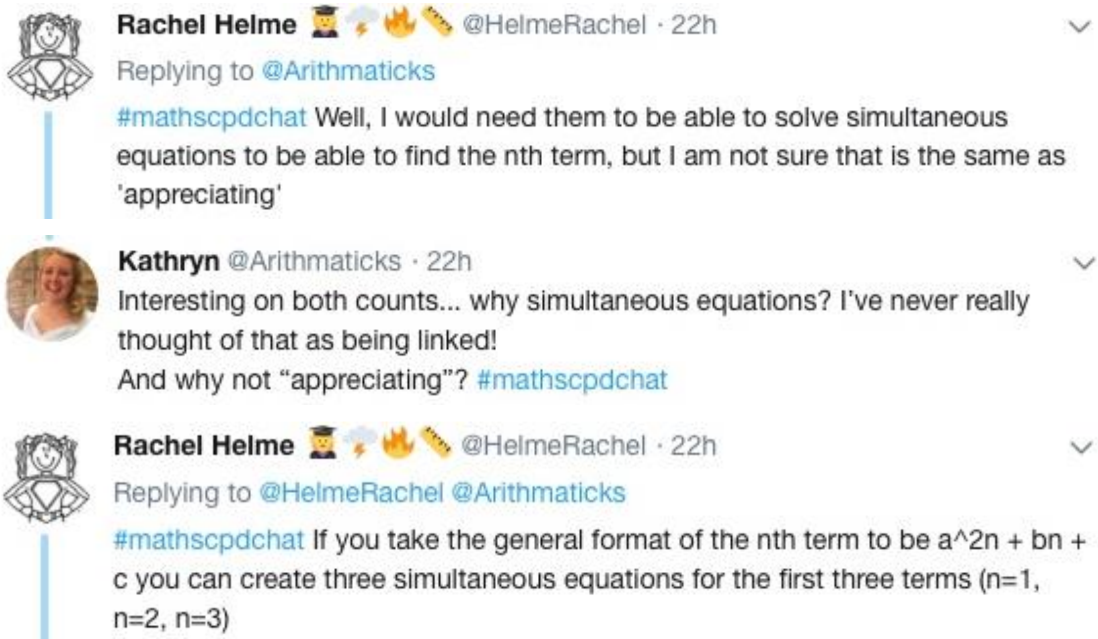
- that **a few given terms of a sequence do not uniquely define a sequence** … they may be terms of several (many) different sequences, each with a different polynomial as its general term.

In what follows, click on any screenshot-of-a-tweet to go to that actual tweet on Twitter.

An interesting ‘conversation’ of tweets, about some issues that arise when considering various ways of finding the *n*th term of a quadratic sequence, followed from this tweet by Kathryn Darwin:


Kathryn @Arithmaticks · 22h
I don't feel like this is something we can just dive right into... there are lots of pre-requisites! How do you make sure students are “mathematically mature” enough to appreciate quadratic sequences? What comes first? #mathscpdchat

including these from Rachel Helme and Kathryn Darwin:


Rachel Helme @HelmeRachel · 22h
Replying to @Arithmaticks
#mathscpdchat Well, I would need them to be able to solve simultaneous equations to be able to find the nth term, but I am not sure that is the same as 'appreciating'

Kathryn @Arithmaticks · 22h
Interesting on both counts... why simultaneous equations? I've never really thought of that as being linked!
And why not “appreciating”? #mathscpdchat

Rachel Helme @HelmeRachel · 22h
Replying to @HelmeRachel @Arithmaticks
#mathscpdchat If you take the general format of the nth term to be a^2n + bn + c you can create three simultaneous equations for the first three terms (n=1, n=2, n=3)

these from Mary Pardoe, stefn dreaves and Colin Foster:

Mary Pardoe @PardoeMary · 22h

Replying to @HelmeRachel @Arithmaticks

This is a VERY useful and clear description (by Colin Foster @colinfoster77) of FIVE ways of finding the nth term of a quadratic sequence, one of which requires pupils to solve simultaneous equations ... #mathscpdchat foster77.co.uk/Foster,%20Math ...

Pros and Cons of Different Ways of Finding the *n*th Term of a Sequence of Numbers

by Colin Foster

A quick informal survey among colleagues revealed a number of different approaches to finding an expression for the *n*th term of a sequence of numbers. We give a lot of attention to this particular skill because of its usefulness in the GCSE coursework investigation, so it seemed worth considering the pros and cons of the different methods.

Method 1. Inspection

The quickest approach to many sequences, though the hardest to learn, is the "Ah, I see what it is!" method! Sometimes the clue comes from the source of the sequence, perhaps the geometry of a pattern, for example. Sometimes, it is possible to see how numbers with not too many factors might have been produced.

n	1	2	3	4	5
u	8	15	24	35	48

For example, in this sequence, 15 suggests itself as 3×5, and 35 as 5×7. Checking this pattern for the other terms quickly confirms that $u = (n + 1)(n + 3) = n^2 + 4n + 3$.

Pupils frequently need to be encouraged that inspection is a 'proper method' and not 'cheating', although they'll need to explain that it wasn't simply inspection of someone else's work!

Method 2. Simultaneous Equations

All the other methods begin by finding the difference between successive values, and then the differences between those successive differences, and so on until a row of differences turn out to be constant. For example,

n	1		2		3		4		5
u	8		15		24		35		48
1st differences		7		9		11		13	
2nd differences			2		2		2		

And so on, where a, b, c, d, etc. are all constants to be determined. (The final constant in each equation can't be zero, otherwise the sequence wouldn't have the required degree, but some or all of the other constants could be zero.)

The simultaneous equations method starts with the appropriate formula (quadratic for our example), and with three unknowns to find (a, b and c) substitutes n and u values for the first three terms, as below.

$$u = a + bn + cn^2$$
$$n = 1 \quad 8 = a + b + c$$
$$n = 2 \quad 15 = a + 2b + 4c$$
$$n = 3 \quad 24 = a + 3b + 9c$$

Then the task is to solve these simultaneous equations to find a, b and c and hence the formula for u.

The main difficulty with this method is solving these simultaneous equations, which for a cubic (or higher) sequence can be heavy work. GCSE pupils are not normally expected to solve simultaneous equations in more than two unknowns, and three is necessary even for a quadratic sequence. For that reason I don't generally encourage pupils down this route.

Method 3. Comparison with the General Case

This begins by finding differences until a row of constant differences are obtained (as above) so that the degree of u is found. Then the differences method is applied to the *general* case of that degree, as below for a quadratic sequence $u = a + bn + cn^2$.

n	1		2		3		4
u	$a+b+c$		$a+2b+4c$		$a+3b+9c$		$a+4b+16c$
1st differences		$b+3c$		$b+5c$		$b+7c$	
2nd differences			$2c$		$2c$		$2c$

stefn dreaves @begbiesan · 22h

Nice comparisons from Colin Foster. Though I see an issue with using a,b,c in the reverse order to what we/students normally expect to see.

ie a is the coeff of the square term, and c is the constant = Much more conventional ...

Colin Foster @colinfoster77 · 22h

Yes, I think I did it the other way round so that when comparing linear/quadratic/cubic 'a' was always the constant term, etc.

and these from Richard Trimble and Anne Watson:

Richard Trimble @RichardTrimble7 · 18h

Replying to @PardoeMary @HelmeRachel and 2 others

Late to the party but must share this lovely method from g'day maths. It has an approach based on the leading diagonal of the differences and feels like vectors. I used it for the first time this year with a weak higher GCSE group and they took to it well.

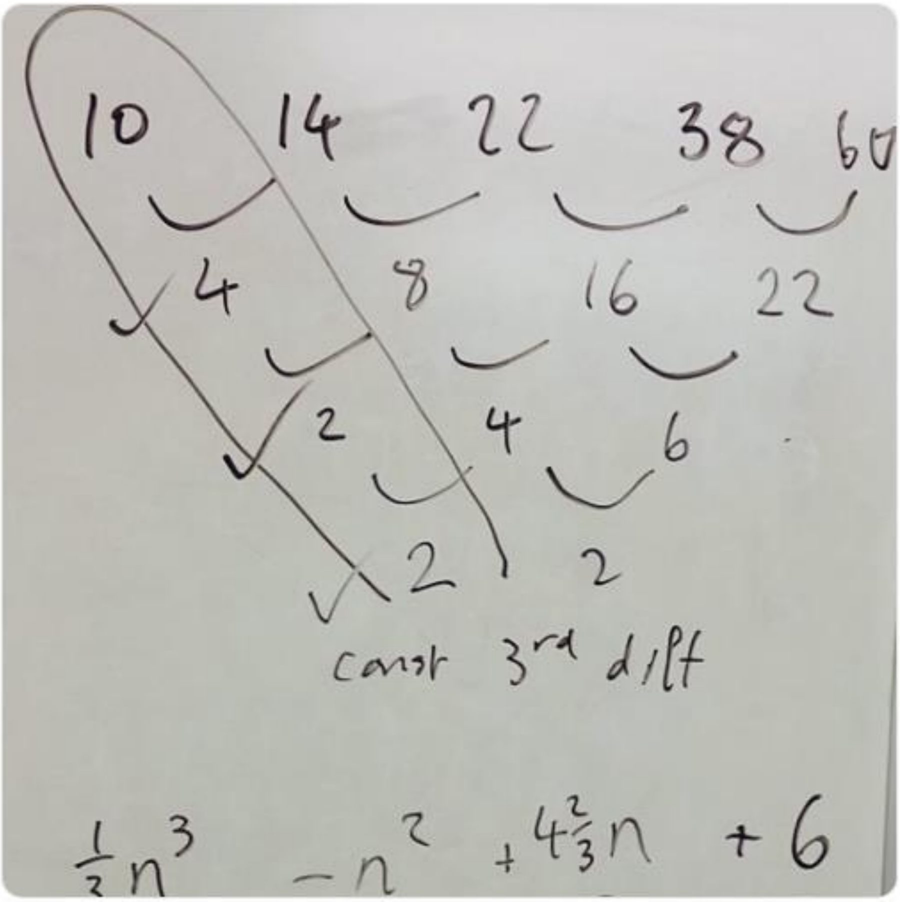
> **1.2 Sequences - Getting Formulas**
> Here is the constant sequence (always 1) and its leading diagonal: Here are the counting numbers (given by \(n\)) and their leading diagonal: Here are
> gdaymath.com

Anne Watson @annemathswatson · 9h

Studying quadratics is a bridge between linear (with too many simplistic assumptions) and higher polynomials - this surely is why they are in the curriculum. If treated as special, opportunities are lost to generalise to higher polynomials.

Richard Trimble @RichardTrimble7 · 20m

That's why I found the approach using the leading diagonal so interesting! A friendly colleague set my class 10,14,22,38 by mistake. We were able to quickly work out the leading diagonal for n^3 and extend the method. With other approaches we'd have needed a lot of heavy lifting.



(to read the discussion-sequence generated by any tweet look at the ‘replies’ to that tweet)

Among the links shared were:

Differences over Differences Methods: Pros and Cons of Different Ways of Finding the *n*th Term of a Sequence of Numbers which is an article by Dr Colin Foster, published in *Mathematics In School, November 2004* (which is a journal of the *Mathematical Association*). In this useful document Colin Foster presents very clearly five different ‘methods’ that pupils in UK secondary schools are ‘taught’ for finding the *n*th term of a quadratic sequence, and he compares and evaluates them. It was shared by Mary Pardoe

New at GCSE: Quadratic sequences which is an article in the *NCETM Secondary Magazine 141, March 2017*. It contains examples of tasks that provide opportunities for pupils to see the structure of visual patterns in various different ways, and thereby arrive at equivalent expressions for the *n*th terms of quadratic sequences derived from sequences of the patterns. Each ‘just seeing’ method is compared with an algebraic method (which is explained). It was shared by Mary Pardoe

Visual Patterns which is a source of images that can be used to form sequences from which quadratic sequences can be derived. It was shared by Martin Brown

Desmos: interpolate which is a Desmos application; users can specify a small number of values of x along with corresponding values of y, and then ‘ask’ the application to determine a function, $y = f(x)$, whose graph passes through the specified points. It was shared by stefn dreaves

G'DAY MATH: Quadratics: 1.2 Sequenes - Getting Formulas which consists of a video overview with lesson materials located below the video. The method demonstrated makes use of the numbers that appear in ‘leading diagonal columns’ when 1st, 2nd, 3rd, … differences for linear, quadratic, cubic, and ‘higher-powers-of-*n*’ sequences are written in rows one-below-the-other with the terms of each row lined-up with the-gaps-between-terms of the row above. Interesting, but likely to be used by pupils without any understanding of why it works, ‘blindly’ as ‘a trick that works’. It was shared by Richard Trimble

Find nth term of a quadratic Sequence using a Classwiz Calculator which is a YouTube video published by The Calculator Guide in 2018. It shows ‘a quick method of finding the *n*th term of a quadratic sequence outside of an exam'. A warning is displayed: 'this method may not give any of the marks in an examination'! It was shared by The Calculator Guide