MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Fractions

3.3 Non-unit fractions: identifying, representing and comparing

Teacher guide | Year 3

Non-unit fractions

Teaching point 1:

All non-unit fractions are made up of more than one of the same unit fraction.

Teaching point 2:

Non-unit fractions are written using the same convention as unit fractions. A non-unit fraction has a numerator greater than one.

Teaching point 3:

When the numerator and the denominator in a fraction are the same, the fraction is equivalent to one whole.

Fractions as numbers

Teaching point 4:

All unit and non-unit fractions are numbers that can be placed on a number line.

Teaching point 5:

Repeated addition of a unit fraction results in a non-unit fraction.

Teaching point 6:

When the numerator and the denominator are the same, the value of the fraction is one.

Comparing fractions

Teaching point 7:

Non-unit fractions with the same denominator can be compared. If the denominators are the same, then the greater the numerator, the greater the fraction.

Teaching point 8:

Non-unit fractions with the same numerator can be compared. If the numerators are the same, then the greater the denominator, the smaller the fraction.

Overview of learning

In this segment children will:

- learn how non-unit fractions are written
- understand how non-unit fractions link to repeated addition of unit fractions of the same denominator
- learn to compare non-unit fractions in different ways, for example by comparing the numerators or comparing the denominators, and learn when to apply each comparison
- explore what it means when the numerators and denominators are the same.

The teaching points in this segment cover three areas of teaching: non-unit fractions, fractions as numbers and comparing fractions.

Non-unit fractions

In segment *3.2 Unit fractions: identifying, representing and comparing*, children learnt how to name unit fractions (e.g. one-third) and write unit fractions using the correct notation (e.g. $\frac{1}{3}$). In this segment, children extend this learning to non-unit fractions. Note that at this stage the focus is on proper fractions and fractions with the same numerator and denominator.

Non-unit fractions are introduced through their connection to unit fractions. They are simply 'multiples' of unit fractions, for example five-sixths is five <u>one</u>-sixths, or five units of $\frac{1}{6}$. It is useful to make this connection to support children in transferring what they already know about unit fractions to non-unit fractions. The dual language of 'five-sixths' and 'five one-sixths' is used within this segment, and also later on in this spine when children are likely to find it helpful to unitise. The clear emphasis on unitising throughout, is vital to prepare children for subsequent learning involving adding fractions, converting mixed numbers into improper fractions and vice versa.

Throughout this unit, children will be exposed to non-unit fractions represented in area models, in linear models and as quantities. Where quantities are used, you will notice that all of the suggested representations for sets of objects are clearly partitioned into equal parts (for example, a set of 12 partitioned into three groups with four in each group). This segment does not go as far as saying that two-thirds of twelve is eight; instead the focus is on seeing three equal parts and highlighting two of them, in the same way as for fractions of shapes and lines. This emphasis on the language of wholes and equal parts (how to identify parts within sets, alongside parts of area and linear representations), builds a strong foundation from which to progress to fractions of quantities in later segments.

Fractions as numbers

The teaching in the middle of this segment progresses to an absolutely critical point. So far, fractions have been used as an operator; talking about one-third *of* this shape or one-fifth *of* this line. However, as well as operators (e.g. $\frac{1}{4}$ *of*), fractions are also *numbers* (e.g. the *number* $\frac{1}{4}$, which also has a value of 0.25). It may seem obvious to adults that fractions are numbers, but it is not unusual for children to think that fractions are just a part of something, and for them to be really quite surprised to discover they are numbers too.

Because fractions are numbers, they can be placed on a number line. Children have already looked at fractions *of* lines in this segment, but this is not the same as a fraction *on* a number line. The distinction will be made between one-fourth of this line (a *part* of a line) and the number one-fourth ($\frac{1}{4}$) that sits at a *point* on a number line. When discussing numbers that are fractions, they are often referred to as 'fractions' rather than 'numbers', for example, in discussing where two-fifths sits in the number system, one might ask, '*Where would we place this* fraction *on the number line?*' Develop the habit of alternating this with language such as *'Where would we place this* number *on the number line?'*, so that children become used to hearing fractions referred to as numbers. Later in this spine, children will also learn how fractions can be used in calculation in the same way as integers.

Comparing fractions

Once children understand that fractions are numbers, they are ready to explore how fractions can be ordered and compared – just as they have already learnt to do with integers. In segment *3.2*, children had an initial introduction to comparing unit fractions of the same whole. This is reviewed once children have met fractions as numbers; they will go back to unit fractions and place them on a number line as an additional model for comparing unit fractions.

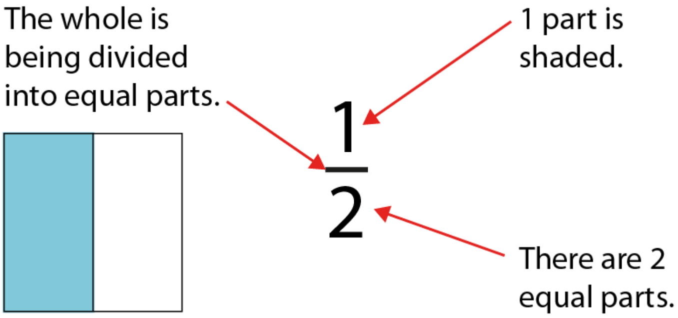
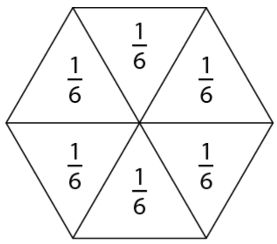
Teaching point 7 then proceeds to comparing non-unit fractions. Non-unit fractions can be easily compared when the denominator is the same, and children will learn that, for example, $\frac{5}{12} > \frac{3}{12}$. Fractions can also be easily compared when they have the same numerator. Children will learn how to compare fractions such as $\frac{4}{9}$ and $\frac{4}{5}$, using a 'same numerator' approach.

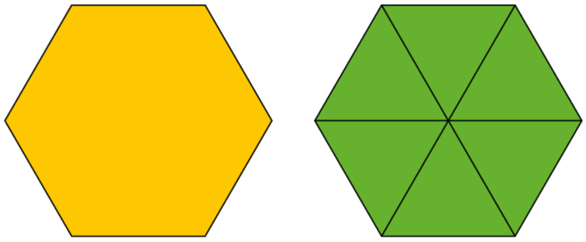
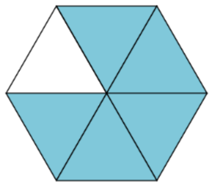
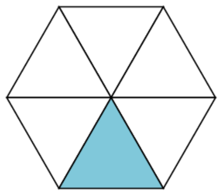
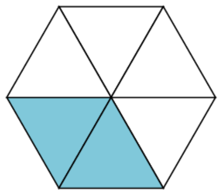
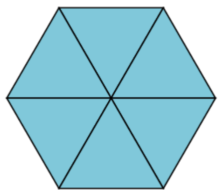
An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

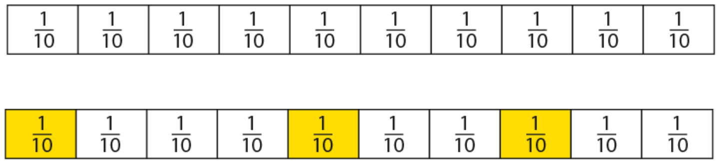
Teaching point 1:

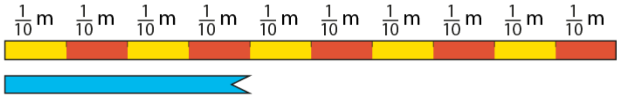
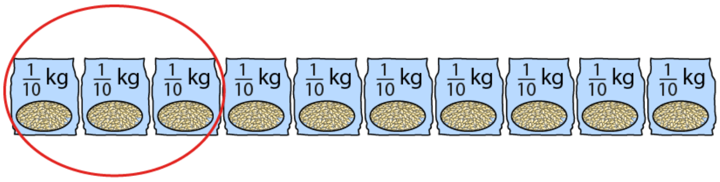
All non-unit fractions are made up of more than one of the same unit fraction.

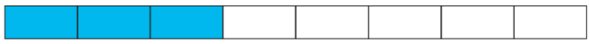
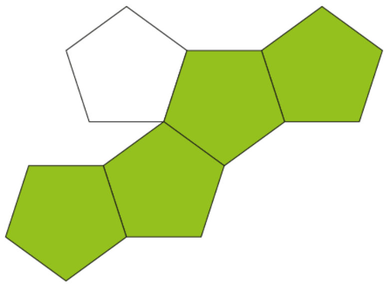
Steps in learning

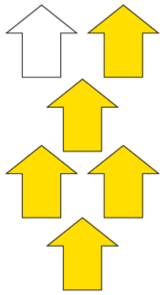
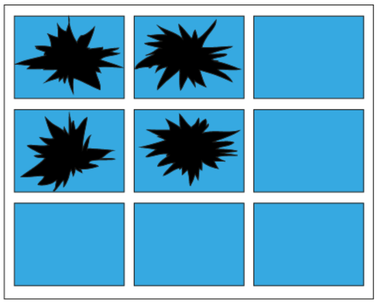
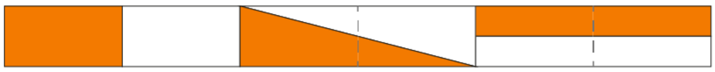
	Guidance	Representations
1:1	In segment *3.2 Unit fractions: identifying, representing and comparing*, children learnt to name, write and compare unit fractions. This segment focuses on non-unit fractions and how they are composed of more than one of a unit fraction. For example, four lots of one-sixth can be referred to as four-sixths. This concept is explored using area, linear and quantity models. Children already have an understanding of fractional notation in relation to equal parts and unit fractions. Begin this segment by reviewing previous learning. Match images of unit fractions to their correct notation, ensuring that a variety of area, linear and quantity models are presented. Recap how a fraction is written (see segment *3.2*, step *2:2*). We draw the fraction bar first to show that we are splitting the whole into equal parts. We then write the denominator, which tells us the total number of equal parts. Finally, we write the numerator to indicate how many parts we have.	
1:2	Start by providing children with a cut-up shape, divided into equal parts (a segmented whole, as shown). Ask children how many equal parts the whole shape has, drawing out that each part is one-sixth. This is revisiting prior learning from segment *3.2 Unit fractions: identifying, representing and comparing*. Using one-sixth as the unit fraction, instruct children as follows:	 • *'Show me one-sixth.'* • *'Now show me four lots of one-sixth.'*

	• *'Show me one-sixth.'* • *'Now show me four lots of one-sixth.'* Explain that 'four lots of one-sixth' is also called 'four-sixths'. Say: *'Now show me five one-sixths…'* *Pattern Blocks* are particularly useful for this exercise, with children comparing green equilateral triangles to a yellow hexagon representing the whole.	
1:3	Show an image of a hexagon with five-sixths shaded and say: *'I have five one-sixths; I have five-sixths.'* Repeat for different numbers of sixths, asking children to complete the following stem sentence each time: ***'I have ___ one-sixths; I have ___-sixths.'*** Now count the sixths: *'One one-sixth, two one-sixths, three one-sixths, four one-sixths, five one-sixths, six one-sixths.'* Note that the corresponding written notation for the non-unit fractions is not introduced at this point. Neither do we make reference yet to six-sixths being equal to one whole. This will be explored explicitly in *Teaching point 4*. Be prepared that some children may notice that three one-sixths is the same as one-half. At this stage we do not explore the concept of equivalence in detail, so acknowledge their comment with an appropriate response such as, *'Yes, sometimes there is more than one way to describe a fraction'*. Equivalence will be explored in segment *3.7 Finding equivalent fractions and simplifying fractions.*	 • *'I have five one-sixths; I have five-sixths.'*   

1:4	Now move to a linear paper model, using a different denominator, such as tenths. Start by counting in fractional steps, using the dual-counting approach. Begin with multiples of unit fractions. Say: *'One one-tenth, two one-tenths, three one-tenths… ten one-tenths.'* Then count in non-unit fractions: *'One-tenth, two-tenths, three-tenths… ten tenths.'* Then say: *'Show me five one-tenths. We call this five-tenths. I have five one-tenths; I have five-tenths.'* Repeat for other non-unit fractions of tenths, using the stem sentence: ***'I have ___ one-tenths; I have ___-tenths.'*** Although there are parallels between this linear model and a number line (which we will meet in *Teaching point 3*), there are also some important differences. In this instance where one-tenth is being used as an operator (for example, *'one-tenth of the rectangle'*), any three one-tenths, for example, can be shaded in order for three-tenths of the bar to be shaded. It doesn't have to be the left-hand three-tenths. When showing non-unit fractions in isolation, make sure that you shade a variety of tenths, as shown opposite.	Linear model:  • *'I have three one-tenths. I have three-tenths.'*

1:5	As one-tenth (and multiples of one-tenth) is a division commonly used in measurements, it is worth exploring this concept further with some linear and quantity models in measurement contexts. Present examples such as those shown opposite. You may wish to model some practically in the classroom. For example, create a metre length of paper and divide it into tenths. Below it, place a length of ribbon measuring four one-tenths of a metre, or 40 cm. Ask: *'How many one-tenths of a metre does the ribbon measure?'*	Linear model – measurement context: *'How many one-tenths of a metre does the ribbon measure?'*  Quantity model – mass context: *'Hannah has 1 kg of rice in total. She has circled the amount of rice she needs for her recipe. How many tenths of the whole kilogram does Hannah need?'* 

1:6	Provide varied practice for children, moving between linear, quantity and area representations. It is important to include area models where the whole is:
	• not a regular shape
	• not one clearly-defined shape (for example, the pentagon shown opposite, which in some ways is more like a quantity model).
	Look at Examples 1 and 2 opposite. Discuss how many parts each whole has been split into. Encourage children to describe the examples using the stem sentence: ***'There are ___ equal parts in the whole. There are ___ parts shaded. ___ is shaded.'***
	At this stage all responses should be verbal and answers should use the spoken name of the fraction (e.g. *'three-eighths'*) rather than the written fraction notation (e.g. $\frac{3}{8}$).
		Example 1:
		
		• *'There are eight equal parts in the whole.'*
		  • *'There are three parts shaded.'*
		  • *'Three-eighths is shaded.'*
		Example 2:
		
		• *'There are five equal parts in the whole.'*
		  • *'There are four parts shaded.'*
		  • *'Four-fifths is shaded.'*

	Introduce Example 3. Ask children to describe the fraction. Note the language children use. Can they accurately identify the whole and the number of parts? Do they use the stem sentence to structure their response? Include real-life problems similar to Example 4 shown opposite. In this example, the whole window is split into nine panes. Carly has broken four of them, so four-ninths of the window will need replacing. It important to include representations where the different parts of the fraction are not congruent, i.e. they do not look the same. Children will have met congruence and incongruence related to equal parts in segment *3.2 Unit fractions: identifying, representing and comparing*. To deepen understanding, introduce dòng nǎo jīn problems like Example 5. Use this example to explore how each of the parts (in this case the sixths) do not always look the same.	Example 3: *'Describe the fraction.'*  Example 4 – real-life context: *'Carly was playing football and accidentally broke some panes of glass in a window. What fraction of the whole window will need new window panes?'*  Example 5 – dòng nǎo jīn: *'What fraction of the whole is shaded orange?'* 

Teaching point 2:

Non-unit fractions are written using the same convention as unit fractions. A non-unit fraction has a numerator greater than one.

Steps in learning

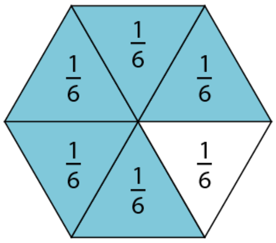
	Guidance	Representations
2:1	Once naming of non-unit fractions has been explored verbally and practically and children have mastered the concepts from *Teaching point 1*, they should have developed a strong sense of unitising, allowing them to see each non-unit fraction as a fraction made up of repeated iterations of a unit fraction. This teaching point moves towards using the corresponding written notation for these non-unit fractions. Return to the area model of a segmented hexagon with equal parts each labelled one-sixth, from step *1:2*. Show five parts shaded. Ask children how many parts are shaded. Use the stem sentence: ***'The whole has been divided into ___ equal parts. ___ of the parts are shaded; that is ___ of the whole.'*** Ask children how they think they could write five-sixths and discuss their responses. This is the first time written notation of non-unit fractions is taught. Introduce the non-unit fraction notation in the same way it was introduced for unit fractions in segment *3.2 Unit fractions: identifying, representing and comparing*. Tell children that we can record the relationship between the part and the whole symbolically, using written notation. Look again at the hexagon image and the labelled diagram below. Demonstrate how to write a fraction, saying the stem sentence and writing the corresponding part of the written notation at the same time:	

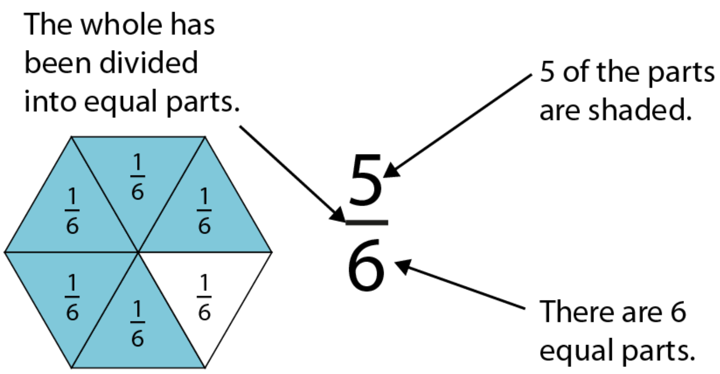
Say	Write
'The whole has been divided…'	The division bar: —
'…into six equal parts.'	The denominator: **6**
'Five of the parts are shaded.'	The numerator: **5**

It is important to clarify children's understanding of the notation using simple questioning around the name of the fraction and its newly introduced notation, such as:

- *'What does the "6" represent?'*
  (The '6' represents the total number of equal parts.)
- *'What does the "5" represent?'*
  (The '5' represents the five that are shaded of the six equal parts.)

Repeat this several times, saying the stem sentence and writing a fraction at the same time, so that children clearly see the links between the written notation and the stem sentences.





- *'The whole has been divided into six equal parts.'*
- *'Five of the parts are shaded.'*
- *'That is five-sixths of the whole.'*

$$\frac{\square}{\square} \rightarrow \frac{\square}{6} \rightarrow \frac{5}{6}$$

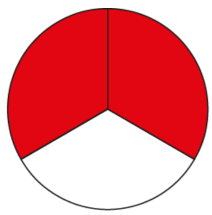
- *'What does the "6" represent?'*
- *'What does the "5" represent?'*

2:2

Return to the familiar examples from *3.2 Unit fractions: identifying, representing and comparing*, but this time with more than one part of the whole shaded, to give the children practice writing non-unit fractions. As before, focus on linking the stem sentence to the written notation. You could ask the children to work in pairs with one child saying the stem sentence while their partner writes the corresponding part of the fraction.

Repeat using different denominators, linear and quantity models, and real-life practical examples (including volume and measurements). For each example, challenge children to write the fraction of each whole that is shaded.

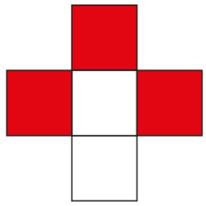
Area models:



- *'The whole has been divided into three equal parts.'*
- *'Two of the parts have been shaded.'*

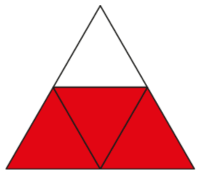
$$\frac{\square}{\square} \rightarrow \frac{\square}{3} \rightarrow \frac{2}{3}$$

- *'What does the "3" represent?'*
- *'What does the "2" represent?'*



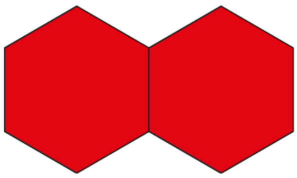
- *'The whole has been divided into five equal parts.'*
- *'Three of the parts have been shaded.'*

$$\frac{\square}{\square} \rightarrow \frac{\square}{5} \rightarrow \frac{3}{5}$$



- *'The whole has been divided into four equal parts.'*
- *'Three of the parts have been shaded.'*

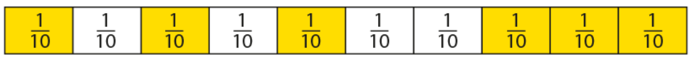
$$\frac{\square}{\square} \rightarrow \frac{\square}{4} \rightarrow \frac{3}{4}$$

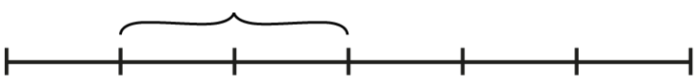


- *'The whole has been divided into two equal parts.'*
- *'Two of the parts have been shaded.'*

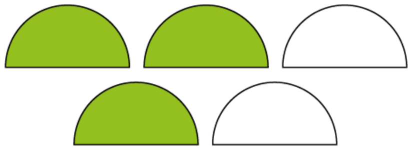
$$\frac{\square}{\square} \rightarrow \frac{\square}{2} \rightarrow \frac{2}{2}$$

Linear models:



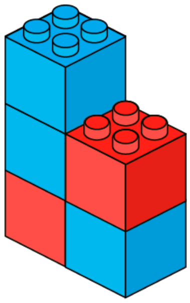


Quantity model:



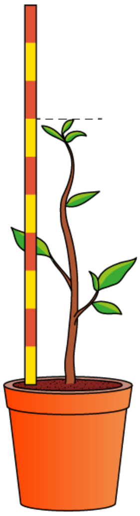
Real-life problems:

- *'Sam has made this model using building bricks.'*

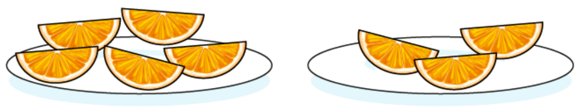
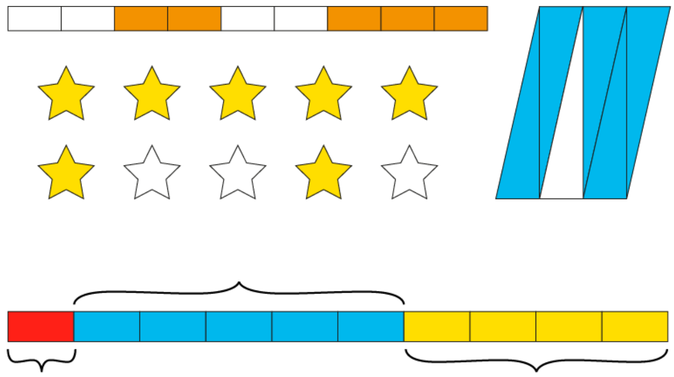
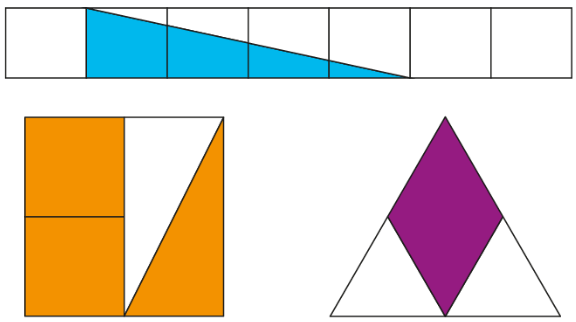


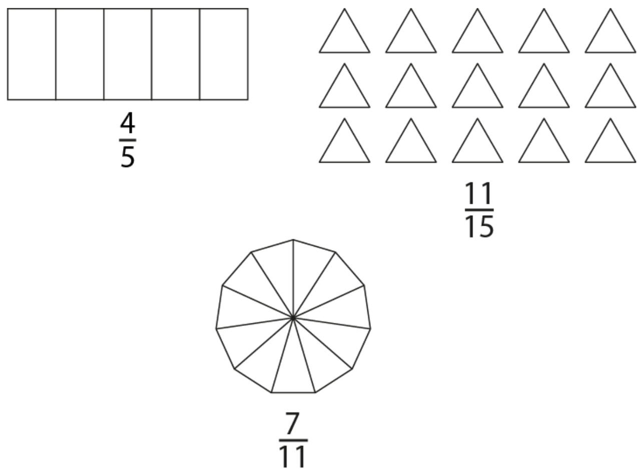
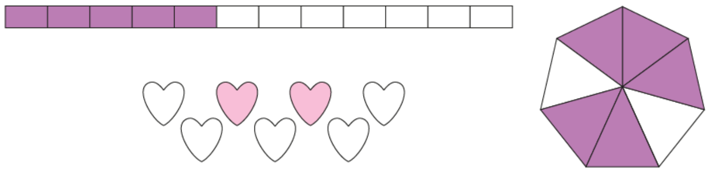
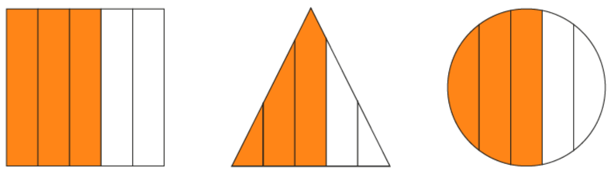
  - *'How many equal parts make up the whole?'*
  - *'What fraction of the whole is red?'*
  - *'What fraction of the whole is blue?'*

- *'Ameera is measuring a plant she has grown. She uses a metre stick. What fraction of one metre does the plant measure?'*



  - *'The whole metre is divided into ___ equal parts.'*
  - *'The plant measures* $\frac{\square}{\square}$ *of the whole metre.'*

		• *'Pawel is sharing his orange with Dominika. What fraction does Pawel have? What fraction does Dominika have?'* Pawel Dominika  • *'The orange is divided into eight equal parts.'* • *'Pawel has five parts; he has five-eighths of the whole.'* • *'Dominika has three parts; she has three-eighths of the whole.'*
2:3	Provide children with varied practice, writing the non-unit notation for each given representation. To promote depth of understanding, use a dòng nǎo jīn problem like the one opposite. Notice how the non-unit fraction is less easy to identify.	*'Write the non-unit fraction that is shaded.'*  Dòng nǎo jīn: • *'What fraction of each whole is shaded? Explain your answer.'*  • *'How do you know you are right?'*

2:4	So far, we have been looking at visual representations of fractions and writing the corresponding fraction notation. Now we are going to start with the written fraction and shade the given fraction on a diagram. At this stage, only use diagrams with the parts clearly indicated by the denominator, avoiding examples that could be further simplified. We want children to focus on seeing the non-unit fractions as more than one part of the whole only. Equivalence and simplification will be explored in segment *3.7 Finding equivalent fractions and simplifying fractions.*	*'Shade the following fractions on these diagrams.'* 
2:5	Provide examples where the children have to decide if the fraction notation and non-unit fraction model match or not, such as those shown opposite. Note that these multiple-choice questions include 'plausible distractors'. The linear bar model represents a common misconception; children who believe this to be the correct answer have probably misunderstood the part–whole relationship and instead think of the numerator and denominator as describing two separate parts. The hearts representation shows five-sevenths as the part that is *not* shaded, instead of the part that is shaded. Encourage children to explain their choice of answer; they may be encouraged to support their reasoning with the stem sentences from step *2.1*: ***'The whole has been divided into ___ equal parts. ___ of the parts have been shaded; that is ___ of the whole.'*** The stem sentence may also be useful in the second example to help children explain why a shape is *not* three-fifths shaded. For example, *'The shape is divided into five parts, but they are not* equal *parts.'*	• *'Tick the representation where $\frac{5}{7}$ is shaded.'*  • *'Tick or cross each diagram to show whether $\frac{3}{5}$ is shaded. Explain your answers.'* 

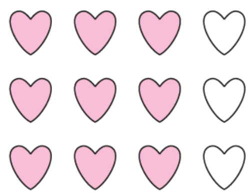
2:6

Once children are confident in describing, identifying and labelling non-unit fractions, including using the correct notation, extend the range of problems to include representations that could be expressed using more than one fraction. At the moment, there is no need to introduce the term *'equivalent fraction'* or to talk about simplifying or converting fractions. We are merely starting to raise awareness that sometimes there is more than one way to write the fraction of a shape or set that is shaded.

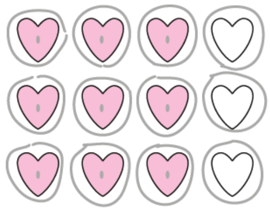
Show children the representation opposite and ask them what fraction of the set has been shaded. Many children will say nine-twelfths, and some will also say three-fourths. Ask children to justify both responses. The stem sentence from step *2.1* will support them in this.

Ask children to relate their justifications back to the representation. For example, if a child says, *'The whole is divided into twelve equal parts, and nine parts are shaded'*, ask *'Where are the twelve equal parts? Can you show me the nine equal parts?'*. Similarly, if a child responds with three-fourths ask, *'Where are the four equal parts? Can you show me the three equal parts?'*. Some children may show this by annotating the diagram, similar to the example opposite.

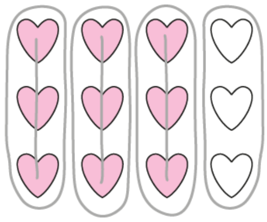
Occasionally, children may suggest fractions that are equivalent, but that can't be easily related to the image, for example, $\frac{18}{24}$. If this happens, still ask *'Where are the twenty-four equal parts? Where are the eighteen parts?'* (You need to split each heart in half to find these parts.) Tell children they will learn more about fractions that can be written in more than one way at a later point. For



- *'The whole is divided into twelve equal parts, and nine of them are shaded.'*



- *'The whole is divided into four equal parts, and three of them are shaded.'*

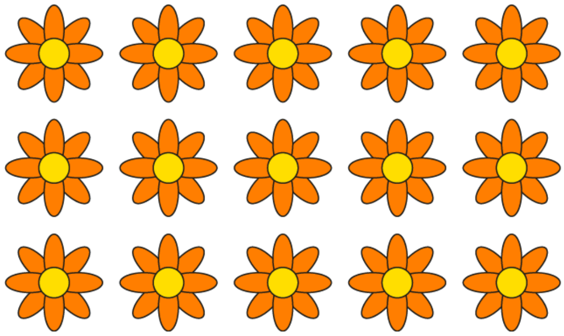


now, focus on making sure that where children are giving equivalent fractions, they can easily relate them to the model being used.

Spend some time discussing further examples where a fraction can be expressed in more than one way. At this stage, avoid linking it to calculating fractions of quantities (for example, *'There are fifteen flowers, we know that one-fifth of fifteen is three, so four-fifths is $4 \times 3 = 12$'*). This will be introduced later in the spine. Instead, encourage children to identify the number of equal parts as expressed by the denominator each time. Here the denominator is '5', so we are looking for five equal parts. As there are five columns of flowers, each column must be one equal part or one-fifth.

Provide further practice using similar fractions and representations, such as those opposite. Children should be fluent in moving between identifying and describing non-unit fractions verbally, and providing corresponding written notation or circling/shading given non-unit fractions.

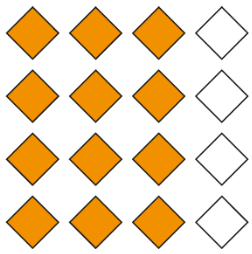
- *'Circle $\frac{4}{5}$ of the flowers.'*



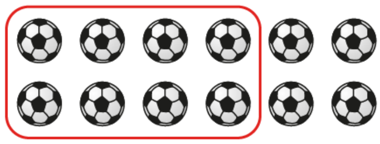
  - *'How else can you express the fraction that you have circled?'*

- *'What fraction of the line is highlighted?'*

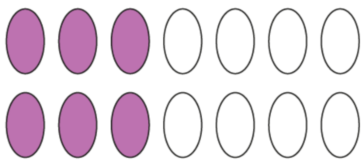
- *'Tick the fraction that matches each representation.'*



$\frac{3}{4}$ $\frac{4}{16}$ $\frac{3}{16}$ $\frac{12}{4}$



$\frac{4}{4}$ $\frac{8}{4}$ $\frac{2}{3}$ $\frac{3}{4}$



$\frac{6}{8}$ $\frac{3}{4}$ $\frac{3}{7}$ $\frac{2}{3}$

<table>
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<td>**2:7**</td>
<td>Once pupils have had the opportunity to consolidate their understanding with all three model types (area, linear and quantity), offer opportunities to further deepen learning. Include some true/false questions that encourage children to prove or disprove a statement. These types of question are extremely useful in developing reasoning skills.

To further deepen understanding of this concept, present dòng nǎo jīn problems like the ones shown opposite. Use the first example to reiterate the need for the parts to be equal.</td>
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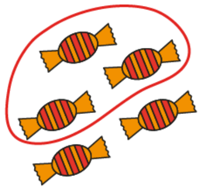
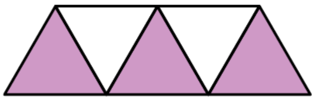
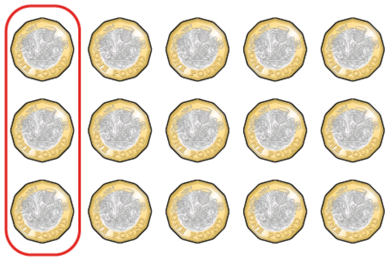
- *'Which representations show $\frac{3}{5}$?'*

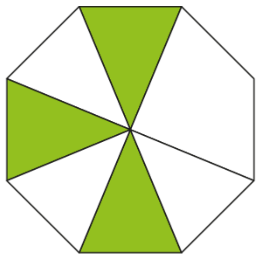






Dòng nǎo jīn:

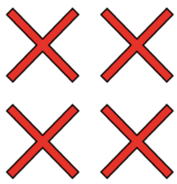
- *'Juan says that $\frac{3}{7}$ of the shape is coloured green. Explain why Juan is wrong.'*



  - *'What fraction is coloured green?'*

- *'This is $\frac{5}{8}$ of the set. Draw $\frac{1}{8}$ of the set.'*

- *'This is $\frac{2}{3}$ of the set. Draw $\frac{1}{3}$ of the set.'*



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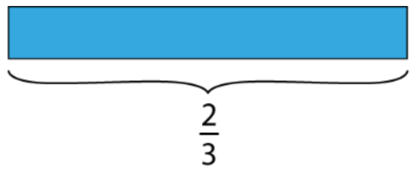
		• *'This is $\frac{2}{3}$ of a piece of ribbon. Draw $\frac{1}{3}$ of the ribbon.'* 

Teaching point 3:

When the numerator and the denominator in a fraction are the same, the fraction is equivalent to one whole.

Steps in learning

Guidance | **Representations**

3:1

In this teaching point, we meet fractions that have the same numerator and denominator. We approach these in the same way as other non-unit fractions; as more than one of the same unit fraction. Throughout this teaching point we prompt children to identify the unit fraction they are working in. The counting that children do here will lead on to the introduction of fractions as numbers in the next teaching point.

Begin by dual-counting fractions in the same way you did in *step 1:2*.

- *'One-fifth, two-fifths, three-fifths, four-fifths, five-fifths.'*

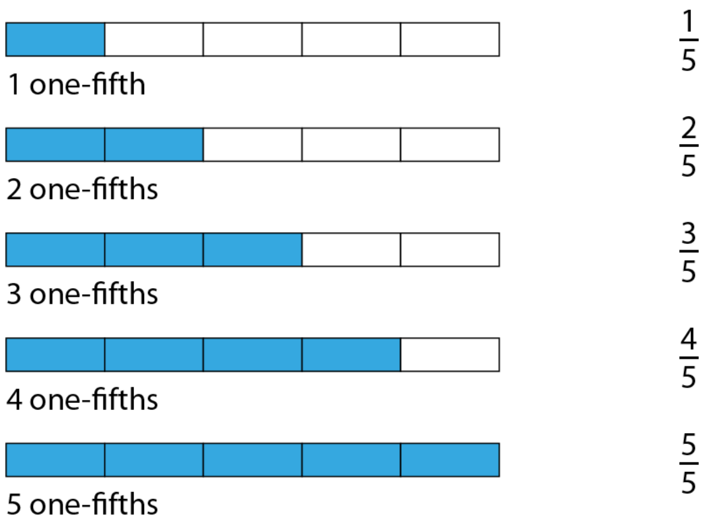
And

- *'One one-fifth, two one-fifths, three one-fifths, four one-fifths, five one-fifths.'*

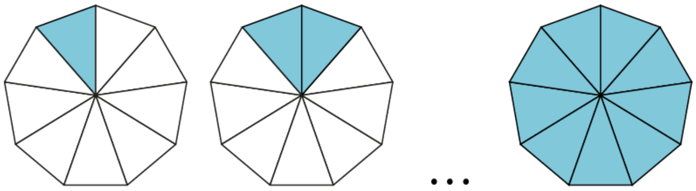
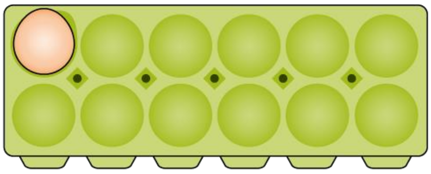
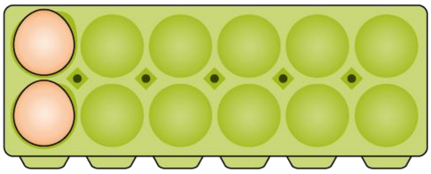
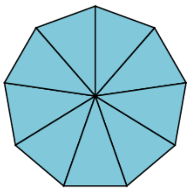
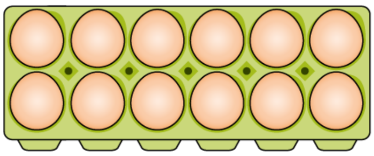
Use the following stem sentence to clarify thinking: ***'We have split our whole into ___ equal parts, so our unit fraction is ___.'***

Now look at the image of five-fifths, and the associated written notation, $\frac{5}{5}$.

Draw children's attention to the fact that if the whole is divided into five equal parts, and we have all five of those parts, then that is the same as having one whole.



- *'We have split our whole into five equal parts, so our unit fraction is one-fifth.'*

3:2	Repeat the steps from step *3:1*, using different models and denominators, for example: • *'One-ninth, two-ninths, three-ninths…'* And • *'One one-ninth, two one-ninths, three one-ninths…'* Make sure each time that the children can identify the unit fraction. Explain that: *'We have split our whole into nine equal parts, so our unit fraction is one-ninth.'* Then ask: *'If the whole is divided into nine equal parts, how many of those parts make one whole?'* Highlight the fact that *nine* one-ninths is the same as one whole. Repeat using other denominators, such as twelfths, as shown opposite.	Ninths:  Twelfths:  $\frac{1}{12}$  $\frac{2}{12}$ … • *'One-twelfth, two-twelfths, three-twelfths…twelve-twelfths.'* • *'One one-twelfth, two one-twelfths, three one-twelfths…twelve one-twelfths.'* • *'We have split our whole into twelve equal parts, so our unit fraction is one-twelfth.'* • *'When we have twelve-twelfths, the whole egg box is full. We have one whole egg box of eggs.'*
3:3	Now look more closely at the three written fractions ($\frac{5}{5}$, $\frac{9}{9}$ and $\frac{12}{12}$) that are equivalent to one whole alongside their associated representations). Ask: *'What do you notice about the numerators and the denominators?'* Challenge children to work with a partner to write a generalisation they can make about fractions where the numerator and denominator are the same. You might want to provide part of a stem sentence as a starting point: '***When the numerator and denominator are the same*** ___***'*** Children will make their own	 $\frac{5}{5}$  $\frac{9}{9}$  $\frac{12}{12}$

suggestions for wording to complete it, but work towards a generalisation such as: ***'When the numerator and denominator are the same, the fraction is equivalent to one whole.'***

3:4

Give children plenty of opportunities for verbal and written practice to check they are developing familiarity with this generalisation. For example, say:

- *'If the whole is divided into fifteen equal parts, fifteen of those parts make one whole.'*
- *'If the whole is divided into two hundred and sixty-nine equal parts, two hundred and sixty-nine of those parts make one whole.'*

Look at missing-numerator examples, such as those provided opposite. Challenge children to write the numerators to make the fractions equivalent to one.

Ask children to explore the fraction problem opposite and to explain their reasoning.

Ask children to add suitable numerators to the fractions in the table, to create:

- a unit fraction
- a fraction that is quite a small part of the whole
- a fraction that is quite a large part of the whole
- a fraction that is equivalent to 1 whole.

Focusing on fractions that form 'quite a small part of the whole' or 'quite a large part of the whole' requires children to find a numerator which is small or large relative to the denominator. This helps to develop their 'fraction sense'. It is the proportional relationship between the numerator and denominator which determines the size of a fraction. This will be explored further in future segments.

Missing-numerator problem:

'Fill in the missing numerator.'

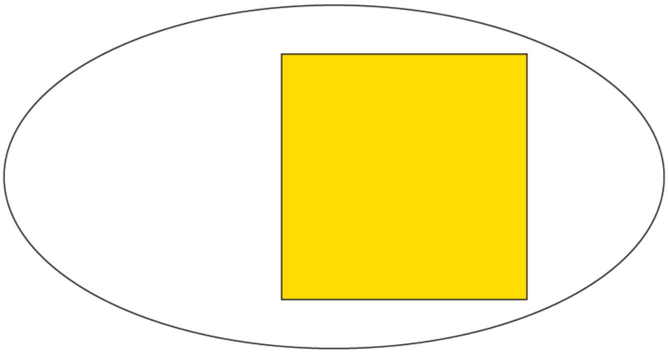
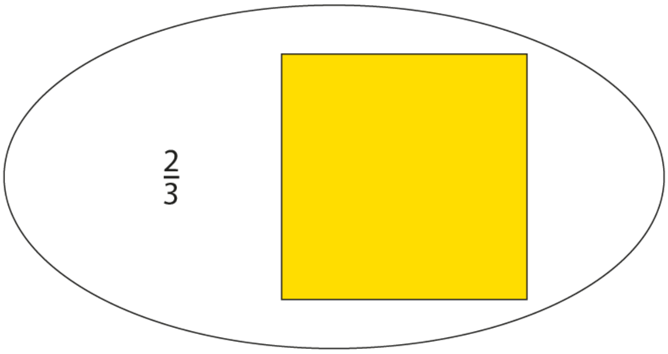
$\frac{\square}{15}$ $\frac{\square}{49}$ $\frac{\square}{6}$ $\frac{\square}{2}$

Fraction problem:

- *'Emily and Jack's mum gives them each an identical currant bun.'*
  - *'Emily cuts her bun into four pieces.'*
  - *'Jack cuts his bun into two pieces.'*
  - *'Emily says: "I have more than you. I've got four pieces and you only have two pieces."'*
  - *'Explain why Emily is wrong.'*

Fractions table:

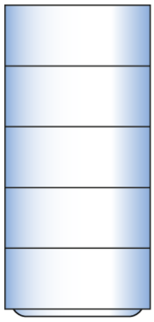
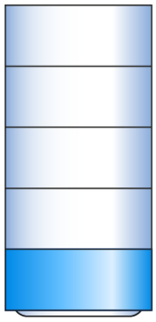
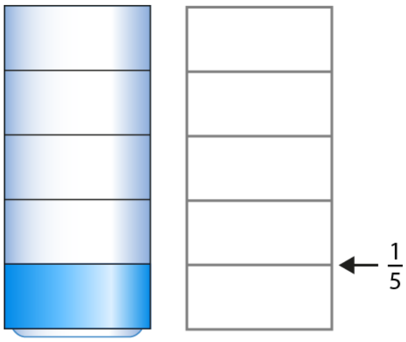
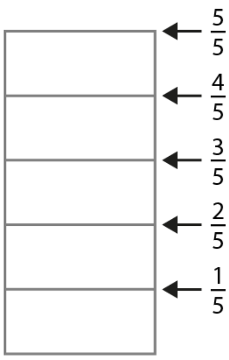
Unit fraction	Fraction that is quite a small part of the whole	Fraction that is quite a large part of the whole	Fraction that is equivalent to 1 whole
$\frac{\square}{18}$	$\frac{\square}{18}$	$\frac{\square}{18}$	$\frac{\square}{18}$
$\frac{\square}{7}$	$\frac{\square}{7}$	$\frac{\square}{7}$	$\frac{\square}{7}$
$\frac{\square}{30}$	$\frac{\square}{30}$	$\frac{\square}{30}$	$\frac{\square}{30}$

3:5	Give children more practice in applying 'fraction sense' and reasoning about the size of the part in relation to the whole.	• *'Meg and Raz are estimating what fraction of the oval is shaded.'*
		
		  • *'Meg's estimate is* $\frac{1}{3}$*.'*
		  • *'Raz's estimate is* $\frac{1}{10}$*.'*
		  • *'Who is more likely to be most accurate? Why?'*
		
		• *'The teacher tells them that the non-shaded part of the oval is* $\frac{2}{3}$*.'*
		
		
		  • *'Whose estimate is correct? How do you know?'*

Teaching point 4:

All unit and non-unit fractions are numbers that can be placed on a number line.

Steps in learning

	Guidance	Representations
4:1	So far, we have been using fractions as an operator, for example talking about one-third *of* this shape or three-fifths *of* this line. Now, introduce an extremely important point: as well as operators, fractions are also numbers. For example, the number $\frac{1}{4}$, which also has a value of 0.25 and sits on the interval between zero and one in the number system. It may seem obvious to us that fractions are numbers, but it is not unusual for children to think that fractions are just a part of something else. This point is introduced through a measurement pouring activity. Take a straight-sided vase, glass or measuring cylinder. Mark it in equal divisions (in the example opposite, it has been split into five equal parts – a wipeable whiteboard pen is good for this). Through discussion, establish that each part is one-fifth of the whole. Tell children you are going to start filling the container and you want them to say, *'Stop!'* when the container is one-fifth full. (You could use rice or a liquid like water or squash to fill the container.) Slowly fill the container until you reach this point. Look at the container and, once agreed that it is one-fifth full, ask one of the children to mark the point the 'stuff' comes up to on a diagram of the container displayed on the board. Repeat this process for two-fifths, three-fifths, four-fifths and five-fifths, pouring the same amount into the	   

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<td>container each time. Mark each new point on your diagram on the board so that you create the image shown. The arrows offer a useful way to emphasise that it is a very specific point on the side of the container that we are referring to. Up until now, a fraction has always described the size of a section or a part, rather than a point.

The children will notice that when the container reaches five-fifths, it is completely full. The numerator and denominator are the same, so $\frac{5}{5}$ is the same as one whole (one whole/full container). It is important to mention this in order to reinforce their learning from Teaching point 3.</td>
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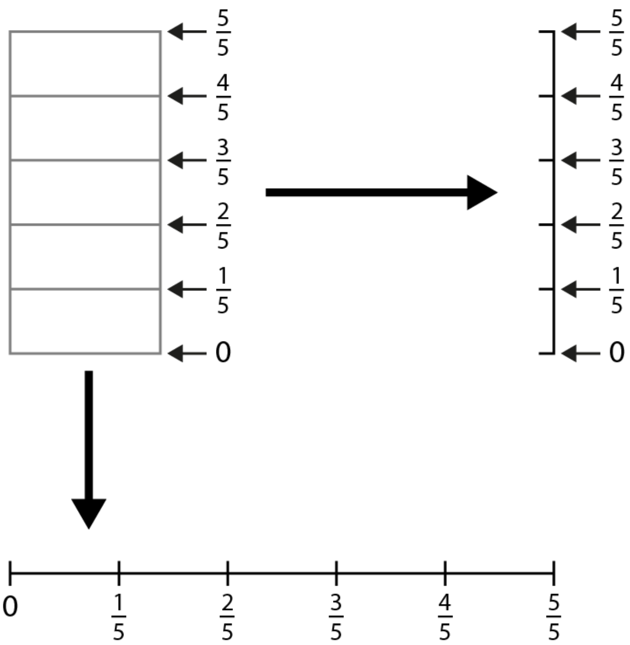
<td>4:2</td>
<td>Look at the completed diagram on the board. Add zero to the base of the diagram, to show where the contents would come to if the container was empty. Show how the markings at the side of the diagram now form a vertical number line. Explain how this number line could also be drawn horizontally, in order to create a more familiar number line. Cross-compare the vertical and horizontal number lines to emphasise how they show the same thing, just in a different orientation.</td>
<td>
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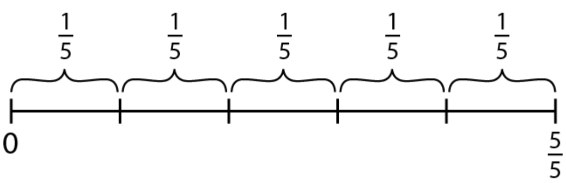
<td>4:3</td>
<td>The next step is to compare the interpretations of fractions that we have met so far. To do this, we compare fractions as operators (parts of a whole) with our new learning about fractions as numbers.

Explain to children that each interval on the number line shown opposite is one-fifth of the whole. As we add additional fifths, we move along the number line</td>
<td>Fractions as operators (part of a whole):

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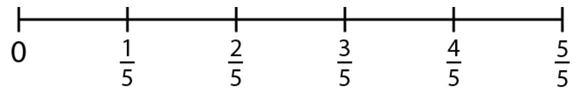
<td>from 0 to $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$ and so on.

In the second example, each *point* on the number line represents a *number*; the number 0, the number one-fifth, the number two-fifths and so on.</td>
<td>Fractions as numbers:

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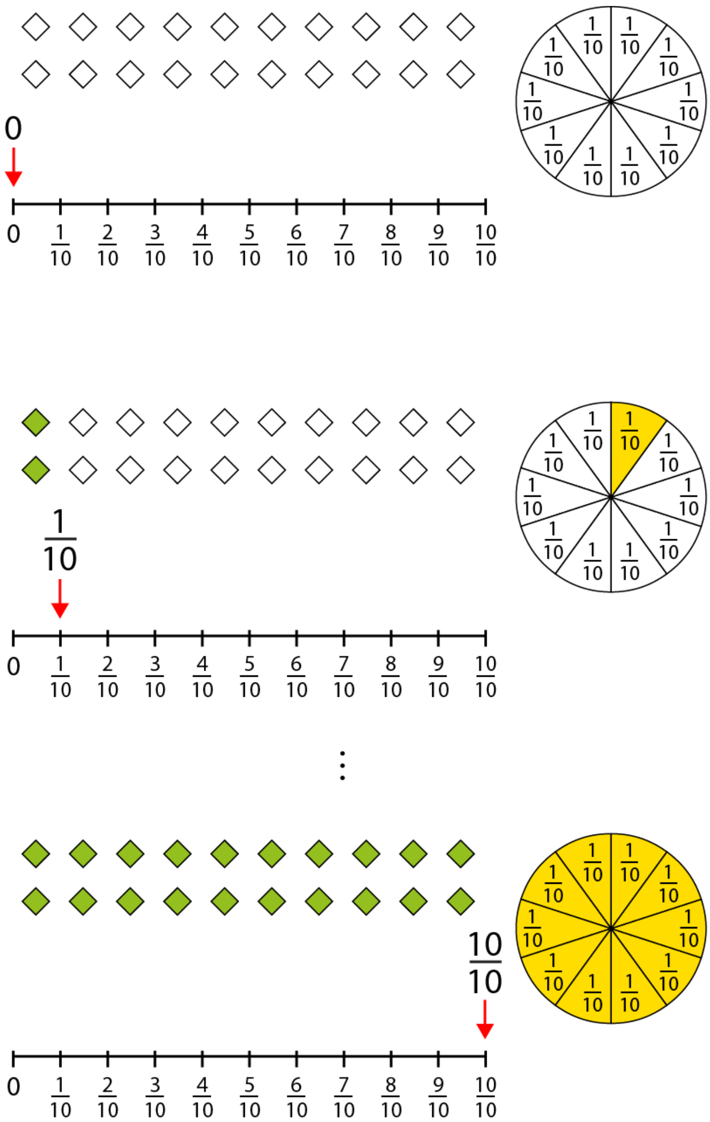
<td>4:4</td>
<td>To build experience with moving between fractions as operators and fractions as numbers, proceed to a different denominator, such as tenths.

Share an area model marked into tenths (such as a circle), alongside a number line, a quantity model and the written fraction notation. Using the associated representations presentation (see 3.3 Representations, slide 55), start by counting in fractional steps, noting how the information in each of the four representations updates with each step.</td>
<td>
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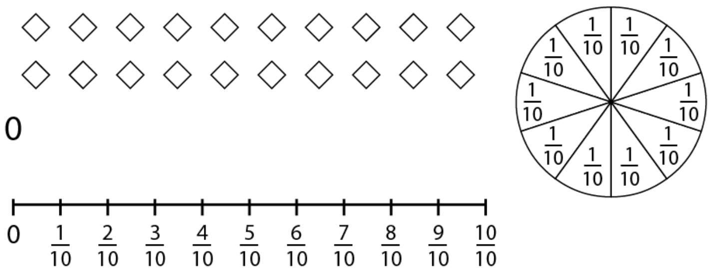
<td>4:5</td>
<td>The children have been introduced to the idea that when a fraction is a fraction of something, it is shown by a section or part, and when a fraction is a number, it is shown as a point on a number line.

Now display the number line, the unshaded area model and the empty quantity model side-by-side again. Give the following instructions, switching</td>
<td>
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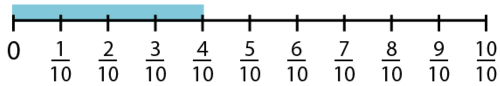
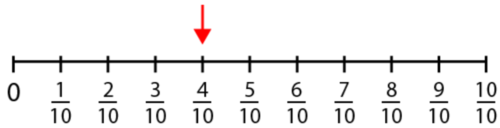
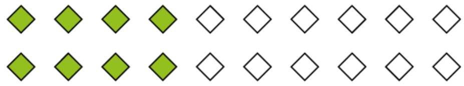
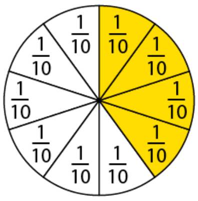
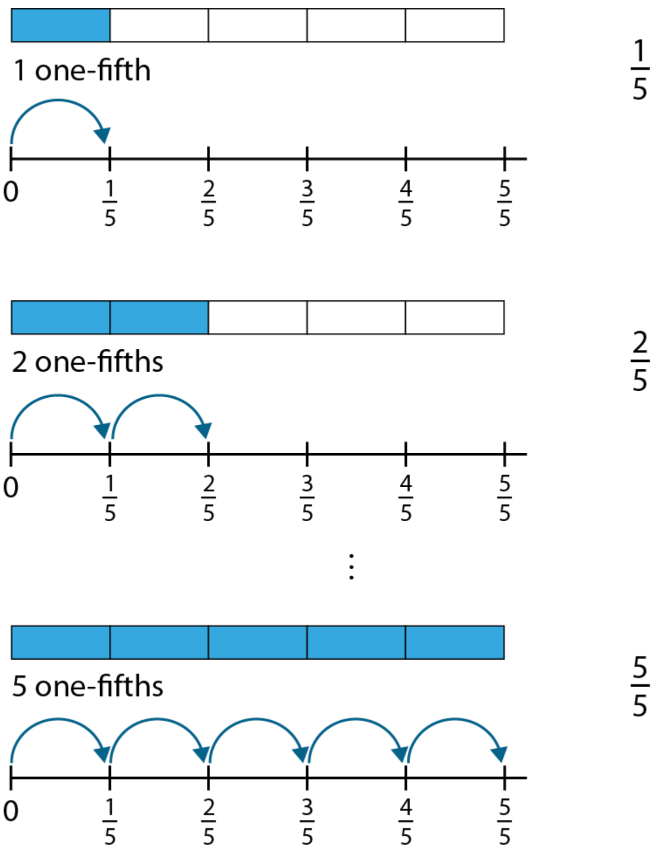
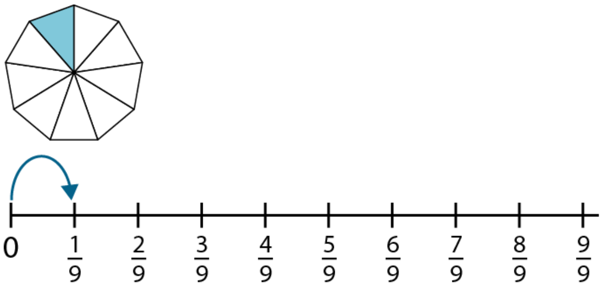
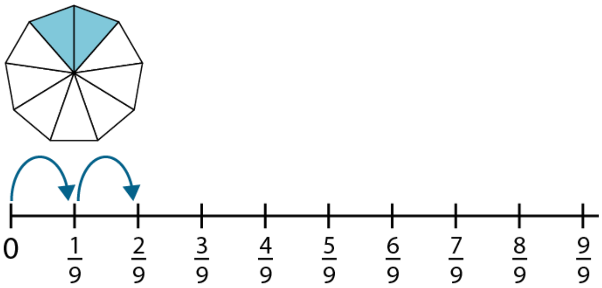
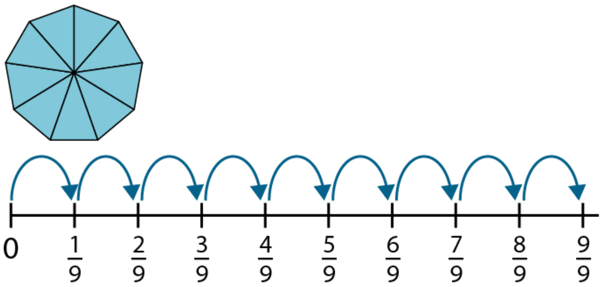
	between using fractions as numbers and fractions as operators. • Shade the first four-tenths of the number line. • Draw an arrow at the number four-tenths on the number line. • Write the fraction notation for the number four-tenths. • Shade four-tenths of the quantity model. • Shade four-tenths of the circle.	  $\frac{4}{10}$  

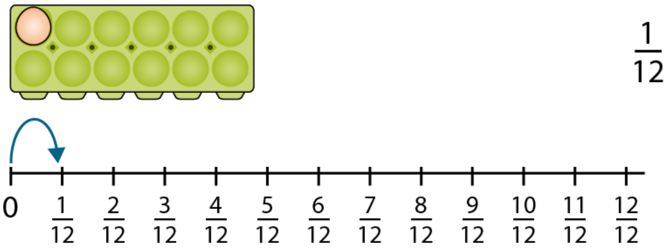
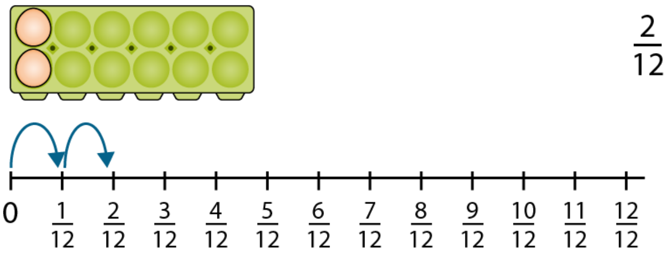
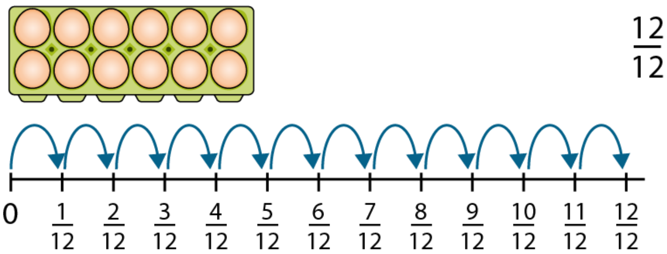
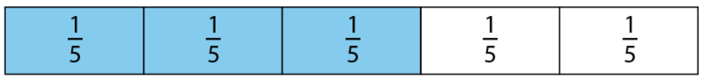
Teaching point 5:

Repeated addition of a unit fraction results in a non-unit fraction.

Steps in learning

	Guidance	Representations
5:1	*Teaching points 1–3* focused on the concept that several (more than one) lots of a unit fraction make a non-unit fraction. For example, three lots of one-eighth (unit fraction) makes three-eighths (non-unit fraction). Now that the children have been introduced to fractions as numbers, unit fractions can be explored using repeated addition. This is key to their understanding and will support their ability to add and subtract 'like' fractions (fractions with the same denominator) in segment *3.4 Adding and subtracting within one whole.* At this point, you may like to return to the images used in *Teaching point 3*, using dual-counting to count up and down. However, this time you may also wish to display the visual image of a number line alongside each of the area models. Dual-count for each example.	 • *'One-fifth, two-fifths, three-fifths, four-fifths, five-fifths'* And • *'One one-fifth, two one-fifths, three one-fifths, four one-fifths, five one-fifths.'*

		 $\frac{1}{9}$  $\frac{2}{9}$ ⋮  $\frac{9}{9}$ • *'One-ninth, two-ninths, three-ninths… nine-ninths.'* And • *'One one-ninth, two one-ninths, three one-ninths… nine one-ninths.'*

		  ⋮  • *'One-twelfth, two-twelfths, three-twelfths…twelve-twelfths.'* And • *'One one-twelfth, two one-twelfths, three one-twelfths… twelve one-twelfths'*
5:2	Look back at the first image from step *5:1*. Ask children: '*What is the unit fraction?'* Confirm: *'Our unit fraction is one-fifth.'* Ask: *'How many one-fifths are there in three-fifths?'* Encouragethe children to use the following sentence to structure their responses: *'There are ___ one-fifths in three-fifths.'* Say to the children: *'I am going to write an equation to show that if I have one-fifth, and another one-fifth, and another one-fifth, then I have three one-fifths.'*	 $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5}$ $\frac{3}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$

Model this on the board by writing:

$$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5}$$

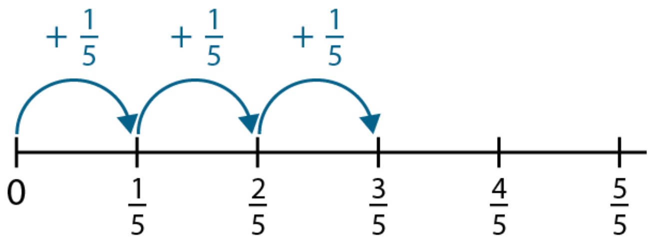
Demonstrate how this can also be written as below, saying the following as you write: *'Three-fifths is made up of one-fifth, and another one-fifth, and another one-fifth.'*

$$\frac{3}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$$

5:3 Next, you may like to introduce a different model for repeatedly adding one-fifths: a number line. As before, look at the representation on the number line, and then construct the equation underneath to highlight the relationship.

Say	Write
'I have one-fifth…'	$\frac{1}{5}$
'…and another one-fifth…'	$+\frac{1}{5}$
'…and another one-fifth.'	$+\frac{1}{5}$
'Altogether I have three-fifths.'	$=\frac{3}{5}$

At this stage, avoid describing this as 'adding fractions' or making generalisations about the numerators and denominators and whether they change or not. We are simply using the written notation of an equation to express something that children *already know*, i.e. that three-fifths is made up of three one-fifths. The repeated addition of unit fractions just moves from a verbal expression of this knowledge to a written expression of it.



$$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5}\ \left(\text{or } \frac{3}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}\right)$$

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<td>5:4</td>
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Now refer back at the other models from step 5:1, showing the area models, number lines and fraction notation. Ask the children to write equations that show how the non-unit fraction is formed by repeated addition of a unit fraction.

For each example, pose the questions:

'What is our unit fraction?'
'What fraction is shaded?'
'How can we write this using repeated addition of the unit fraction?'

Share the answers on the board, making sure that the addition is represented in both formats, for example:

$\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{4}{9}$

And

$\frac{4}{9}=\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}$

Provide further varied practice by presenting a range of models and representations for children to investigate.
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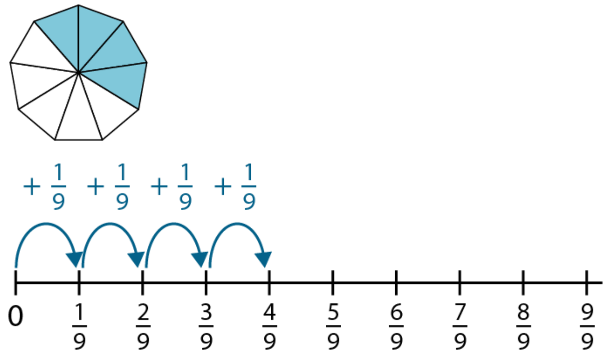

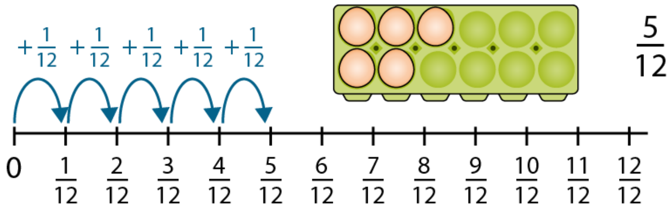

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<td>5:5</td>
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Cuisenaire® rods can be used to further reinforce the concept. The rods allow children to explore the part–whole relationships in a practical, concrete way.
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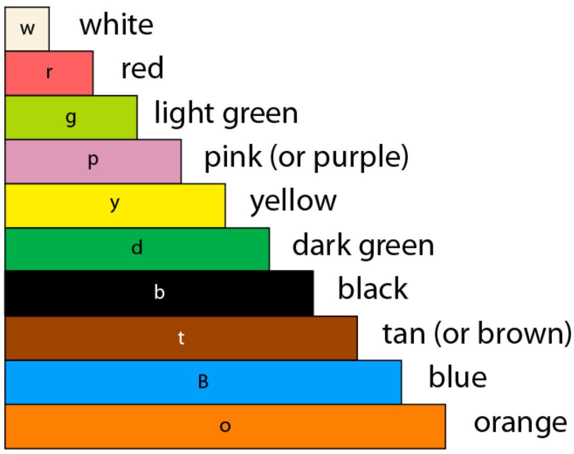

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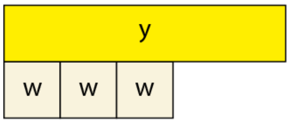
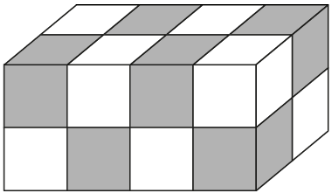
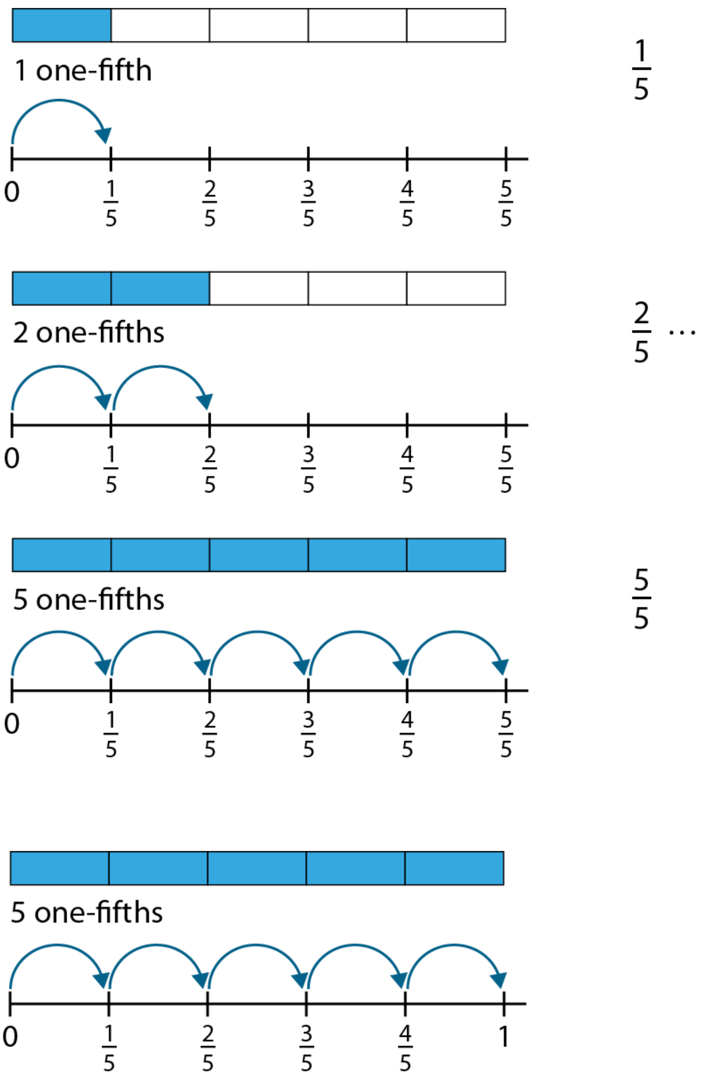
		Example 1: *'The yellow rod represents the whole. What fraction of the whole is represented by the white rods?'*  • *'There are five white rods in the whole.'* • *'Each white rod represents one-fifth of the whole.'* • $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5}$ Example 2: *'The blue rod represents the whole. What fraction of the whole is represented by the light green rods?'*  • *'There are three light green rods in the whole.'* • *'Each light green rod represents one-third of the whole.'* • $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$
5:6	Provide varied practice, using both area models and number lines as supporting prompts. Progress to children expressing a non-unit fraction as a repeated addition of unit fractions without the use of scaffolded models. To promote depth of understanding, present a dòng nǎo jīn problem.	$\frac{2}{8} = \frac{\square}{8} + \frac{\square}{8}$ $\frac{3}{8} = \frac{\square}{\square} + \frac{\square}{\square} + \frac{\square}{\square}$ $\frac{4}{8} =$ $\frac{5}{8} =$ $\frac{\square}{\square} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$ $\frac{5}{\square} = \frac{\square}{8} + \frac{\square}{8} + \frac{\square}{8} + \frac{\square}{8} + \frac{\square}{8}$

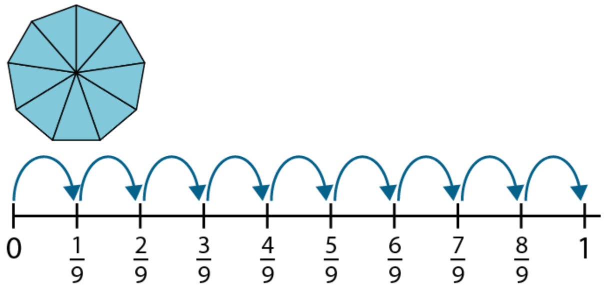
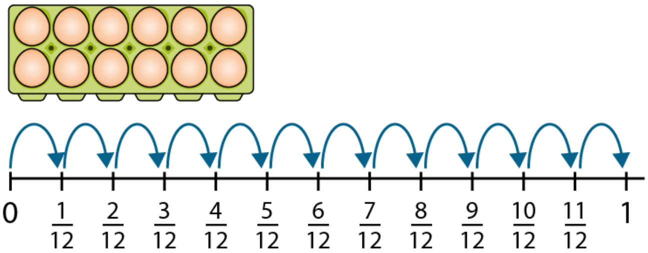
		$\frac{\square}{9} = \frac{1}{\square} + \frac{1}{\square} + \frac{1}{\square} + \frac{1}{\square} + \frac{1}{\square}$ Dòng nǎo jīn: *'Stan makes a repeating pattern with some white and grey cubes.'*  • *'Write a repeated addition of a unit fraction to show what fraction of his model is made of grey cubes.'* • *'Then write a repeated addition of a different unit fraction to show what fraction of his model is made of grey cubes.'*

Teaching point 6:

When the numerator and the denominator are the same, the value of the fraction is one.

Steps in learning

	Guidance	Representations
6:1	In *step 3:3*, children learnt that when a fraction has the same numerator and denominator, the fraction is equivalent to one whole. In this teaching point, we will explore this further but in the context of fractions as numbers. Look at an area model of a fraction alongside its corresponding number line, for example the fifths model used earlier in this segment in *step 5:1*. As previously, count up to five-fifths. When you reach five-fifths, ensure children are engaged with the completed area model and that they recognise the fraction reached on the number line is five-fifths. Revise with the children: *'What have we learnt already about fractions that have the same numerator and denominator?'* Return to the generalisation from *Teaching point 3*: ***'When the numerator and denominator are the same, the fraction is equivalent to one whole.'*** Explain that this also means that the fraction has a value of one, and so we can write it as: $\frac{5}{5} = 1$ Count up in fifths again, this time using a number line (such as the one opposite), that shows one, rather than five-fifths. Say: *'One-fifth, two-fifths, three-fifths, four-fifths, one.'*	 - *'One-fifth, two-fifths, three-fifths, four-fifths, one.'*

6:2	Repeat using other representations, such as those from *Teaching point 5*, but with number lines that now display '1'.	 
6:3	Next to one another, present the written fraction notations for the examples you have encountered so far in this teaching point. Generalise as follows: ***'When the numerator and denominator are the same, the fraction has a value of one.'*** Challenge children for ideas of how else they could complete: $\frac{\square}{\square} = 1$ Work through lots of examples, thoroughly emphasising that any fraction with the same numerator and denominator has a value of one. The children may well suggest increasingly large numbers as the numerator and denominator, but should still be encouraged to acknowledge that, for example, $\frac{100}{100} = 1$.	$\frac{5}{5} = 1$ $\frac{9}{9} = 1$ $\frac{12}{12} = 1$ $\frac{\square}{\square} = 1$

6:4	Present a range of fractions and summarise the learning so far from this teaching point by sorting fractions.	*'Sort these numbers into the correct sorting circles.'* $\frac{1}{4}$ $\frac{6}{6}$ $\frac{8}{11}$ $\frac{2}{3}$ $\frac{15}{15}$ $\frac{1}{100}$ **Unit fractions** **Non-unit fractions < 1** **Non-unit fractions = 1**
6:5	Children tend to generally be quite comfortable with the idea that, e.g. $\frac{5}{5}=1$, $\frac{9}{9}=1$ and $\frac{100}{100}=1$. However, they often find it more challenging to recognising that $\frac{5}{5}=\frac{9}{9}$. Address this explicitly now. Reiterate for children that they already know that $\frac{5}{5}=1$ and $\frac{9}{9}=1$. Display the equations opposite on the board. Ask: • *'What does this tell us about the relationship between five-fifths and nine-ninths?'* (It shows us that they must also be equal. Show this by writing '=' in the circle opposite.) • *'So, which of these numbers is bigger?'* (Referring to the fractions as numbers. Confirm that neither is bigger – they both have the same value.) In earlier segments, some children may have noticed that the same image can be represented by more than one fraction, for example, that $\frac{3}{12}$ can be shown as $\frac{1}{4}$). But this is the very first time they have written an equivalence between two fractions with different denominators. Becoming comfortable with the idea that two different fractions can be numerically equal is	$\frac{5}{5}=1$ $\frac{9}{9}=1$ $\frac{5}{5} \bigcirc \frac{9}{9}$

	central to being able to work confidently with fractions. Check their understanding by providing lots of other pairs of fractions; each fraction should have a numerator and denominator that equals one.	
6:6	Allow children to apply their understanding through a variety of question types, including: • missing-number problems • true or false-style questions • missing-symbol problems • word problems. All these types of questions can be used in both group work and independent practice. You can explore this concept further using a dòng nǎo jīn problem like the one shown opposite. Allow children to discuss their ideas.	Missing-number problems: • *'Complete the missing numbers.'* $1 = \frac{\square}{5}$ $\quad$ $\frac{10}{\square} = 1$ $\frac{7}{7} = \frac{\square}{8}$ $\quad$ $\frac{5}{\square} = \frac{6}{6}$ • *'What could the missing numbers be?'* $\frac{4}{4} = \frac{\square}{\square}$ $\frac{\square}{10} = \frac{20}{\square} = \frac{\square}{30} = \frac{40}{\square} = \frac{\square}{50}$ Missing-symbol problems: *'Fill in the missing symbols (<, >, or =).'* $1 \bigcirc \frac{3}{3}$ $\quad$ $\frac{10}{10} \bigcirc 1$ $\frac{4}{5} \bigcirc 1$ $\quad$ $1 \bigcirc \frac{4}{9}$ $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} \bigcirc 1$ $\quad$ $1 \bigcirc \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ True or false-style questions: *'Is each statement true or false? Justify your answers.'* $\frac{3}{3} < \frac{6}{6}$ $\quad$ $\frac{8}{8} > \frac{2}{2}$ $\frac{7}{7} < \frac{9}{9}$ $\quad$ $\frac{11}{11} < \frac{5}{5}$

Word problems:

'Boris eats $\frac{1}{5}$ of the cake. Maisy eats $\frac{1}{5}$ of the cake. Sanjay and Silas eat $\frac{1}{5}$ of the cake each, and Jack also has $\frac{1}{5}$. Is there any cake left? Explain your answer.'

Dòng nǎo jīn:

'Use the numerals 4, 9 and 10 to make these equations correct. You can only use each numeral once.'

$$1 > \frac{\square}{6}$$

$$\frac{8}{\square} < 1$$

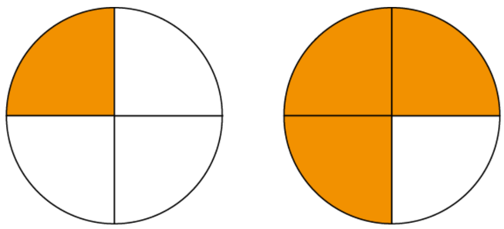
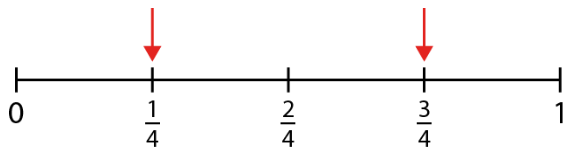
$$1 > \frac{9}{\square}$$

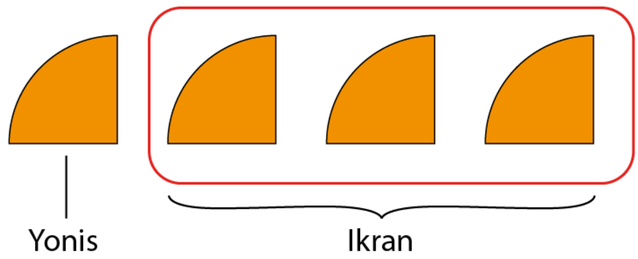
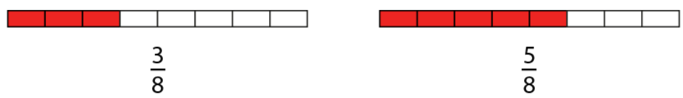
Teaching point 7:

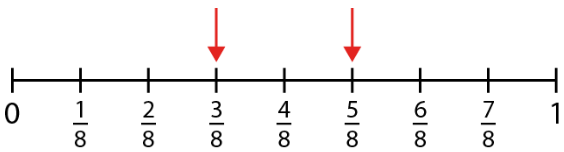
Non-unit fractions with the same denominator can be compared. If the denominators are the same, then the greater the numerator, the greater the fraction.

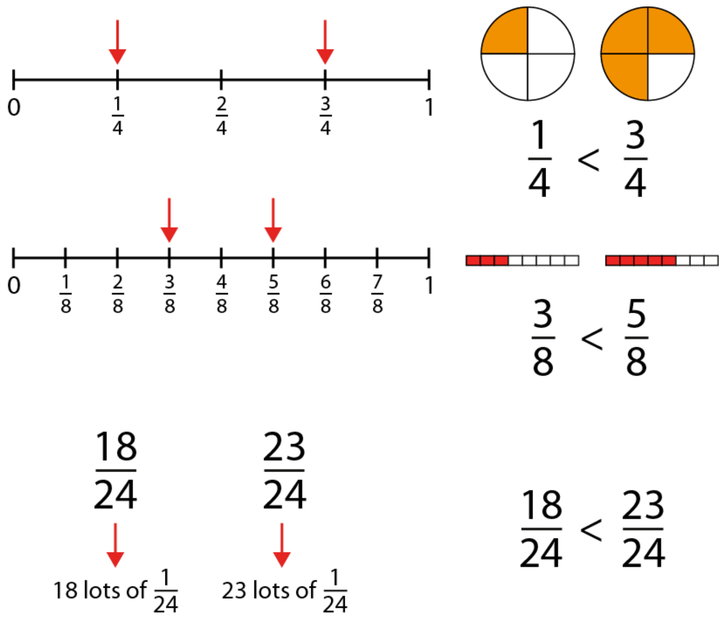
Steps in learning

	Guidance	Representations
7:1	In segment *3.2 Unit fractions: identifying, representing and comparing*, children were introduced to comparing unit fractions using area models. They learnt that in order to compare fractions using area models, the wholes need to be the same size. *Teaching points 7* and *8* will now extend and build on this prior learning, both by revisiting unit fractions now that the children have learnt that fractions are numbers, and to extend comparison of fractions to non-unit fractions. Present children with a simple story, for instance: *'Yonis has one-quarter of an orange and his sister Ikran has three-quarters of an orange.'* Challenge children to find different ways to convince you that Ikran has more orange than Yonis. Gather and discuss their answers as a group. • Some children may have drawn representations of the orange and shaded one-quarter and three-quarters to visually prove that three-quarters is greater than one-quarter. • Some children may have positioned one-quarter and three-quarters on a number line. (Recognising fractions as numbers is still quite a new concept to them, so this is the least likely scenario.) • Some children may have verbally reasoned that because three-quarters is three lots of one-quarter, three-quarters must be greater than one-quarter.	

	• Some children may have drawn four segments of an orange and shaded or circled three segments to show that three-quarters of an orange is more than one-quarter of an orange.	
7:2	Summarise the ideas offered by the children in step *7:1*, and introduce any of the solutions that weren't suggested. For *Method 1 – on a diagram*, note that any area model can be used; it doesn't have to be circles. For *Method 3 – verbal reasoning*, introduce some formalised stem sentences that all children will be able to learn and use. Encourage them to write the fraction notation in their completed stem sentences. • ***'$\frac{\square}{\square}$ is ___ lot of $\frac{1}{\square}$'*** • ***'$\frac{\square}{\square}$ is ___ lots of $\frac{1}{\square}$'*** • ***'I know that ___ is less than ___…'*** ***'…so $\frac{\square}{\square}$ is less than $\frac{\square}{\square}$'***	• *'Yonis has one-quarter of an orange and his sister Ikran has three-quarters of an orange.'* Method 1 – on a diagram  $$\frac{1}{4} < \frac{3}{4}$$ Method 2 – on a number line  $$\frac{1}{4} < \frac{3}{4}$$

		Method 3 – verbal reasoning • '$\frac{\boxed{1}}{\boxed{4}}$ *is one lot of* $\frac{\boxed{1}}{\boxed{4}}$' • '$\frac{\boxed{3}}{\boxed{4}}$ *is three lots of* $\frac{\boxed{1}}{\boxed{4}}$' • *'I know that <u>one</u> is less than <u>three</u>…'* '*…so* $\frac{\boxed{1}}{\boxed{4}}$ *is less than* $\frac{\boxed{3}}{\boxed{4}}$' Method 4 – with a quantity model  $$\frac{1}{4} < \frac{3}{4}$$
7:3	Repeat the exercise from the previous step, but this time using a different denominator. Present a new scenario, such as the one opposite, and ask children to find more than one way to justify their answer. Examples of potential responses are provided. Where verbal reasoning is used, encourage children to use the given sentence structure format. This will ensure that all children develop experience in using clear chains of reasoning to support a conclusion. (It is less likely that children will use a quantity model to explain this).	• *'Polly has done* $\frac{3}{8}$ *of her homework. Ola has done* $\frac{5}{8}$ *of her homework. Convince me that Ola has completed more of her homework than Polly has.'* Method 1 – on a diagram  $$\frac{3}{8} < \frac{5}{8}$$

	Repeat this exercise with other pairs of fractions that have the same denominator, until the children are confident explaining which fraction is larger using any of the three methods opposite.	Method 2 – on a number line  $$\frac{3}{8} < \frac{5}{8}$$ Method 3 – verbal reasoning • '$\frac{3}{8}$ *is three lots of* $\frac{1}{8}$' • '$\frac{5}{8}$ *is five lots of* $\frac{1}{8}$' • *'I know that three is less than five…'* *'…so* $\frac{3}{8}$ *is less than* $\frac{5}{8}$'
7:4	The next step is to introduce a pair of fractions with a larger denominator, such as $\frac{18}{24}$ and $\frac{23}{24}$. Ask children: *'Which is greater?'* Ask them to consider the methods they have learnt so far in *Teaching point 7* and debate which is the easiest in order to compare these two fractions. Discuss their responses and ideas, including the efficiencies and inefficiencies of the various methods. Some children may attempt to represent the fractions using an area model, quantity model or number line. However, now that the denominator is larger, these options are going to take a lot longer to create. They may possibly be able to visualise an area or set model, or a number line. Ultimately, the most efficient method is to use the	*'Which is greater:* $\frac{18}{24}$ or $\frac{23}{24}$*?'* • '$\frac{18}{24}$ *is eighteen lots of* $\frac{1}{24}$' • '$\frac{23}{24}$ *is twenty-three lots of* $\frac{1}{24}$' • *'I know that twenty-three is greater than eighteen…'* *'…so* $\frac{23}{24}$ *is greater than* $\frac{18}{24}$'

	verbal chain of reasoning, as this is no more complex than it was before. Because both fractions are made up of several one twenty-fourth unit fractions, this means we can compare them easily; the one with more unit fractions (or larger numerator) is the greater.	
7:5	By this stage, children should have confidence in their understanding that non-unit fractions are made up of more than one unit fraction of the same kind. All children should have encountered a variety of opportunities and experiences in comparing fractions diagrammatically. Now arrive at the following generalised statement: ***'When we compare fractions with the same denominator, the greater the numerator, the greater the fraction.'*** Assess further pairs of fractions with the same denominators, referring back to this generalised statement each time. Encourage children to reason using the idea that each non-unit fraction is made of several (two or more) of the same unit fraction and that whichever has more unit fractions is the greater.	
7:6	Finally, give children varied practice in ordering and comparing two or more fractions with the same denominator. Encourage them to use the generalisation repeatedly, including when you are discussing questions as a class. Be aware though, that while working on becoming fluent in the language of the generalisation, children do not lose focus on *why* they are using the statement and what it represents. Make sure that alongside repeating the generalised statement, they can also explain that the larger non-unit fraction in each pair is made	

up of *more* iterations of the unit fraction than the smaller non-unit fraction.

Varied practice should include the following:

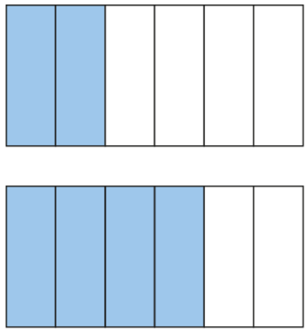
- Shading and comparing using the language of *less than* and *greater than*. Models such as those provided opposite will support children's understanding and allow them to compare easily.
- Real-life problems offer valuable opportunities for children to relate fractions problems to real-life contexts.
- Missing-symbol problems. Ask children to apply the generalisation from step *7:5* to compare pairs of fractions using the appropriate symbols.
- Ordering. Ask children to compare three or more fractions, placing them in ascending order. Model how we can use the < symbol between fractions to show how each number compares to those to either side.

In each instance, it is important to use a range of denominators for children to compare each time. Notice how prior learning from previous teaching points is also referred to in these examples.

Children should consolidate and further deepen their understanding through practice with a range of questions, such as this dòng nǎo jīn problem.

Shading and comparing:

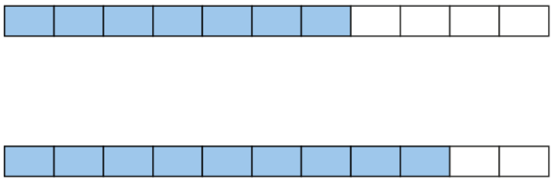
'Shade the fractions, compare them and then complete the comparison statements.'



$\frac{2}{6}$

$\frac{4}{6}$

'$\frac{2}{6}$ *is less than* $\frac{4}{6}$'



$\frac{7}{11}$

$\frac{9}{11}$

'$\frac{9}{11}$ *is greater than* $\frac{7}{11}$'

$\frac{2}{9}$

$\frac{5}{9}$

'$\frac{2}{9}$ *is less than* $\frac{5}{9}$'

Real-life problems:

- *'Class 3B have used $\frac{5}{12}$ of a box of pens. Class 3C have used $\frac{4}{12}$ of a box of pens. Which class has used more?'*
- *'Hibba swam $\frac{6}{10}$ km. David swam $\frac{3}{10}$ km. Who swam further?'*

Missing-symbol problems:

'Fill in the missing symbols (<, >, or =).'

$\frac{3}{7} \bigcirc \frac{2}{7}$ $\frac{4}{11} \bigcirc \frac{9}{11}$ $\frac{12}{25} \bigcirc \frac{18}{25}$

$\frac{17}{23} \bigcirc 1 \bigcirc \frac{23}{23}$

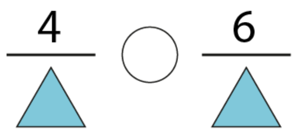
Ordering:

'Arrange the following numbers in order, from smallest to largest, separating each with the symbol <.'

$\frac{7}{11}$ $\frac{3}{11}$ $\frac{10}{11}$ 1 $\frac{4}{11}$

Dòng nǎo jīn:

- *'Which symbol should always go in the circle? Explain your answer.'*



- *'Which number(s) could go in the missing-number box to make this statement true?'*

$\frac{1}{4} > \frac{1}{\square} > \frac{1}{10}$

Teaching point 8:

Non-unit fractions with the same numerator can be compared. If the numerators are the same, then the greater the denominator, the smaller the fraction.

Steps in learning

8:1

Guidance

In *Teaching point 7*, children focused on comparing fractions where the denominators were the same. This teaching point will now explore comparing fractions where the numerators are the same.

Background understanding for this teaching point will draw on learning from segment *3.2 Unit fractions: identifying, representing and comparing, Teaching point 5,* where children compared unit fractions using diagrams before arriving at the following generalisations:

- ***'When comparing unit fractions, the greater the denominator, the smaller the fraction.'***
- ***'When we compare fractions, the whole has to be same.'***

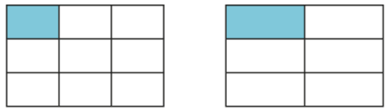
Begin by looking at area models of unit fractions. Ask children to write the notation for the unit fractions that are represented and then compare them using the < or > symbols. Note that the symbols < and > should only be used between numbers and not between images.

Remind children of the generalisation they met in segment *3.2* – that in order to compare fractions, the whole has to be the same.

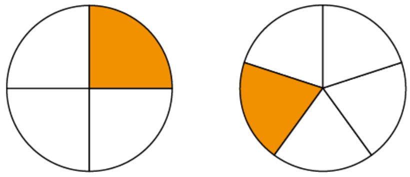
Progress quickly to revising comparing unit fractions by looking at the denominator. Encourage children to use this generalisation from segment *3.2* to tackle the comparison problems opposite:

Representations

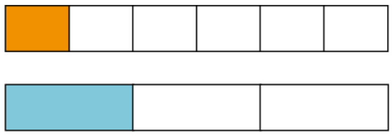
- *'Write the fractions that are represented, then use < or > in each circle to compare each pair of fractions.'*



$$\frac{1}{9} < \frac{1}{6}$$



$$\frac{1}{4} > \frac{1}{5}$$



$$\frac{1}{6} < \frac{1}{3}$$

- *'Compare these fractions using the denominator. Fill in the missing symbols (<, >, or =).'*

$\frac{1}{6}$ ○ $\frac{1}{9}$ $\frac{1}{8}$ ○ $\frac{1}{3}$

$\frac{1}{10}$ ○ $\frac{1}{4}$ $\frac{1}{7}$ ○ $\frac{1}{12}$

<table>
<tr><td></td><td>‘When comparing unit fractions, the greater the denominator, the smaller the fraction.’</td><td></td></tr>
<tr><td>8:2</td><td>
Since encountering unit fractions for the first time, children have subsequently learnt that fractions are numbers and that every fraction has a position on the number line. Before we progress to comparing pairs of fractions with a common numerator, it is important to first look at the position of unit fractions on a number line to support our comparison between unit fractions.
Think back to the pouring activities and images from segment 3.2 Unit fractions: identifying, representing and comparing where children investigated filling different unit fractions of a vase. Show children the empty container again. Tell them the container is one-half full and ask them to point to roughly where the fluid would come up to. Mark this rough point on a drawing of an empty container on the board.
Repeat this for one-third, one-quarter and one-tenth, showing the estimated position first on the container and then marking it on the drawing on the board.
Show how, by adding ‘0’ and ‘1’ to the drawing of the container to signify one empty and one full container, we have identified the positions of the unit fractions on a zero–one number line. The number one-half is half way between zero and one. The number one-third is one-third of the way from zero to one, and so forth. It is, of course, possible to position them by splitting the number line into equal parts, but it is important that children use their developing ‘fraction sense’ to position them.
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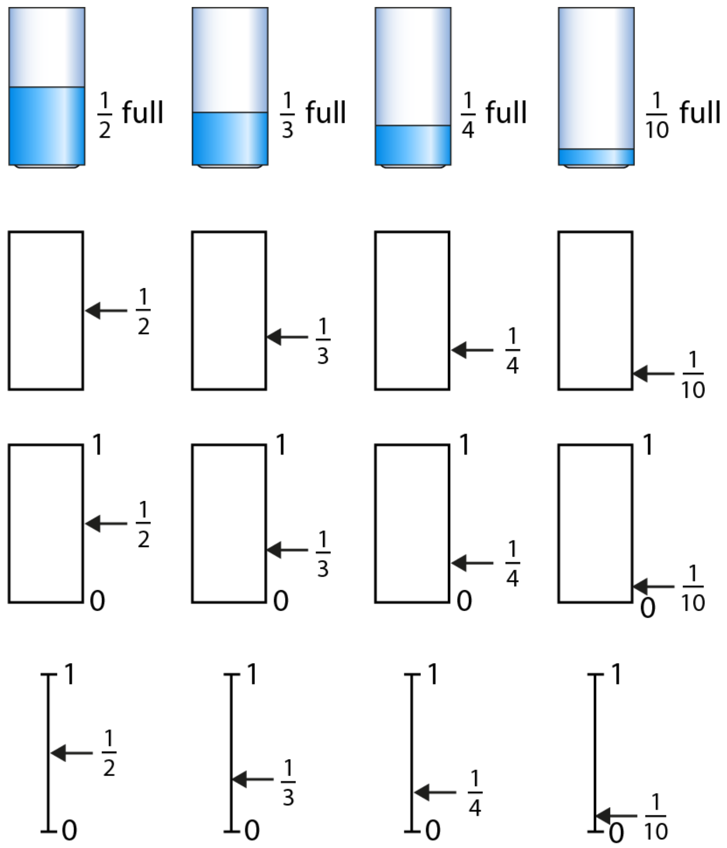

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8:3

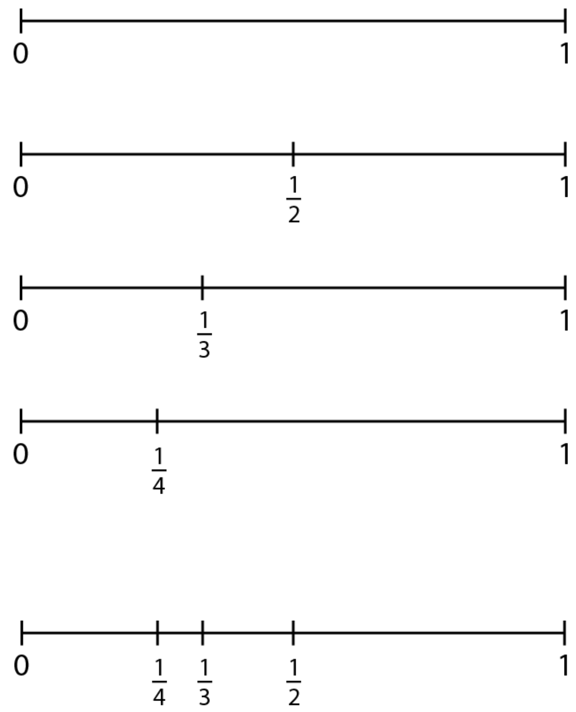
By this stage, children know that a fraction is a number and have seen some unit fractions marked on vertical number lines. We can now look at unit fractions on a horizontal number line. Show children several number lines with '0' and '1' marked at either end. This could be an image on the board or a screen, or a physical example such as several equal lengths of string strung up across the classroom. If using string number lines, children could have cards to hang or peg on the lines.

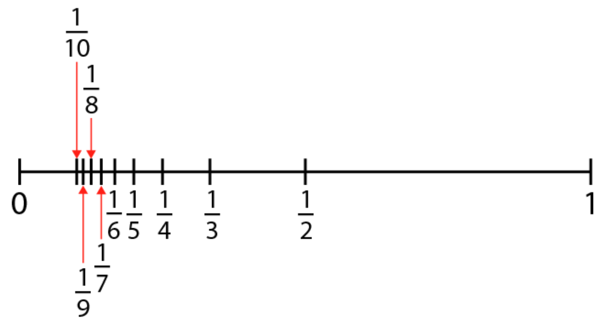
Ask children to explain where each unit fraction should be placed on the number line. Start with one-half and ask where they would position it. Prompt children to think back to the pouring activity. The number one-half is half way along the number line.

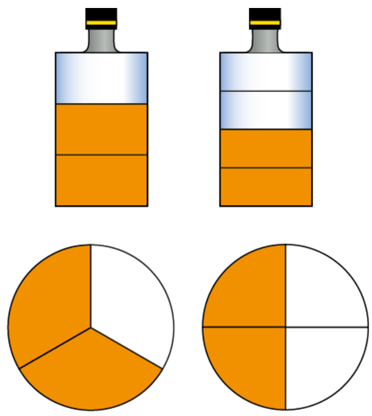
Take the next number line and ask whether one-third would be closer to zero or to one. Use the following stem sentence to help: ***'The whole is divided into ___ equal parts and we have ___ of them.'***

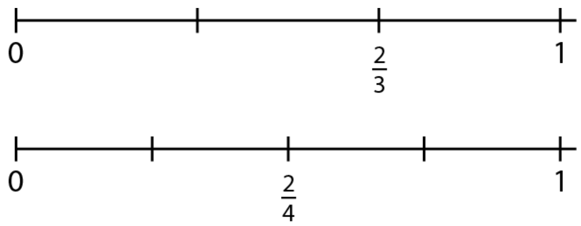
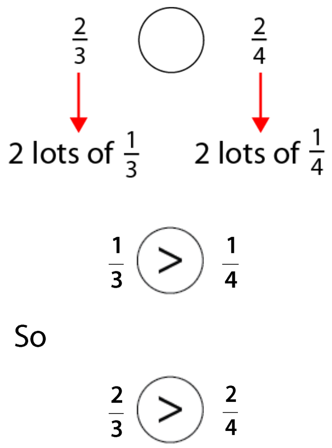
Once the children visualise the line cut into three equal parts, they should be able to identify that the number one-third is one-third of the way along the number line. Working roughly is fine, and indeed to be encouraged – it isn't necessary to measure or divide the line or string into three exact parts.

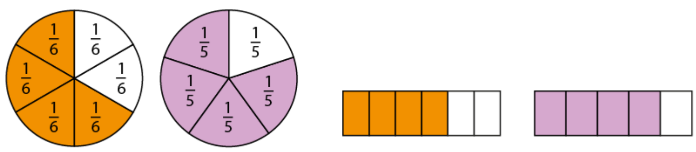
Continue to place more unit fractions, discussing your reasoning for the placement of each fraction. Merge the number lines into a single unit fraction number line as shown opposite. (Note that children often find it easier to place unit fractions on an empty number line, rather than one with other unit fractions already placed.)

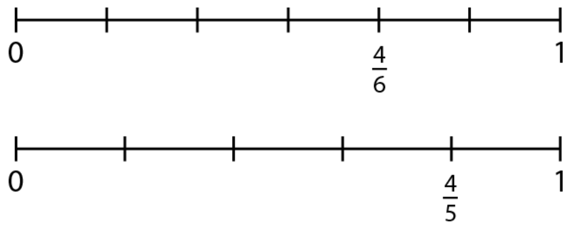


8:4	We are now ready to combine the separate number lines into a single unit fraction number line. If you are working with string, you may want to move to looking at a unit-fraction number line on-screen at this stage. Look at the number line opposite. Discuss what the children notice about where the fractions placed. They should see that the larger the denominator, the closer the fraction is to zero. Link this back to the previous learning on comparing unit fractions in steps *8:1* and *8:2* and recap the generalisation that: ***'When comparing unit fractions, the greater the denominator, the smaller the fraction.'*** The children may also notice that the fractions are quite closely grouped. This will be the first time that children have seen a number line with unequal spacing. It can appear quite unusual to them at first, so allow for some time to discuss it. Ask children to reason where various other unit fractions would go, for example $\frac{1}{20}$, $\frac{1}{100}$, $\frac{1}{327}$ and $\frac{1}{1{,}000{,}000}$. Explain that every unit fraction smaller than one-tenth will be located within (what is usually represented as) the relatively small space between zero and one-tenth. Children are generally amazed at this and are often intrigued that so many numbers can fit in such a small space in the number line, so do take some time to engage with this thoroughly.	

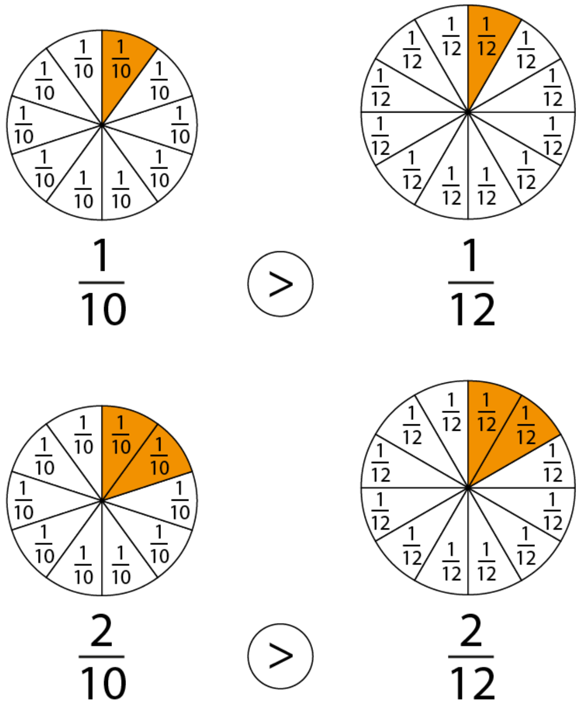
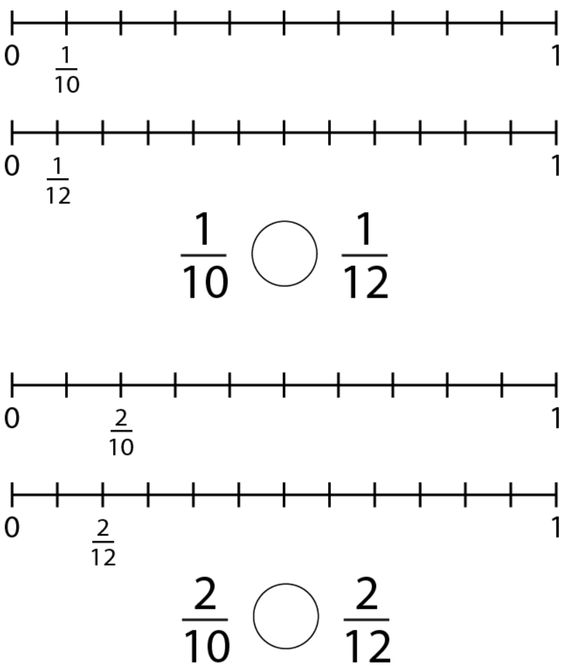
8:5	Now that the relative size of unit fractions has been explored on both area models and number lines, allow children to compare unit fractions using the number line and inequality symbols, < and >. Children can now utilise several strategies to explain why, for example $\frac{1}{8} < \frac{1}{6}$. Strategies include: • Using a number line: one-eighth is nearer to zero than one-sixth is. • Considering the size of the parts: In one-eighth the whole is divided into eight equal parts. In one-sixth the whole is divided into six equal parts. The more parts there are, the smaller each part is. So, $\frac{1}{8} < \frac{1}{6}$. • Using the generalisation: ***'When comparing unit fractions, the greater the denominator, the smaller the fraction.'***	Missing-symbol problems: *'Fill in the missing symbols (<, >, or =).'* $\frac{1}{6} \bigcirc \frac{1}{9}$ $\frac{1}{8} \bigcirc \frac{1}{3}$ $\frac{1}{10} \bigcirc \frac{1}{4}$ $\frac{1}{7} \bigcirc \frac{1}{12}$ Ordering: *'Arrange the following numbers in order, from smallest to largest.'* $\frac{1}{9}$ $\frac{1}{7}$ $\frac{1}{15}$ $\frac{1}{3}$ $\frac{1}{10}$ Missing-number problems: *'Fill in the missing digits to make each statement true.'* $\frac{1}{6} < \frac{1}{\square}$ $\frac{1}{\square} > \frac{1}{8}$
8:6	We can now advance to comparing non-unit fractions with the same numerator. Describe a scenario, such as the one given opposite, to the children. Present the fraction numbers on the board. Ask the children to discuss with a partner, different ways to convince you that Izzy has more juice in her bottle than Olivia has, and that two-thirds is indeed greater than two-quarters. Discuss the validity of the suggestions children present. Their strategies may include creating a diagram; this could be an image which bears a resemblance to a bottle, like the first image, or one that represents two-thirds and two-quarters in a different way, such as the circular diagrams opposite. Or they may choose to position $\frac{2}{3}$ and $\frac{2}{4}$ on a number line.	• *'Izzy and Olivia each have an identical juice bottle. Izzy's bottle is $\frac{2}{3}$ full. Olivia's bottle is $\frac{2}{4}$ full. Izzy has more juice than Olivia. Suggest different ways to prove that $\frac{2}{3} > \frac{2}{4}$.'* Method 1 – on a diagram: 

	Some children may opt for a verbal reasoning strategy to explain why two-thirds must be greater than two-quarters.	Method 2 – on a number line:  Method 3 – verbal reasoning: *'$\frac{2}{3}$ is two lots of $\frac{1}{3}$.'* *'$\frac{2}{4}$ is two lots of $\frac{1}{4}$.'* *'I know that $\frac{1}{3}$ is greater than $\frac{1}{4}$, so I know that two lots of $\frac{1}{3}$ is greater than two lots of $\frac{1}{4}$.'*
8:7	Focus on the verbal reasoning strategy. Tell children that we will unpick how we can verbally justify our conclusion that $\frac{2}{3} > \frac{2}{4}$. Remind children that two-thirds is two one-thirds, or two lots of one-third. Also remind them that two-quarters is two one-quarters, or two lots of one-quarter. Children already know that to compare unit fractions, we can simply compare the denominators: the greater the denominator, the smaller the fraction. Explain that because we know that one-third is greater than one-quarter, we also know that two-thirds is greater than two quarters. Spend time here securing this important reasoning, including discussion around the size of the unit parts in each fraction. Reiterate that each of the two parts in two-thirds is bigger than each of the two parts in two-quarters, so in total two-thirds must be greater than two-quarters.	

	Summarise the reasoning: *'$\frac{2}{3}$ is two lots of $\frac{1}{3}$.'* *'$\frac{2}{4}$ is two lots of $\frac{1}{4}$.'* *'I know that $\frac{1}{3}$ is greater than $\frac{1}{4}$, so I know that two lots of $\frac{1}{3}$ is greater than two lots of $\frac{1}{4}$.'* Giving children experience in verbalising chains of reasoning like this – where they make statements about several things they know and then use their statements to form a new logical conclusion – will lay the foundations for children to write written proofs in sequences of equations later in their mathematics education. At secondary level, they will be expected to argue from a set of assumptions to a conclusion, using set rules of deduction. To conclude this step, display all three methods together (diagram, number line and verbal reasoning), as a visual reminder of the different strategies they've used to compare two fractions with the same numerator.	
8:8	To consolidate this part of the learning, repeat the exercise using a different pair of fractions with the same numerators and small denominators. For example, four-sixths and four-fifths. Expose children again to all three different methods.	Method 1 – on a diagram:  $\frac{4}{6} < \frac{4}{5}$ $\frac{4}{6} < \frac{4}{5}$

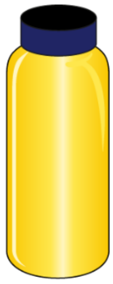
		Method 2 – on a number line:  $$\frac{4}{6} < \frac{4}{5}$$ Method 3 – verbal reasoning: *'$\frac{4}{6}$ is four lots of $\frac{1}{6}$.'* *'$\frac{4}{5}$ is four lots of $\frac{1}{5}$.'* *'I know that $\frac{1}{5}$ is greater than $\frac{1}{6}$, so I know that four lots of $\frac{1}{5}$ is greater than four lots of $\frac{1}{6}$.'*
8:9	Now explain that you are going to compare another pair of fractions but this time with much larger denominators, such as seven-twentieths and seven-thirtieths (see the example opposite). Some children may attempt to divide a circle or rectangle into 20 and 30 equal parts, but may well give up! Also be prepared for children who start drawing 'notches' on a number line. Leave pupils to consider the usefulness and efficiency of the various methods. Then work through the question together, expressing each fraction as several unit fractions and arriving at the reasoning that because one-twentieth is greater than one-thirtieth, this means that seven-twentieths is greater than seven-thirtieths.	$\frac{7}{20}$ (>) $\frac{7}{30}$ ↓ ↓ 7 lots of $\frac{1}{20}$ 7 lots of $\frac{1}{30}$ $\frac{1}{20}$ (>) $\frac{1}{30}$ So $\frac{7}{20}$ (>) $\frac{7}{30}$ *'$\frac{7}{20}$ is seven lots of $\frac{1}{20}$.'* *'$\frac{7}{30}$ is seven lots of $\frac{1}{30}$.'* *'I know that $\frac{1}{20}$ is greater than $\frac{1}{30}$, so I know that seven lots of $\frac{1}{20}$ is greater than seven lots of $\frac{1}{30}$.'*

	Summarise the reasoning using the following stem sentences (as in the examples on the previous page): • ***'$\frac{\square}{\square}$ is ___ lots of $\frac{\square}{\square}$'*** • ***'$\frac{\square}{\square}$ is ___ lots of $\frac{\square}{\square}$'*** • ***'I know that $\frac{\square}{\square}$ is greater than $\frac{\square}{\square}$ …'*** ***'…so I know that ___ lots of $\frac{\square}{\square}$ is greater than ___ lots of $\frac{\square}{\square}$'*** When the denominators are large numbers, drawing diagrams or number lines is less feasible, but the use of the chain or reasoning through the generalised statement can still be easily applied.	

8:10	To ensure children are truly confident with the idea that we can use our knowledge of unit fractions to compare non-unit fractions with the same numerator, display the two sequences of images opposite. Each time an additional part is revealed, ask children which fraction is the greater now. It should really help to cement the idea that when the numerators are the same, the fraction with a denominator of ten will always be larger than the fraction with a denominator of twelve.	

8:11	Consolidate learning by showing the following activity, and completing the inequalities. Ask the children to convince you at each step. For example, *'How can you convince me that three-tenths is greater than three-twelfths?'* The stem sentences from *step 8.9* provide an excellent speaking frame to support their reasoning and you can refer back to the images you have just looked at as a visual aid. You can now introduce the generalisation: ***'When we compare fractions with the same numerator, the greater the denominator, the smaller the fraction.'*** Look at other 'same numerator' examples and give children plenty of practice using the generalisation alongside the reasoning they developed earlier in this teaching point.	$\frac{1}{10} \bigcirc \frac{1}{12}$ $\frac{2}{10} \bigcirc \frac{2}{12}$ $\frac{3}{10} \bigcirc \frac{3}{12}$ $\frac{4}{10} \bigcirc \frac{4}{12}$ ⋮ $\frac{10}{10} \bigcirc \frac{10}{12}$
8:12	Offer further varied practice that specifically allows children to apply the generalisation: ***'When we compare fractions with the same numerator, the greater the denominator, the smaller the fraction.'*** Practice should include the following: • missing-symbol problems with the same numerators • missing-number problems • ordering a set of fractions with the same numerator.	Missing-symbol problems: *'Fill in the missing symbols (< or >).'* $\frac{2}{5} \bigcirc \frac{2}{3}$ $\frac{1}{5} \bigcirc \frac{1}{3}$ $\frac{7}{20} \bigcirc \frac{7}{8}$ $\frac{3}{10} \bigcirc \frac{3}{100}$ $\frac{5}{12} \bigcirc \frac{5}{11}$ $\frac{4}{7} \bigcirc \frac{4}{10}$ Missing-number problems: *'Fill in the missing numbers. Can you find more than one solution for each statement?'* $\frac{9}{19} < \frac{20}{\square}$ $\frac{8}{\square} > \frac{8}{13}$ $\frac{\square}{9} < \frac{8}{\square}$ $\frac{\square}{6} > \frac{2}{\square} > \frac{\square}{10}$

		Ordering: • *'Arrange the following numbers in order, from smallest to largest, separating each with the symbol <.'* $\frac{3}{7}$ $\frac{3}{8}$ $\frac{3}{13}$ $\frac{3}{5}$ $\frac{3}{11}$ • *'Arrange the following numbers in order, from largest to smallest, separating each with the symbol >.'* $\frac{7}{11}$ $\frac{7}{9}$ $\frac{7}{15}$ $\frac{7}{8}$ $\frac{7}{7}$
8:13	These deeper dòng nǎo jīn problems require pupils to combine learning from *Teaching points 7* and *8*. • *Example 1:* Once children have had the opportunity to tackle this independently, work through the problem as a group. Use the examples and reasoning opposite. • *Example 2:* To consolidate their understanding, ask children to compare the lists of fractions in this and *Example 1*. All can be compared using the reasoning from within this segment. (Note that using common denominators is an inefficient way to compare examples. As children have not yet met this concept, they are unlikely to use it. However, if any children have been introduced to this concept outside of school and attempt to use it, direct them back to comparing the fractions using the reasoning and generalisations learnt in this segment.) • *Example 3:* In order to answer the juice problem opposite, children need to first determine what fraction of the juice Sally and Freddy have left. They can then compare the remaining amounts. • *Example 4:* Similar logic can be used to order these fractions.	Dòng nǎo jīn: Example 1 • *'Arrange the following numbers in order, from smallest to largest, separating each with the symbol <.'* $\frac{3}{7}$ $\frac{5}{7}$ $\frac{3}{10}$ • *'Because I am comparing fractions with the same denominator, I know that* $\frac{3}{7} < \frac{5}{7}$*.'* • *'Because I am comparing fractions with the same numerator, I know that* $\frac{3}{10} < \frac{3}{7}$*.'* • *'So, the order must be* $\frac{3}{10} < \frac{3}{7} < \frac{5}{7}$*.'* Example 2: *'Order the following sets of numbers from smallest to largest, separating each with the symbol <.'* $\frac{3}{11}$ $\frac{7}{8}$ $\frac{3}{8}$ $\frac{5}{5}$ $\frac{4}{12}$ $\frac{4}{20}$ $\frac{5}{12}$ $\frac{3}{20}$

	Recognising that each of these non-unit fractions are one part away from a whole, is the key to ordering them using reasoning. Take advantage of any opportunities to relate fraction problems to real-life contexts.	*Example 3:* • *'Sally and Freddy each have a bottle of juice the same size.'*  • *'Sally has drunk $\frac{2}{3}$ of her juice.'* • *'Freddy has drunk $\frac{4}{5}$ of his juice.'* • *'Who has more juice left?'* • *'Sally has $\frac{1}{3}$ of her juice left.'* • *'Freddy has $\frac{1}{5}$ of his juice left.'* • *'$\frac{1}{3} > \frac{1}{5}$ so, Sally has more juice left.'* Example 4: *'Arrange the following numbers in order, from smallest to largest, separating each with the symbol <.'* $\frac{7}{8}$ $\frac{9}{10}$ $\frac{1}{2}$ $\frac{14}{15}$ $\frac{2}{3}$ $\frac{5}{6}$ $\frac{4}{5}$