National Centre for Excellence in the Teaching of Mathematics

Professional development resource

Unit 2 - Understanding and identifying proportional contexts
Lesson 2e: Using the DNL to explore relations

Lesson summary

In this lesson we look at numerical relations that arise from a variety of contexts. We represent the relations in several ways, starting with the relatively familiar table, mapping diagram and Cartesian graph. However, for most of the lesson we consider how relations can be represented on a double number line (DNL) and we compare the resulting DNLs to help us differentiate between, and identify the characteristics of, ratio and non-ratio relations.

In the second lesson, we consider how the DNL can be used to analyse and solve progressively more abstract ratio (and non-ratio) problems, with a focus on multiplication as scaling.

Zoom Taxis

Tariff

£3 basic fare
plus
£2 per mile

Focus of student learning

In this lesson we use the DNL to look at ratio and non-ratio relations arising from various contexts.

- How readily do students identify and differentiate between ratio and non-ratio relations?
- How readily can students use the DNL to represent relations?
- How readily can students identify different features of the DNL?
- How readily can students use the DNL to identify and differentiate between ratio and various non-ratio relations?

Lesson preparation

- It is important to try the **Pre-task** with the class a few days before teaching Lesson 2e.
- The file **UNIT-2ef-slides.pdf** contains
  - slides of the **Pre-task**
  - slides of the Stage 1 task and diagrams
  - slides of the Stage 2 tasks and diagrams
  - [slides of Lesson 2f]
  - a number-relations challenge (for us rather than the students?).

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Lesson 2e Pre-task

Pre-task

Painting fence posts

Kelly works at a large garden centre.
Today her job is to paint fence posts.

After 8 minutes Kelly has painted 3 posts.

a. Predict how many posts Kelly will have painted after 20 minutes.
Give a reason for each prediction.

b. It turns out that after 20 minutes Kelly has painted 9 posts.
Explain how this number might have come about.

c. After half an hour Kelly has painted 14 posts.
Explain how this number might have come about.

d. Predict how many posts Kelly will have painted after 1 hour.
Give a reason for each prediction.

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Commentary: Lesson 2e Pre-task

The purpose of this task is to *harvest* students' thinking, in order to help with the planning of Lesson 2e.

The task can be done as homework or individual classwork though it is probably best suited to a whole-class discussion - in which case show each part in turn, ie only after the previous part has been tackled.

You might want to spend 10 - 20 minutes on the task.

The task involves thinking about the possible relationship between two variables - the *number of minutes* that have elapsed since starting to paint fence posts and the *number of posts* that have been painted. In an idealised situation these numbers would be in proportion but the actual numbers given in the task are not.

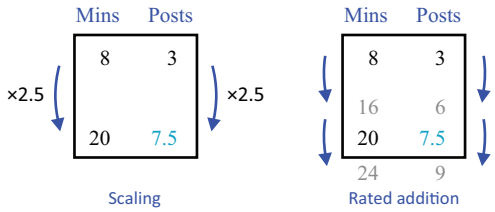
Thus the task can be used to probe whether students

- are aware that a ratio relation could exist between the numbers
- can recognise that the given numbers are not in proportion
- can find plausible reasons why the given numbers are not in proportion.

a. Some students might focus on the given numbers (rather than the context) and come up with the additive solution 15 [8–5=3, 20–5=15; or 8+12=20, 3+12=15].

Others might be aware that there could be a ratio relation. If so, do they

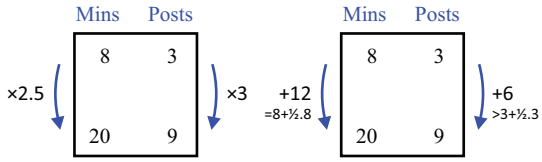
- express this in everyday or mathematical terms
- try to find the exact answer (7½ posts) and if so, how
- estimate, and if so, how
- consider practical constraints that modify the relation?



b. It seems that Kelly has speeded up [9 > 7½; or 3×3=9, but 3×8>20 (below); or 12=8+½.8, but 6>3+½.3 (below)].

Perhaps she has overcome teething problems, or become more efficient with practice.

In the last 12 minutes Kelly has painted 6 posts and so is working at an average rate of 1 post every 2 minutes. This is somewhat faster than her averate rate in the first 8 minutes (3 posts rather than 4 in 8 minutes).



Note: We could 'flip' the above (minimal) ratio table, like this, so that it more closely matches the DNLs that are used in these lessons, where the values of each variable are shown *horizontally*. The given ratio table matches the order in which the numbers 8, 3, 20 happen to have been given in the *Painting fence posts* task.

Mins	8	20
Posts	3	?

c. Kelly has painted 5 posts in the last 10 minutes, so she is continuing at an average rate of 1 post every 2 minutes.

d. Students might come up with a variety of answers, for example
29 post (assuming she continues at her average rate of 1 post every two minutes)
30 posts, say (perhaps she has become even more efficient)
28 posts, say (perhaps she is getting tired, or bored, or has taken a break).

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The Lesson

Stage 1.
As a prelude to exploring the Double Number Line (DNL) we look at three forms of representation that are probably more familiar: the table, mapping diagram, and Cartesian graph.

Zoom Taxis

Tariff

£3 basic fare
plus
£2 per mile

1 a) i. What does a 6 mile journey cost with *Zoom Taxis*?

ii. What does a 4 mile journey cost?

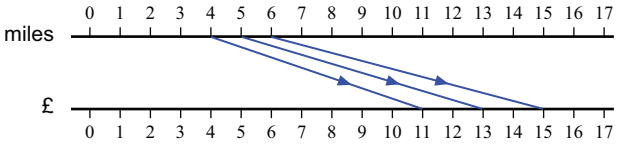
iii. What does a 5 mile journey cost?

b) Imagine representing the journeys in part a) in

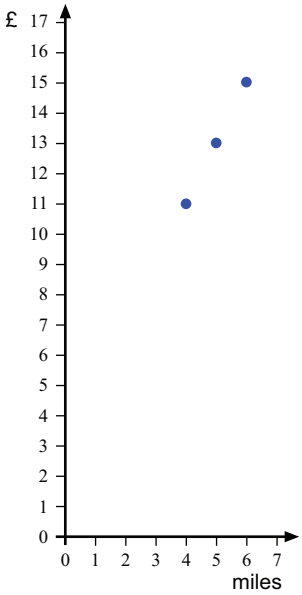
an ordered table,

miles	£
4	11
5	13
6	15

a mapping diagram, and



a Cartesian graph:



Compare the three representations.

i. Where would the given information (about £3 and £2) appear in each?

ii. What are the strengths and weaknesses of the representations?

Stage 2.

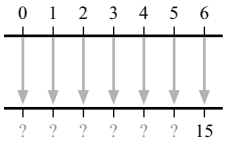
The DNL is another representation, with some similarities to each of the above. Along with the Cartesian graph, it is particularly well suited to representing ratio relations, but in a less abstract way. It also provides a powerful model for the notion of *multiplication as scaling*.

For the rest of this lesson (and the next) we focus on the DNL and consider the degree to which it can be used easily and effectively to represent different contexts. We consider ratio and non-ratio contexts in the hope that the DNL will help students to differentiate between them.

From now on we are going to use a Double Number Line (DNL) to represent situations like *Zoom Taxis* journeys. To keep things relatively simple, the situations all involve the mapping 6 → 15.

The DNL has two horizontal axes (like the mapping diagram). *However* (unlike the mapping diagram), the mapping arrows are all vertical (although we don't usually bother to draw them). This means that the two axes are likely to be numbered differently.

We usually number the top axis *uniformly*, as here. →



So the problem we have to solve is this:

How do we number the bottom axis?

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Lesson commentary

Stage 1.

You will need to judge how much time to devote to this first stage - it provides a rich activity which could well form a lesson in its own right (especially if students are given the opportunity to construct the representations themselves). For the current lesson, the key ideas are i. that we can represent mathematical relations in all sorts of ways and ii. that these may be more or less effective in conveying specific properties of the relation. Thus it is not necessary to tease out all of the features referred to below.

1 a) This acts as a check on whether students understand the 'story'.
It also introduces the mapping 6 → 15 which features in all of the tasks in this lesson.

b) **The table**: if the information in the table is ordered (as here), students might notice that as the number of miles increases by 1, the number of £s increases by 2. (Why?)

And they might notice that when the number of miles is 0, the number of pounds is 3.
(Some might argue that this pair of values is not allowed as it does not represent a proper journey).

The mapping diagram: some students might notice that the arrows fan out and that if the gap between a pair of arrows is 1 on the top line, it will be 2 on the bottom line - Why?

And they might notice that 0 would maps onto 3, the basic fare (if this pair of values is 'allowed').

The Cartesian graph: students might realise that we get points that lie on a straight line. (This applies to whole numbers of miles: we don't know how fractional values are dealt with by *Zoom Taxis* - if the mileage is rounded up or down to the nearest whole mile, say, we would have a step function.)

And they might notice that if the horizontal distance between a pair of points is 1, the vertical distance is 2 (ie the gradient of the line is 2, the number of £s per mile); and that the line cuts the £ axis at the intercept 3, the basic fare.

It can be argued that the table, mapping diagram and Cartesian graph are progressively more abstract representations.

The table provides numerical information more explicitly than the other two representations but it is not so easy to see the more general picture.

The mapping diagram and graph use geometric features to convey properties of the relation. These can be quite salient (eg the steepness of the line which the points form on the Cartesian graph, indicates the magnitude of the cost per mile). However, these features can also seem quite artificial - they don't necessarily emerge 'naturally' from the context, in which case students will need to learn how to read them.

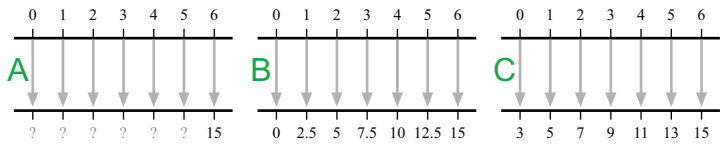
The two scales on the Cartesian axes don't have to be the same (in contrast to the mapping diagram). This has practical advantages. However, it can affect the 'steepness' of a line which might mislead students about the nature of the mathematical relationship, though the gradient, of course, remains the same.

Stage 2.

The Double Number Line (DNL) is commonly used just for ratio relations, ie relations of the form $x \to kx$.
In such cases, the scale is linear on both axes and the zeros are aligned*. To highlight these characteristics we deliberately consider other contexts, where the characteristics of the DNL differ. It is thus important to find time near the end of the lesson to discuss these differences, *which means you might have to omit one or two contexts*.

When our given diagram (A) represents a ratio relation, then the bottom axis would be numbered as in B. Diagram C shows the numbering for *Zoom Taxis*.

[In *formal* mathematical terms, B shows a *linear* relation, while C is *affine*.]



Two of the relations in the Stage 2 tasks are empirical (concerning a baby's weight and the time needed to cook potatoes). The other relations can be summarised algebraically (for our benefit) like this:

$x \to 2x + 3$ (*Zoom Taxis*), $x \to x + 9$, $x \to 2.5x$, $x \to 90 \div x$, $x \to \frac{5}{12}x^2$, and $x \to 2.5x$ again.

*Similarly, on a Cartesian graph, ratio relations are shown by a *straight* line that goes through the *origin*.

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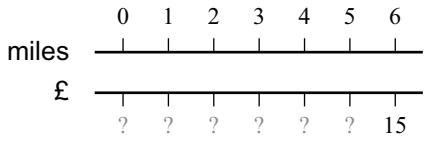
The Lesson continued

Stage 2 continued.

2 This DNL represents journeys with *Zoom Taxis*.

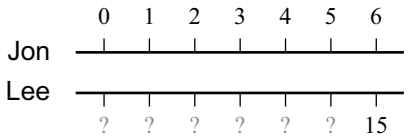
Number the bottom axis.

[Reminder: the tariff is £3 plus £2 per mile.]



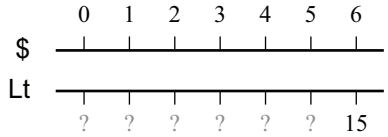
3 a) When Jon was 6 years old, his cousin Lee was 15 years old.

Number the axis showing Lee's age.



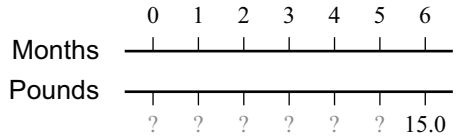
b) Six US dollars ($) are worth 15 Lithuanian litai (Lt).

Number the Lt axis.



c) This DNL records the weight of baby Meg for her first 6 months.

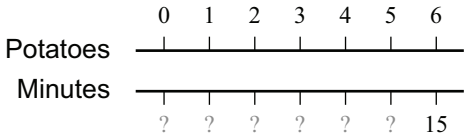
Write down possible numbers for the pounds axis.



d) Some medium size potatoes are put in a large pan of boiling water.

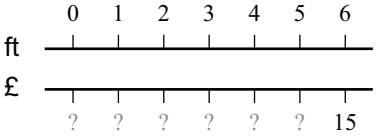
It takes 15 minutes to cook 6 potatoes.

Write down numbers for the minutes axis.



e) A 6 ft length of fine sterling silver wire costs £15.

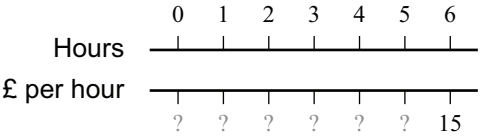
Number the £ axis.



f) Jenny is offered £90 to tidy Mr Dobb's garden.

So if the job takes her 6 hours she would get £15 per hour.

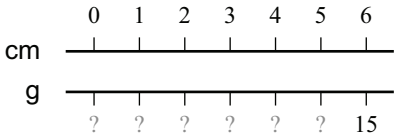
What if it takes her less time?
Number the £ per hour axis.



g) This DNL is for the weight of circular discs cut from a sheet of card.

A 6 cm wide disc weighs 15 g.

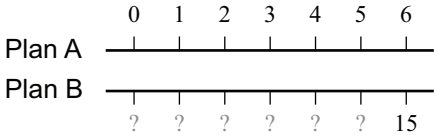
Number the grams axis.



h) Ken has made two plan drawings of his bedroom.

A line that is 6 cm long on plan A is 15 cm long on plan B.

Number the Plan B axis.



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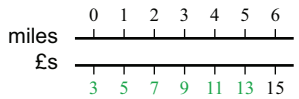
Lesson commentary continued

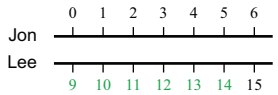
Stage 2 continued.

2 This task is fairly straightforward and there is no need to spend long on it initially. However, *later*, when comparing all the DNLs, it is important to bring out the fact that the bottom scale is 'uniform' (ie linear) but the 0s don't line-up.

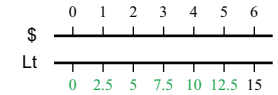
Note: we have labelled the various axes in an informal way. For example, *Jon* should be taken to mean *Jon's age in years*, while *Months* stands for *Number of months*.



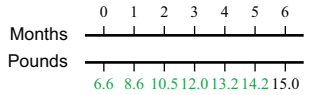
3 a) This is very straightforward, although some students might initially think the relation is multiplicative (eg, 'When Jon was 3, Lee was 7½'). The DNL has similar characteristics to the *Zoom Taxis* DNL, which again are worth highlighting later.



b) This is our first ratio relation. *Look for a variety of approaches*: thus some students might think the relation is additive (eg '6 maps onto 15, so 5 will map onto 14'); others might use an 'along-the-lines' division strategy to correctly determine that, for example, 3 maps onto 15÷2=7.5 and 2 maps onto 15÷3=5 (which gives us the *rated addition* strategy '3+2=5 maps onto 7.5+5=12.5').

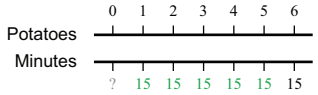


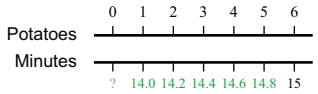
c) There is lots of scope here for discussing the 'uniformity' or otherwise of the bottom scale. Some might think that weight gain is linear (eg 9 lb at birth, gain 1 lb per month). Others might argue that weight-gain fluctuates. The example (right), where the monthly gain slows, is fairly realistic.



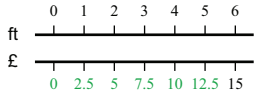
d) This is one of those trick questions (along with boiling eggs, and drying identical towels on a washing line) where there is essentially no relation between quantity and time.

However, one could argue that we need to allow for the time needed to heat each potato that is put in the pan. The second DNL provides a simplified model of what this might look like.

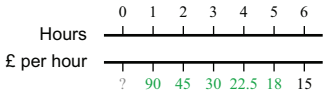




e) Here we have a ratio relation again, so the DNL will be the same as the $-Litai DNL. Do students notice this - and that the context is similar to the *Buying ribbon* context in Lesson 2a?

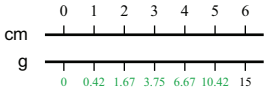


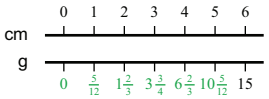
f) Here we have an inverse ratio relation. The bottom scale is very different - the numbers decrease. Also, they don't change by a constant amount (in contrast to the *Burning candle* scenario in Lesson 2c, or a story like this: A cafe has 21 tables; when 6 are in use, how many are free?).



g) Here we have a square relation. The numbers are difficult and it is not essential to find them all (though it is a nice challenge to try to do so with a calculator). The key point is that the bottom scale is not linear.

[The decimal numbers, right, have been rounded to 2DP.]





h) This is another ratio context, so the DNL will be the same as for the exchange context (dollars/litai) and the silver wire. However, it is likely that more students will think the relation is additive for this geometric context.

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Adapting the lesson

- These tasks have a strong diagnostic element and should be accessible to most students. However, the tasks in Stage 2 are likely to fill more than one lesson, so you need carefully to choose which ones to use and which to leave out (or to defer to another lesson). However, make sure to include task a) and the ratio tasks b), e) and h), as these are important for Lesson 2f.
- Also, you might want to spend longer on Stage 1 by devoting a separate lesson to it.
- It is also worth finding time (perhaps in a later lesson) to compare the DNL with the representations in Stage 1, particularly the Cartesian graph.

Suggestions for lesson study

You might want to consider these aspects of students' thinking:

- How readily do students notice characterstics of the various DNLs (eg whether the lower scale is 'uniform' or whether the 0s line-up)?
- How do students number the bottom scale? Do they work along the lines or between the lines?
- Do students find some contexts (and in particular, some ratio contexts) easier thans others?

You might also want to consider pedagogic aspects such as these:

- How did we (and how, if we teach the lesson again, could we find further ways to) elicit (and share) examples of students' thinking?
- How did we (and how, if we teach the lesson again, could we find further ways to) build on and interrogate students' thinking?

Research background to the lesson

Representing a mathematical situation in different ways (eg, verbally, symbolically, spatially) can enrich students' understanding. In this pair of lessons we focus on a particular representation (or model), the double number line. Students will have had some experience of the DNL in Lesson 2a where it is used as a model of situations that bear a direct resemblance to it, eg finding the cost of different lengths of ribbon. Here, the DNL is somewhat more abstract, and our aim is not so much to use it as a scaffolding device to make tasks easier to solve, but rather to use it to reveal the structure of a situation, so that students can explore and develop a greater insight into the nature of ratio relations.

Students' ability to recognise ratio relations (and to solve ratio tasks) can be highly dependent on the particular numbers involved and on the context. Consider the *Eels* item here (right), which can be said to involve the multiplier ×1.5. Written-test data gathered in 2008-09 by the ICCAMS project found that 39 % of a representative sample of Y9 students could solve this item. This compares to only 13% for an item involving the same multiplier but in the context of similar geometric figures, and to 74% for another *Eels* item involving the simple, whole number multiplier ×2. [In 1976, the facilities for a representative Y9 sample were 50%, 20% and 83% respectively.]

> Eels are fed sprats, the number depending on their length.
> If a 10 cm eel is fed 12 sprats, how many should a 15 cm eel be fed to match?

In this pair of lessons, we consider a range of contexts, some but not all of which involve ratio. To keep things relatively simple (at least to start with), the tasks all involve the mapping 6 → 15, so that the ratio tasks all involve the non-trivial, but not too challenging multiplier ×2.5. (Of course, it is easy to change the numbers if it is felt that students would benefit from working with tasks based on other multipliers.)

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