11 Geometry

Mastery Professional Development

Solutions to Exemplified Key Ideas

Click the heading to move to that page. Please note that these materials are principally for professional development purposes; solutions are provided to support this aim.

11.1.1.2 Understand and use the conventions of bearing notation to describe direction

Appreciate why bearings are measured from the same point (north) and in the same direction (clockwise)

Example 1:

Responses may vary but should demonstrate an understanding that:

a) Fenn doesn't specify which direction they should be facing as a starting point, nor in which direction they should turn. His friends could therefore walk in any direction based on these directions.

b) This is an improvement as his friends know which way they initially need to be facing in relation to the bandstand. However, Fenn still hasn't specified if they are turning clockwise or anticlockwise. Therefore, there are still two different directions they could walk and only a 50 per cent chance of finding Fenn.

Example 2:

There are multiple acceptable answers; an example for each question is given below:

a) Any two airports where one is due east of the other, e.g., **London City** is due east of **Cardiff**.

b) Any two airports where one is south west of the other, e.g., **Plymouth** is south west of **Bristol**.

c) Any three airports where the angle formed between them is 90° e.g. the angle between **Leeds Bradford**, **Bournemouth** and **Exeter** is 90°.

d) Any three airports where the angle formed between them is 225°, e.g. ,the angle between **Manchester**, **Bristol** and **Plymouth** is 225°.

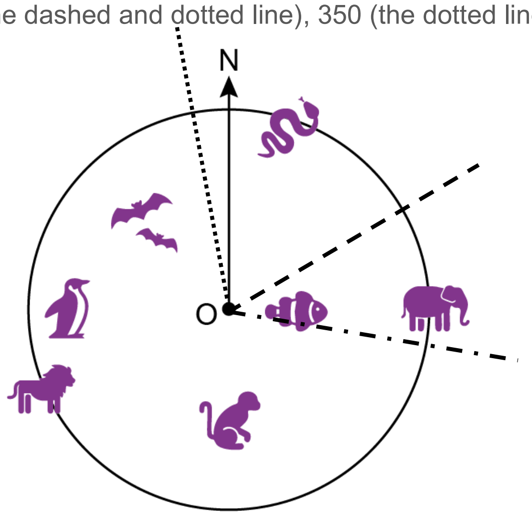
e) Any two airports where the bearing from the first to the second is 090°, e.g., the bearing from **Exeter** to **Bournemouth** is 090°.

f) Any two airports where the bearing from the second to the first is 225°, e.g., the bearing of **Plymouth** from **Bristol** is 225°.

Example 3:

a)	Lion 250	Elephant 090	Penguin 270	Fish 090
	Bats 320	Snake 010	Monkey 180	

b) As indicated by the diagram below, accept any animal marked on bearings of: 000 (the north line), 060 (the dashed line), 100 (the dashed and dotted line), 350 (the dotted line).



Understand how the relationship between start and end points can be expressed using a bearing

Example 4:

a) 150° b) 150° c) 150°

d) Responses should demonstrate an understanding that, despite the different wording, the starting point and end point are the same in each part. Therefore, all three have the same bearing of 150°.

Example 5:

a) from b) to c) to d) to

e) from f) from g) from

Example 6:

a) The bearing from Jane to Richard (or Joe to Helen) is 090°.

b) The bearing from Jane to Joe (or Richard to Helen) is 180°.

c) The bearing from Joe to Richard is 045°.

Example 7:

a) Students' answers will vary due to the nature of estimation, but it is important that each of their pairs of answers is consistent with the bearings being on a straight line (i.e., that there is a difference of 18 between the two values). Similarly, it is important that their answers to C/D and E/F must be the same. Suggested pairs of responses are given below.

A 16 and B 34 C 07 and D 25 E 07 and F 25

G 11 and H 29 I 18 and J 36

b) The difference between each pair of values is always 18.

Express and find directions using bearings

Example 8:

a) (i) City, Stansted, Southend (ii) none

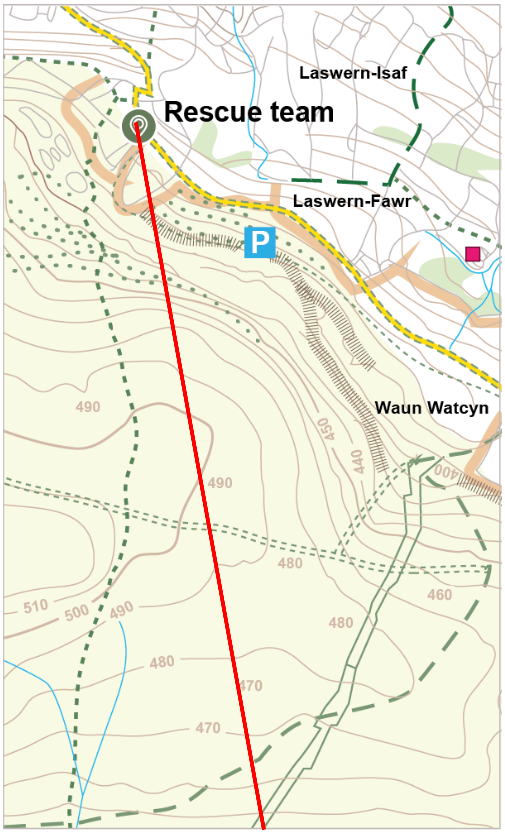
(iii) none (iv) Gatwick, Heathrow, Luton

b) Gatwick or Heathrow

c) Stansted

Example 9:

a) The casualty could be positioned anywhere along the line shown on the map below:



b) 1: 205°, 2: 172°, 3: 101°, 4: 204°, 5: 308°.

Use angle facts to find bearings

Example 10:

No, Tyriq is not correct.

For part a, he could either:

- Draw a 'south' line from the church so that he is measuring an obtuse angle to the school, which he needs to add on to 180°, or
- Use the north line at the church but measure the angle between the church and school in an anticlockwise direction, which he then needs to subtract from 360°.

For part b, he could either:

- Draw a 'south' line from the school so that he is measuring an acute angle to the shop, which he needs to add on to 180°, or
- Use the north line at the school but measure the angle between the shop and school in an anticlockwise direction, which he then needs to subtract from 360°.

Example 11:

a) (i) 165°	b) (i) 70°	c) (i) 240°
(ii) 195°	(ii) 40°	(ii) 090°

Example 12:

a) 116° b) 193° c) 115° d) 313°

Example 13:

a) Dawn, Ethan and Oz.

b) 330°

c) Using a bearing of 090°:

- Anya to Bailey
- Faith to Oz
- Faith to Cordelia
- Oz to Cordelia
- Ethan to Dawn

Using a bearing of 180°:

- Anya to Ethan
- Bailey to Dawn

d) Responses may vary but should demonstrate an understanding that Giles is not correct. Since Ethan lies somewhere between due south and due west of Cordelia, his bearing from Cordelia should be between 180° and 270° . Giles may have mistakenly referred to angle CDE, rather than the bearing in relation to the north line.

11.1.2.4 Understand the addition and subtraction of vectors both algebraically and geometrically

Understand that a displacement can be described by a vector

Example 1:

a) Mr Caldicott's b) Neither

Example 2:

a) (5, 4) b) $\begin{pmatrix}0\\1\end{pmatrix}$ c) (5, 4) d) $\begin{pmatrix}0\\1\end{pmatrix}$

e) Responses may vary but should demonstrate an understanding that the two ants have finished at the same coordinate with the same resultant vector. (Note that students may not be familiar with the term 'resultant vector' at this stage in their learning so may not use it in their answers.)

Understand that directed line segments with the same magnitude and direction are the same vector

Example 3:

a) $\underline{\mathbf{a}}$ b) $\underline{\mathbf{a}}$ c) $\underline{\mathbf{b}}$ d) $\underline{\mathbf{a}}$ e) $\underline{\mathbf{b}}$ f) $2\underline{\mathbf{a}}$

g) $2\underline{\mathbf{a}}$ h) $\underline{\mathbf{a}} + \underline{\mathbf{b}}$ i) $\underline{\mathbf{a}} + \underline{\mathbf{b}}$ j) $2\underline{\mathbf{a}} + 2\underline{\mathbf{b}}$ k) $2\underline{\mathbf{a}} + \underline{\mathbf{b}}$ l) $\underline{\mathbf{a}} + 2\underline{\mathbf{b}}$

Know that $-\underline{a}$ is a vector with the same magnitude as $\underline{a}$, but pointing in the opposite direction

Example 4:

a) $-\underline{\mathbf{a}}$ b) $-\underline{\mathbf{a}}$ c) $-\underline{\mathbf{b}}$ d) $\underline{\mathbf{b}}$ e) $-\underline{\mathbf{b}}$

f) $-2\underline{\mathbf{b}}$ g) $2\underline{\mathbf{b}}$ h) $\underline{\mathbf{a}} + \underline{\mathbf{b}}$ i) $\underline{\mathbf{a}} - \underline{\mathbf{b}}$ j) $-\underline{\mathbf{a}} - \underline{\mathbf{b}}$

k) $-\underline{\mathbf{a}} + \underline{\mathbf{b}}$ l) $\underline{\mathbf{a}} - \underline{\mathbf{b}}$ m) $-\underline{\mathbf{a}} + \underline{\mathbf{b}}$ n) $\underline{\mathbf{a}} - \underline{\mathbf{b}}$

Example 5:

a) Oz to Bailey, Ethan to Oz, Dawn to Cordelia.

b) $-$**a**

c) Using **a**: Anya to Oz, Bailey to Cordelia, Faith to Ethan, Oz to Dawn.

Using $-$**a**: Oz to Anya, Cordelia to Bailey, Ethan to Faith, Dawn to Oz.

Using **b**: Faith to Anya, Oz to Bailey, Ethan to Oz, Dawn to Cordelia.

Using $-$**b**: Anya to Faith, Bailey to Oz, Oz to Ethan, Cordelia to Dawn.

d) Responses may vary but should demonstrate an understanding that Giles is correct. A regular hexagon is made up of six congruent equilateral triangles, with opposite sides being parallel to each other. This means that Anya to Bailey = Anya to Oz + Oz to Bailey = **a** + **b**.

Know that the sum of two (or more) vectors is described as the resultant

Example 6:

a) B

b) Teachers should facilitate a discussion to compare students' answers, supporting them to understand that correct answers will recognise that the combination of two vectors creates a single vector, known as the resultant.

c) $\begin{pmatrix} -1 \\ 7 \end{pmatrix}$

Example 7:

a) Accept any responses where the pairs of corresponding horizontal/vertical movements sum to the horizontal/vertical movement in the resultant vector:

(i) $x_1 + x_2 = 8$, $y_1 + y_2 = 9$ (ii) $x_1 + x_2 = 2$, $y_1 + y_2 = -5$

(iii) $x_1 + x_2 = -2$, $y_1 + y_2 = -5$ (iv) $x_1 + x_2 = 0$, $y_1 + y_2 = 0$

b) Responses may vary but should demonstrate an understanding that the **difference** between corresponding horizontal/vertical movements now needs to be the same as the horizontal/vertical movement in the resultant vector.

c) Responses may vary but should demonstrate an understanding that the three corresponding horizontal/vertical movements now need to sum to the horizontal/vertical movement in the resultant vector. This means that any adjustment to either of their original vectors must be reflected in their new vector **c**. Alternatively, they could keep their answers to part a, and use the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for **c** each time

Example 8: Responses may vary but should demonstrate an understanding that: a) Charlie has represented the two vectors as adjacent sides of a triangle, with the third side representing the resultant vector. Charlie's reasoning is correct, and Dani's is not, because a vector describes magnitude and direction, not position. b) Lou's solution is the same as Charlie's due to the commutative property of addition. Both Lou's and Charlie's vectors have the same magnitude and direction.
Example 9: a) $2\underline{\boldsymbol{t}}$ b) $-\underline{\boldsymbol{t}}$ c) $\frac{1}{2}\underline{\boldsymbol{s}}$ d) $\underline{\boldsymbol{s}} - \underline{\boldsymbol{t}}$ e) $\underline{\boldsymbol{t}} - \underline{\boldsymbol{s}}$ f) $\frac{1}{2}\underline{\boldsymbol{s}} - \underline{\boldsymbol{t}}$ g) $\frac{4}{3}\underline{\boldsymbol{t}} - \frac{3}{4}\underline{\boldsymbol{s}}$ h) $2\underline{\boldsymbol{t}} - \frac{5}{4}\underline{\boldsymbol{s}}$
Apply the properties of shapes to vector calculations *Example 10:* Responses may vary, but could include either or both of the following chains of reasoning: • Vector $\overrightarrow{PQ}$ is parallel and equal in length to $\overrightarrow{OR}$ (or $\overrightarrow{PQ}$ can be thought of as a translation of $\overrightarrow{OR}$) and therefore we know that it is also $\underline{\boldsymbol{b}}$. Vector $\overrightarrow{OQ}$ is therefore $\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$. • Vector $\overrightarrow{RQ}$ is parallel and equal in length to $\overrightarrow{OP}$ (or $\overrightarrow{RQ}$ can be thought of as a translation of $\overrightarrow{OP}$) and therefore we know that it is also $\underline{\boldsymbol{a}}$. Vector $\overrightarrow{OQ}$ is therefore $\underline{\boldsymbol{b}} + \underline{\boldsymbol{a}}$.
Example 11: $\overrightarrow{OS} = \frac{1}{2}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$ or equivalent.
Example 12: a) $\overrightarrow{PR} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$ b) $\overrightarrow{RS} = \frac{1}{2}(\underline{\boldsymbol{a}} - \underline{\boldsymbol{b}})$ c) $\overrightarrow{SO} = -\frac{1}{2}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$
Example 13: B Responses may vary but should demonstrate an understanding that $\overrightarrow{RT} = \overrightarrow{RQ} + \frac{1}{2}\overrightarrow{QP}$. Since $\overrightarrow{OP}$ and $\overrightarrow{RQ}$ are parallel and equal in length, both are $\underline{\boldsymbol{a}}$. Since $\overrightarrow{OR}$ and $\overrightarrow{PQ}$ are parallel and equal in length, both are $\underline{\boldsymbol{b}}$. T is the midpoint of PQ, and so $\overrightarrow{QT}$ is $-\frac{1}{2}\underline{\boldsymbol{b}}$.

11.1.2.5 Use vectors to construct geometric arguments and proofs

Use their knowledge that directed line segments with the same magnitude and direction are the same vector

Example 1:

a) Responses may vary depending on the route selected but may demonstrate an understanding that:

   (i) Using the properties of a regular hexagon, where opposite sides are both parallel and equal, we know $\overrightarrow{DC}$ has the same magnitude and direction as $\overrightarrow{FA}$. Therefore, $\overrightarrow{DC} = \underline{\boldsymbol{b}}$.

   (ii) Similarly, $\overrightarrow{BC}$, $\overrightarrow{FE}$ and $\overrightarrow{OD}$ all have the same magnitude and direction. $\overrightarrow{OD} = \underline{\boldsymbol{a}}$, and so $\overrightarrow{BC} = \overrightarrow{FE} = \overrightarrow{OD} = \underline{\boldsymbol{a}}$.

   (iii) $\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DC} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$.

   (iv) Since $\overrightarrow{FA} = \underline{\boldsymbol{b}}$, $\overrightarrow{AF} = -\underline{\boldsymbol{b}}$. $\overrightarrow{AE} = \overrightarrow{AF} + \overrightarrow{FE} = \underline{\boldsymbol{a}} - \underline{\boldsymbol{b}}$.

   (v) $\overrightarrow{BO}$ and $\overrightarrow{OE}$ have the same magnitude and direction as $\overrightarrow{AF}$. $\overrightarrow{BE} = \overrightarrow{BO} + \overrightarrow{OE} = -\underline{\boldsymbol{b}} - \underline{\boldsymbol{b}} = -2\underline{\boldsymbol{b}}$.

b) $\overrightarrow{AB}$, $\overrightarrow{FO}$ and $\overrightarrow{ED}$ can all be expressed as $\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$. Students may notice a variety of things, but should demonstrate an understanding that these lines are all parallel to each other and equal in length.

Demonstrate that vectors are parallel if one is a scalar multiple of another

Example 2:

a) Responses may vary but should demonstrate an understanding that $\overrightarrow{BE} = 2(-\underline{\boldsymbol{b}}) = 2(\overrightarrow{AF})$. This tells us that $\overrightarrow{BE}$ has the same direction as, and is therefore parallel to, $\overrightarrow{AF}$ (albeit twice its magnitude).

b) Students might choose various routes or express their answers in different ways, particularly with the prompt 'convince a friend'. Explanations might include statements such as:

   - $\overrightarrow{ED} = \overrightarrow{EO} + \overrightarrow{OD}$.
   - Since $\overrightarrow{EO}$ has the same magnitude and direction as $\overrightarrow{AF}$, we can conclude that $\overrightarrow{EO} = \underline{\boldsymbol{b}}$.
   - $\overrightarrow{OD} = \underline{\boldsymbol{a}}$ and therefore $\overrightarrow{ED} = \overrightarrow{EO} + \overrightarrow{OD} = \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}}$.
   - $\overrightarrow{FC} = \overrightarrow{FO} + \overrightarrow{OC}$.
   - Since $\overrightarrow{FE}$ has the same magnitude and direction as $\overrightarrow{OD}$, we can conclude that $\overrightarrow{FE} = \underline{\boldsymbol{a}}$.
   - Since $\overrightarrow{DC}$ has the same magnitude and direction as $\overrightarrow{FA}$, we can conclude that $\overrightarrow{DC} = \underline{\boldsymbol{b}}$.
   - Therefore, $\overrightarrow{FO} + \overrightarrow{OC} = \overrightarrow{FE} + \overrightarrow{EO} + \overrightarrow{OD} + \overrightarrow{DC} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}} = 2(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}) = 2\overrightarrow{ED}$.
   - *This tells us that $\overrightarrow{FC}$ has the same direction as, and is therefore parallel to, $\overrightarrow{ED}$ (though is twice the magnitude).*

Example 3:

a) Students' demonstrations may vary but should recognise that $\overrightarrow{CD} = 12\underline{\boldsymbol{a}} + 4\underline{\boldsymbol{b}} = 4(3\underline{\boldsymbol{a}} + \underline{b}) = 4\overrightarrow{AB}$. This tells us that $\overrightarrow{CD}$ has the same direction as, and is therefore parallel to, $\overrightarrow{AB}$ (though it is four times the magnitude).

b) No, they are not parallel to each other. Explanations may vary but should demonstrate an understanding that vectors are parallel if one is a scalar multiple of another. $15\underline{\boldsymbol{a}} + 4\underline{\boldsymbol{b}} =$ is not a multiple of $12\underline{\boldsymbol{a}} + 4\underline{\boldsymbol{b}}$.

Example 4:

Exact responses may vary but should demonstrate an understanding that:

a) $\overrightarrow{OX}$ and $\overrightarrow{ON}$ have been labelled as given in the question. Since N is the midpoint of $\overrightarrow{OY}$ we can also conclude that $\overrightarrow{NY} = \overrightarrow{ON} = \underline{\boldsymbol{q}}$.

b) The student has reasoned that vector $\overrightarrow{NM}$ is the sum of vectors $\overrightarrow{NX}$ and $\overrightarrow{XM}$. They are essentially creating a path, with the orange part of the calculation comprising the known vectors from point N to point X, via point O. We are told that vector $\overrightarrow{OX} = \underline{\boldsymbol{p}}$ and, since $\overrightarrow{ON} = \underline{\boldsymbol{q}}$, $\overrightarrow{NO}$ is easily identifiable as $-\underline{\boldsymbol{q}}$. The blue part of the calculation is the yet-to-be found vector from point X to point M.

c) To find vector $\overrightarrow{XM}$, the student needs to find vector $\overrightarrow{XY}$ and then halve it (since we are told that point M is the midpoint of XY). A valid path of known vectors for $\overrightarrow{XY}$ is via O and N, so $\overrightarrow{XY} = -\underline{\boldsymbol{p}} + 2\underline{\boldsymbol{q}}$.

d) $\overrightarrow{NM} = \overrightarrow{NX} + \overrightarrow{XM} = \left(-\underline{\boldsymbol{q}} + \underline{\boldsymbol{p}}\right) + \frac{1}{2}\left(-\underline{\boldsymbol{p}} + 2\underline{\boldsymbol{q}}\right) = -\underline{\boldsymbol{q}} + \underline{\boldsymbol{p}} - \frac{1}{2}\underline{\boldsymbol{p}} + \underline{\boldsymbol{q}} = \frac{1}{2}\underline{\boldsymbol{p}}$.

e) The vectors are parallel because vector $\overrightarrow{OX}$ is a multiple of $\overrightarrow{NM}$. $2\overrightarrow{NM} = \overrightarrow{OX}$.

Know how to prove three points lie on a straight line

Example 5:

Responses may vary but should demonstrate an understanding that the vectors are parallel because vector $\overrightarrow{RT}$ is a multiple of $\overrightarrow{RS}$ or $\overrightarrow{ST}$.

Students can find vector $\overrightarrow{RT}$ using a path from R via O and Q ($\overrightarrow{RT} = \overrightarrow{RO} + \overrightarrow{OQ} + \overrightarrow{QT}$):

- $\overrightarrow{RO} = -\underline{\boldsymbol{x}}$ (since we are told that $\overrightarrow{OR} = \underline{\boldsymbol{x}}$)
- $\overrightarrow{OQ} = \underline{\boldsymbol{y}}$ (given in the question rubric)
- $\overrightarrow{QT} = \frac{1}{2}\underline{\boldsymbol{y}}$ (since the ratio of OQ:QT is 2:1, $\overrightarrow{QT}$ must be half of $\overrightarrow{OQ}$)
- Therefore, $\overrightarrow{RT} = \overrightarrow{RO} + \overrightarrow{OQ} + \overrightarrow{QT} = -\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}} + \frac{1}{2}\underline{\boldsymbol{y}} = \frac{3}{2}\underline{\boldsymbol{y}} - \underline{\boldsymbol{x}}$.

Students can find vector $\overrightarrow{RS}$ using a path from R via P, which requires them to first find the vector $\overrightarrow{PQ}$:

- $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \underline{\boldsymbol{y}} - 2\underline{\boldsymbol{x}}$ (since R is the midpoint of OP, and $\overrightarrow{RO} = -\underline{\boldsymbol{x}}$, we therefore know that $\overrightarrow{PO} = -2\underline{\boldsymbol{x}}$; $\overrightarrow{OQ} = \underline{\boldsymbol{y}}$ is given in the question rubric)
- $\overrightarrow{PS} = \frac{3}{4}\overrightarrow{PQ} = \frac{3}{4}(\underline{\boldsymbol{y}} - 2\underline{\boldsymbol{x}}) = \frac{3}{4}\underline{\boldsymbol{y}} - \frac{3}{2}\underline{\boldsymbol{x}}$ (since the ratio of PS:SQ is 3:1)
- $\overrightarrow{RP} = \underline{\boldsymbol{x}}$ (since R is the midpoint of OP, we know $\overrightarrow{OR} = \overrightarrow{RP}$)
- $\overrightarrow{RS} = \overrightarrow{RP} + \overrightarrow{PS} = \underline{\boldsymbol{x}} + \left(\frac{3}{4}\underline{\boldsymbol{y}} - \frac{3}{2}\underline{\boldsymbol{x}}\right) = \frac{3}{4}\underline{\boldsymbol{y}} - \frac{1}{2}\underline{\boldsymbol{x}}$.

If $\overrightarrow{RT} = \frac{3}{2}\underline{\boldsymbol{y}} - \underline{\boldsymbol{x}}$ and $\overrightarrow{RS} = \frac{3}{4}\underline{\boldsymbol{y}} - \frac{1}{2}\underline{\boldsymbol{x}}$, then we have shown that $\overrightarrow{RT} = 2\overrightarrow{RS}$, and that they are therefore parallel. Because the two parallel vectors start at the same point (R) we can say that R, S and T lie on a straight line.

11.1.3.3 Understand enlargement as a transformation of vectors

Understand the effect the centre of enlargement has on the position of an enlarged image

Example 1:

a) Responses may vary but should demonstrate an understanding that the size and orientation of the image is the same for all three enlargements, but their positions are different.

b) A: Enlargement 3
B: Enlargement 1
C: Enlargement 2

c) Enlargement 1: Origin, (0,0)
Enlargement 2: (2,3)
Enlargement 3: Point C (3,2)

Appreciate the effects of an enlargement by scale factor between 0 and 1

Example 2:

a) Image 1: A   Image 2: D   Image 3: R   Image 4: Y

b) Responses will vary depending on students' choice of centre of enlargement. See part c below for a list of descriptions of potential images.

c) For all the images, the new points of the moon shape would be $\frac{1}{3}$ of the distance between the corresponding points on the object and the centre of enlargement. For the given centres of enlargement, the following images would be produced:

- B: vertically aligned with images 1 and 2; positioned $\frac{2}{3}$ of a square to the right of image 1.
- C: vertically aligned with images 1 and 2; positioned $\frac{2}{3}$ of a square to the left of image 2.
- P: vertically aligned with image 3 and horizontally aligned with image 4.
- Q: horizontally aligned with, but one square above, image 3.
- S: horizontally aligned with, but one square below, image 3.
- X: vertically aligned with, but two squares to the left of, image 4.
- Z: vertically aligned with, but 2 squares to the right of, image 4; it would be touching the object.

Example 3:

a) (i) C   (ii) B   (iii) D   (iv) A   (v) C   (vi) B

b) Students' responses will vary, depending on what they notice. Some possible observations include:

- The answers to parts (i) and (v) are the same, so the scale factor from object A to image B is the same as the scale factor from object B to image C.
- The answers to parts (ii) and (vi) are the same, so the scale factor from object B to image A is the same as the scale factor from object C to image B.
- The scale factor from object A to image B is 2, and the scale factor of object B to image A is $\frac{1}{2}$.
- The scale factor from object A to image C is 4, and the scale factor of object B to image A is $\frac{1}{4}$.
- When the image is smaller than the object, the scale factor is between 0 and 1.
- Whan the image is bigger than the object, the scale factor is greater than 1.

• The scale factor from an object to an image is the reciprocal of the opposite enlargement (for example, where the object/image switched around).
Appreciate the effects of an enlargement by a negative scale factor *Example 4:* a) $\overrightarrow{OA} = \begin{pmatrix}0\\0\end{pmatrix}$ $\overrightarrow{OB} = \begin{pmatrix}0\\3\end{pmatrix}$ $\overrightarrow{OC} = \begin{pmatrix}2\\0\end{pmatrix}$ $\overrightarrow{OA'} = \begin{pmatrix}0\\0\end{pmatrix}$ $\overrightarrow{OB'} = \begin{pmatrix}0\\-6\end{pmatrix}$ $\overrightarrow{OC'} = \begin{pmatrix}-4\\0\end{pmatrix}$ b) Image 2: (0,0) Image 3: (4,5)

11.1.3.5 Understand the relationship between lengths, areas and volumes in similar shapes

Appreciate the connection between similarity and proportionality *Example 1:* Responses may vary but should demonstrate an understanding that: • For resize 1, the height has been maintained but the base has been stretched. • For resize 2, the length of the base has been maintained, but the height has been stretched. • For resize 3, both the height and length of the base have been stretched by the same scale factor. • Same: any stretched dimensions appear to have been doubled. • Different: only resize 3 resulted in a picture that is similar to the original. As resizes 1 and 2 were only stretched in one dimension, the image became distorted because it was no longer proportional.
Understand that the ‘area scale factor’ is the square of the ‘length scale factor’ *Example 2:* A and B are correct. C is incorrect. Explanations may vary but should demonstrate an understanding that, for A and B, just one length is doubled so it is effectively adding a second rectangle of the same dimensions. Therefore, the area is doubled. For C, both dimensions are doubled and so the area has been increased by a factor of 4. Students might imagine the three extra rectangles added above to the right and then diagonally to the right, to create a rectangle that is four times as big.
Example 3: C is the area of the larger frame. Responses may vary but should demonstrate an understanding that the length scale factor from the smaller frame to the larger frame is 2.5. Therefore, the area scale factor is 2.5^2, which is 6.25. $12 \times 6.25 = 75$.
Example 4: Responses may vary but should demonstrate an understanding that the area scale factor is not the same as the length scale factor. Where the length has been increased by a factor of 2, the area will be increased by a factor of 2^2, and so the area of the smaller shape is four times smaller than the area of the larger shape. The missing area is therefore 5.5 cm^2.

Example 5:

a) D is correct. A and B could be argued to be correct, but it has not been specified which circle is the object and which is the image.

b) Responses may vary but should demonstrate an understanding that:

- Statement A would be true if it is an enlargement from circle A ***to*** circle B.
- Statement B would be true if it is an enlargement from circle B ***to*** circle A.
- Statement C should be changed to specify that the radius of circle A is ***3*** times bigger than the radius of circle B.

Example 6:

Students' demonstrations may vary but should show an understanding that the area scale factor from the larger shape to the smaller shape is $\frac{20}{45} = \frac{4}{9}$.

The length scale factor therefore is $\sqrt{\frac{4}{9}} = \frac{2}{3}$.

This means that the missing length is $3.6 \times \frac{2}{3} = 2.4$ cm.

Understand that the 'volume scale factor' is equal to the cube of the 'length scale factor'

Example 7:

a) (i) 20 cm (ii) 400 cm^2

b) (i) 10 cm (ii) 100 cm^2

c) Responses may vary but students should notice that, since all cubes are similar, there is a relationship between corresponding answers in parts a and b. The volume of the cube in part a is eight times the volume of the cube in part b. The length of the side in part a is twice that of the cube in part b; the area of the face in part a is four times that of the face in part b. In other words, the length, area and volume scale factors of x, x^2 and x^3 are also evident between the larger cube and each of the smaller cubes.

d) (i) 100 (ii) 10

Example 8:

a) 64 000 cm^3

b) The ratio of the smallest to the middle-sized packing cube is 1:2.

c) The ratio of the smallest to the largest packing cube is 1:27.

d) The ratio of the middle-sized to the largest packing cube is 8:27.

Example 9:

a) B and C are correct.

b) A: The volume of sphere A is $\frac{1}{27}$ of the volume of sphere B.

D: A circle with the same radius as sphere B would have an area that is **9** times that of a circle with the same radius as sphere A.

Example 10:

a) Medium box volume = 2 160cm^3

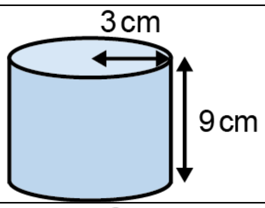
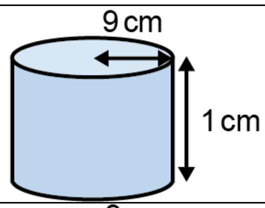
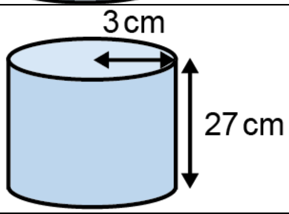
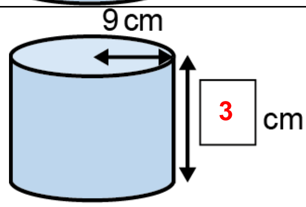
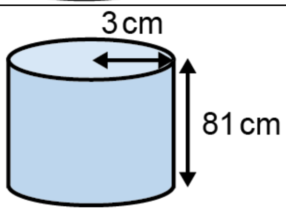
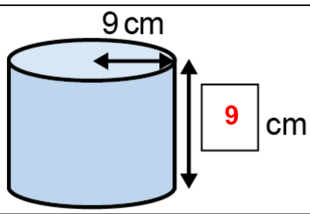
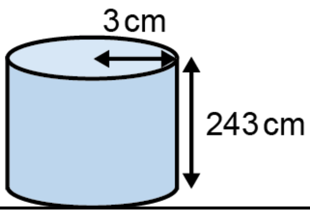
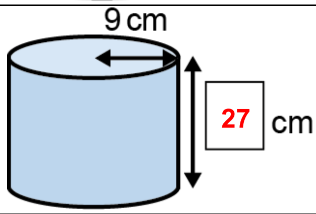
Large box volume = 4 320cm^3

b) Responses may vary but should demonstrate an understanding that Phoebe is not correct. The volume will be $2 \times 2 \times 2 = 2^3 = 8$ times bigger.
Example 11: Length scale factor: $\frac{5}{6}$ Volume scale factor: $\left(\frac{5}{6}\right)^3 = \frac{125}{216}$ Missing volume: $864 \times \frac{125}{216} = 500$ cm³

Example 12:

a) *If the pipes are cut from the same diameter, then the two pipes would need to be the same length (and would also be congruent). If one pipe is cut from each possible diameter, then the 18 cm diameter pipe would need to be three times longer than the 6 cm pipe.*

b), c) and d) The answers are in the completed table below.

	Cross-sectional area = 9π cm²	**Cross-sectional area = 9 times bigger**
Volume = 81π cm³		
Volume = 3 times bigger		
Volume = 9 times bigger		
Volume = 27 times bigger		

11.1.4.3 Use and apply the key characteristics of transformations to analyse situations where transformations are combined

Know how the commutative property applies when combining two transformations of the same type

Example 1: Responses may vary but should demonstrate an understanding that:

a) Caitlin has translated the object first by vector $\binom{4}{-2}$ to show $\boldsymbol{T_1}$, then $\boldsymbol{T_1}$ was translated by vector $\binom{-2}{4}$.
Heidi has translated the object first by vector $\binom{-2}{4}$ to show $\boldsymbol{T_1}$ then translated $\boldsymbol{T_1}$ by vector $\binom{4}{-2}$.
Both result in the same position for $\boldsymbol{T_2}$, the final image.

b) Accept any two column vectors $\binom{a}{b}$ and $\binom{c}{d}$ such that $\boldsymbol{a}+\boldsymbol{b}=2$ and $\boldsymbol{c}+\boldsymbol{d}=2$ .Students should notice that the resultant is the same, as long as these conditions are kept.

Example 2:

a) Vertices at (1, 7), (4, 7), (4, 6), (2, 6), (2, 5), (1, 5).

b) Agree. Responses may vary but should demonstrate an understanding that when the centre of rotation remains constant, the resultant rotation will remain the same.

c) Same size and orientation. Different position.

d) Responses may vary but should demonstrate an understanding that they will not get the same final image, because the centre of rotation is not the same for each rotation Ben completes.

Example 3:

a) 90° clockwise about the origin.

b) Responses may vary but should demonstrate an understanding that the final image will be in a different position. When the order of the reflections is reversed ($y = x$ then y –axis), the rotation that takes the object to Re_2 will be 90° anticlockwise about the origin. A reflection in the line $y = x$ then in the y –axis is the same as a rotation $90°$ clockwise about the origin.

Understand when a composition of two transformations can be combined as a single transformation

Example 4:

a) (i) B (ii) D (iii) C

(i) Rotation 90° clockwise about (1, 7) (ii) Rotation 180° about (4, 3) (iii) Not possible

Example 5:

a) (ii)

b) B: Yes C: Yes D: No

c) B: Translation by column vector $\binom{10}{0}$ C: Rotation 90° anticlockwise, about (1, 7)

Example 6:

a) Responses may vary but should demonstrate an understanding that where the vertices in both objects are consistent, the corresponding vertices in shape D’ match those of shape D. The two vertices to the far left of shape D’ are in a different position because of the shape being modified.

b) Responses may vary but should demonstrate an understanding that it is because of the shape in *Example 6* having a line of symmetry which means that the orientation isn't as easily identifiable. It isn't the case that they have undergone this transformation. c) Translation by column vector $\binom{10}{0}$. d) Rotation 90° anticlockwise, about (1, 7) ***or*** reflection in the line $x = 1$. e) Responses may vary but should demonstrate an understanding that it is because of the shape in *Example 6* having a line of symmetry.
Example 7: a) Yes. Rotation 90° clockwise, about (-2, -2). b) Not possible. c) Yes. Reflection in the line $x = 0$, y-axis.
Use the general case to identify combinations of transformations that can be described by a single transformation *Example 8:* a) (i) $(-x, y)$ (ii) (y, x) (iii) $(-y, -x)$ (iv) $(y, -x)$ (v) $(-y, x)$ (vi) $(-x, -y)$ b) (i) Rotation 90° clockwise, about (0, 0) (ii) Rotation 180°, about (0, 0) (iii) Reflection in the line $y = x$

11.2 Reasoning with the properties of a circle

11.2.2.1 Identify and reason with lines associated with a circle (including segments, chords and tangents)

Identify lines and regions associated with a circle

Example 1:

Responses may vary but should demonstrate an understanding that:

a) Diagram B is correct because the chord and diameter meet at the same point on the circumference, and so the tangent is marked at the point the chord and the tangent meet the circumference.

b) Diagram A: a circle with a diameter and a tangent marked at the point where the diameter meets the circumference. A radius and a chord intersecting the radius are also shown. The area of the minor sector formed by the chord is shaded.

   Diagram C: a circle with a diameter and a radius marked. At the point where the radius meets the circumference, a tangent and a chord are shown. The area of the minor sector formed by the chord is shaded.

c) Students' observations will vary but may include:

   - the right angles formed where a diameter or radius meets the tangent
   - the other angles formed between two straight line segments (including the boundaries of the sectors in A and C)
   - lengths that are equidistant, such as radii or either half of the chord bisected by a radius in A.

Example 2:

a) (i) Angle BAC is subtended by the chord BC **or**

   Angle ABC is subtended by the chord AC **or**

   Angle ACB is subtended by the chord AB.

   (ii) The chord BC subtends the angle BAC **or**

   The chord AC subtends the angle ABC **or**

   The chord AB subtends the angle ACB.

b) Responses may vary but should demonstrate an understanding that including point O (the centre of the circle) allows both the angle at the centre and the angle at the circumference to be identified for each chord (effectively doubling the number of angles stated). For example, for chord AC, angle AOC is the angle at the centre, and angle ABC is the angle at the circumference.

Identify triangles formed within circles and use their properties

Example 3:

a) Angle AOB $= 100°$

b) Responses may vary but should demonstrate an understanding that the answer would remain the same, since the radius of a circle is equal to any point on the circumference. A triangle formed by two radii and a chord is therefore isosceles, with the two radii as equal sides.

c) Responses may vary but should demonstrate an understanding that if point O were not the centre of the circle, there would not be enough information to answer parts a and b.

Example 4:

a) Six.

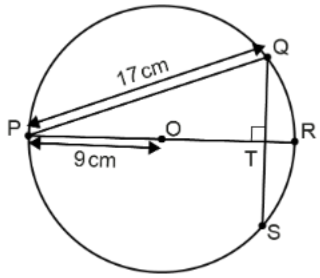
b) All will be isosceles. Each triangle includes two radii which are equal in length.

c) Six.

d) All three which include O will be isosceles; there is not sufficient information to determine whether or not the three that do not include O are isosceles.

Example 5:

a)



b) OR = 9 cm, OQ = 9 cm, OS = 9 cm, PR = 18 cm, PS = 17 cm.

c) QS = 16 cm.

Recognise that drawing extra lines can help expose underlying geometry

Example 6:

a) Responses may vary but should demonstrate an understanding that EG = FH because they are both diameters of the same circle.

b) Rectangle.

c) There are several approaches. Perhaps the simplest is to note that OE = OF = OG = OH as they are all radii. This means that lines EG and HF are equal in length and bisect each other, and hence EFGH is a rectangle. Each vertex in a rectangle is 90°.

   Alternatively, students may consider the properties of angles:

   - Since OE=OF=OG=OH as they are all radii, isosceles triangles will be created.
   - Angle EOF = angle HOG because vertically opposite angles are equal.
   - Angle EOH = angle FOG because vertically opposite angles are equal.
   - Additionally, angle EOF + angle FOG = 180°.
   - If angle EOF = x then angle FOG =$180° - x$.
   - It can then be established that angle EFO = $90° - \frac{x}{2}$ and angle GFO = $\frac{x}{2}$, giving an angle sum of 90°.

Know that a tangent to a circle is perpendicular to its radius

Example 7:

Angle JOK = 120°

11.2.2.2 Use chains of reasoning to show that the angle at the centre is twice the angle at the circumference

Recognise that a relationship exists between the angle at the centre and the angle at the circumference when both are subtended from the same two points

Example 1:

Responses to parts a–c will vary based on individual drawings.

d) The angle subtended at the centre of a circle by an arc is twice the angle subtended at the circumference by the same arc.

Example 2:

a) Any valid diagram where the angle formed by the radii to points X and Y is 120°.

b) Students' diagrams will vary as point Z may be anywhere on the circumference.

c) 60°.

d) 60°.

e) The angle XZ_2Y is always 60°.

f) Students' explanations may vary, but should capture the essence of the property that the angle subtended at the centre of a circle by an arc is always twice the angle subtended at the circumference by the same arc.

Know that the angles formed at the centre and circumference can be used to identify isosceles triangles within circle diagrams

Example 3:

a) Accurate construction with the same features as the example shown.

b) Triangle POQ, triangle POR, triangle PQR, triangle QRO.

c) Responses should demonstrate an understanding that triangles POQ, POR and QRO are all isosceles, each with two sides equal to the radius of the circle.

d) Responses could include:

- QPO = PQO
- POQ = 180 – (QPO + PQO)
- RPO = PRO
- POR = 180 – (RPO + PRO)
- QOR = 360 – (QOP + ROP)
- QPR = QPO + RPO
- OQR = ORQ = 0.5 × (180 – QOR)

e) Responses may vary based upon students' individual measurements, but should demonstrate an understanding that the orange angle at the centre is twice the blue angle at the circumference.

Example 4:

a) Responses may vary but should demonstrate an understanding that OL, OM, and ON are radii and therefore are equal in length. Any triangle formed with O and two of the points is isosceles

b) Angle LOM $= a$

Angle LMO = angle MLO= $\frac{180-a}{2}$

Angle LON $= b$

Angle OLN = angle ONL $= \frac{180-b}{2}$

Angle MON $= 360 - (a + b)$

Angle MLN = angle MLO + OLN $= \frac{180-a}{2} + \frac{180-b}{2} = 180 - \frac{1}{2}a - \frac{1}{2}b$

Apply geometrical properties to show that the angle at the centre of a circle is twice the angle at the circumference

Example 5:

	Geometric property	Reasoning
1	$OX = OY$	Radius of the circle.
2	$OXY = OYX = x$	Isosceles triangle base angles are equal.
3	$XOY = 180 - 2x$	Angles in a triangle sum to $180°$.
4	$XOW = 2x$	An exterior angle of a triangle equals the sum of the two opposite interior angles **or** the angle sum of a straight line is 180.
5	$OZ = OY$	Radius of the circle.
6	$OZY = OYZ = y$	Isosceles triangle base angles are equal.
7	$ZOY = 180 - 2y$	Angles in a triangle sum to $180°$.
8	$WOZ = 2y$	An exterior angle of a triangle equals the sum of the two opposite interior angles **or** the angle sum of a straight line is 180.
9	$XOZ = 2x + 2y = 2(x + y)$	Angle at the centre = 2 × angle at the circumference.

Example 6:

a) 50°.

b) Responses may vary but should demonstrate an understanding that angle w is twice angle OVU or angle OUV.

Recognise that there are three different geometric configurations required for a full proof, and understand these additional proofs

Example 7:

Students may or may not be convinced about the size of the angle, but should be convinced that this is a different geometric configuration and therefore the rule cannot be extrapolated.

Example 8:

a) The triangle is isosceles because OA and OC are radii of the circle.

b) OAC = x and ACO = x. Therefore AOC = $180° - 2x$ (angle sum of a triangle) and DOB = $2y$ (DOB and BOC are angles forming a straight line).

c) Yes. It is isosceles because OB and OC are both radii of the circle.

d) OBC = y and BCO = y. Therefore BOC = $180° - 2y$ (angle sum of a triangle) and DOA = $2x$ (DOA and AOC are angles forming a straight line).

e) AOB = $2y - 2x$

f) ACB = $y - x$

Example 9:

a) Responses may vary, but students should be encouraged to see the similarities between this configuration and the configuration in *Example 5*.

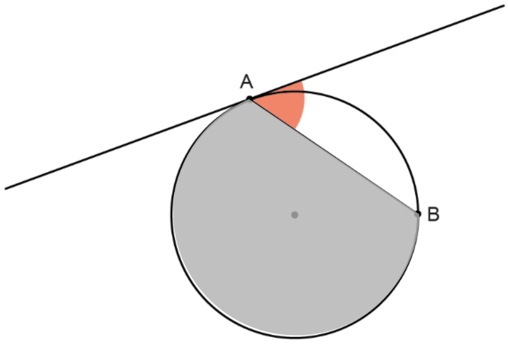
b) Yes, the angle subtended at the centre is twice the angle subtended at the circumference.

11.2.2.7 Use chains of reasoning to show the alternate segment theorem

Identify the segments and angles formed by a chord

Example 1:

a) Responses may vary but should demonstrate an understanding that no angles have been formed yet - just a line segment.
b) Responses may vary but should demonstrate an understanding that two angles have been formed between the chord and the tangent.
c)



Example 2:

a) Chord.

b) Agree. Responses should demonstrate an understanding that Alexandra is referring to the size of the angles subtended at the circumference by the original chord.

c) Responses should recognise that the angles formed in the first segment are acute whereas the angles formed in the second segment are obtuse.

Example 3:

a) Angle AOB is isosceles because OA and OB are radii of the circle.

b) If OAB = 40° then ABO = 40° since AOB is isosceles. Angle BOA = 100° (angle sum of a triangle), angle ACB = 50° (angle subtended at the circumference is half the angle subtended at the centre).

c) Angle ACB will remain as 50° in both diagrams because the position of AB and therefore angle AOB is static.

d) In both diagrams, the angles OAB and ABO will remain as 40° and angle BOA =100°. In the first diagram (on the left), he will now be able to ascertain other angles since the indication of BCA being isosceles results in symmetry. CAB = ABC = 65°. Since OAB = ABO = 40°, OBC = OAC = 25°. In the second diagram (on the right) there is insufficient information to determine any additional angles.

e) If ACB = $\boldsymbol{\theta}$, then AOB = $\boldsymbol{2\theta}$ and angle OAB = ABA = $\frac{180-2\theta}{2}$.

Recognise instances of angles formed in alternate segments by a chord and tangent

Example 4:

a) E, H, I, J

b) Responses may vary but should demonstrate an understanding that:

A: the angle ACB and the angle formed by the tangent and the chord AB in the alternate segment.

B: no alternate segment relationship.

C: the angle shaded as shown at C and the angle BAC.

D: the angle shaded as shown at A and the angle ACB.

F: the angle shaded as shown at C but finishing at the chord BC, and angle BAC.

G: the angle shaded as shown at A and angle ACB.

K: the angle shaded as shown at B and angle BAC.

L: the angle shaded as shown at B and angle BAD OR the angle shaded as shown at A and the angle formed by the tangent and the chord BC.

M: the angle shaded as shown at A and the angle formed by the tangent and the chord BC.

Recognise and use the triangles that can be identified within circle diagrams that show alternate segments

Example 5:

Responses may vary but could include:

- Triangle ORQ is isosceles since OR and OQ are radii.
- Angles ORQ and OQR are equal.
- The angle between the tangent and radius OR is 90°.
- The circle theorem that states the angle at the centre is twice the angle at the circumference, can then be applied, with angle QOR being the angle at the centre.

Know how to prove the alternate segment theorem

Example 6:

a) OVU is isosceles as OV and OU are radii.

b) OV and the tangent are perpendicular.

c) OUV = $\frac{180° - 2y}{2} = 90° - y$

d) (i) OVU = $90° - x$

(ii) VUO = $90° - x$

(iii) UOV = $2x$

e) $2y = 2x$ so $y = x$

f) This shows that angle UPV = y and the angle marked as x between the chord and the tangent is also equal to y.

Example 7:

If I know …	I can find out …	By using …
Angle x	Angle OVU	A radius meets a tangent at 90°, and angles on a straight line sum to 180°.
Angle OVU	Angle OUV	Base angles in an isosceles triangle are equal.
Angle OUV	Angle UOV	Sum of angles in a triangle is 180°.
Angle UOV	Angle y	Angle at the circumference is half of the angle at the centre.

Recognise when there are multiple or unusual instances of the alternate segment theorem, and apply the theorem accurately

Example 8:

Responses may vary but should demonstrate an understanding that chord AC forms an angle of 67° with the tangent at A, and it is angle ABC that is subtended by this chord in the alternate segment. The pair of 62° angles are linked to the chord AB, which also meets the tangent at A and subtends angle ACB.

Example 9:

a) If P were moved to different positions on the circumference, the angle at P could be a right angle or an obtuse angle.

b) AB is a diameter.

c) AB is perpendicular to the tangent. The angle AOB is 180°, the angle subtended at the circumference is half the angle subtended at the centre, so the angle APB is $180° \div 2 = 90°$.

d) There are two things to attend to: the angle on a straight line (on the tangent) and the angles in a triangle, both of which sum to 180°. Since the acute angle in the alternate segment has already been established as being equal to the angle between the tangent and the chord, the blue (larger acute) angles are equal. The yellow (smallest acute) angle is common to both the triangle and the straight line. This means that the red (obtuse) angles must be equivalent.

11.2.3.1 Appreciate that the equation of a circle emerges from the use of Pythagoras' theorem

Use Pythagoras' theorem to determine the distance from the origin to points on the circumference of a circle

Example 1:

a) An arc (quarter-circle).

b) 5.

c) (5, 0), (0, 5), (−5, 0), (0, −5), (−3, 4), (−4, 3), (−3, −4), (−4, −3), (3, −4), (4, −3)

Students' explanations may vary but should demonstrate an understanding that because the centre of rotation is the origin, the integer coordinate points that the line passes through will be symmetric across the x-axis and y-axis.

d) A circle. e) Responses may vary but should demonstrate an understanding that Pythagoras' theorem could be used.
Example 2: a) A = (5, 0), B = (0, −5), C = (−5, 0). b) Responses may vary but should demonstrate an understanding that you could estimate one of the values in each coordinate pair and use Pythagoras' theorem to determine the second value. c) Students' responses should demonstrate an understanding that the coordinates in their answer to part a would change to A = (12, 0), B = (0, −12), C = (−12, 0). Students should indicate that their approach for part b would remain the same, but that they would need to use $x^2 + y^2 = 12^2$.
Know how to identify the radius of a circle with centre at the origin, given its equation *Example 3:* a) C: 3 b) Responses may vary but should demonstrate an understanding that the equation of a circle with centre (0, 0) is $x^2 + y^2 = r^2$. Putting $x^2 + y^2 = 9$ into this form gives $x^2 + y^2 = 3^2$ therefore r =3.
Example 4: a) B: $\sqrt{2}$ b) Responses may vary but should demonstrate an understanding that the equation of a circle with centre (0, 0) is $x^2 + y^2 = r^2$. Putting $x^2 + y^2 = 2$ into this form gives $x^2 + y^2 = (\sqrt{2})^2$ therefore $r = \sqrt{2}$.
Example 5: a) B: 1.6 and C: $\sqrt{2.56}$ b) Responses may vary but should demonstrate an understanding that the equation of a circle with centre (0, 0) is $x^2 + y^2 = r^2$. Putting $x^2 + y^2 = 2.56$ into this form gives $x^2 + y^2 = (\sqrt{2.56})^2$ $\sqrt{2.56} = 1.6$ therefore $r = \sqrt{2.56} = 1.6$.
Example 6: *7*
Determine the equation of a circle with centre at the origin, given its radius *Example 7:* $x^2 + y^2 = 5^2$ or equivalent
Example 8: C: $x^2 + y^2 = 2$
Example 9: Two of the equations describe the circle. A: $x^2 + y^2 = 4\sqrt{9}$ (not fully simplified) and D: $x^2 + y^2 = 12$ (fully simplified).

Know how to use the equation of a circle to determine the coordinates of a point that lies on the circle *Example 10:* $y = 12$ or $y = -12$
Know how to use the equation of a circle to determine whether a point lies inside, outside, or on the circle *Example 11:* Responses may vary but should demonstrate an understanding that the point (5, 12) lies inside the circle because a radius of 17 squares to 289 but $5^2 + 12^2 = 169$.
Determine the equation of a circle with centre at the origin, given a point that lies on the circle *Example 12:* $x^2 + y^2 = 5^2$
Determine the equation of a circle with centre at the origin, given the endpoints of the diameter *Example 13:* a) $x^2 + y^2 = 25^2$ b) $y = 24$ or $y = -24$
Example 14: a) $x^2 + y^2 = 25^2$ b) $x = 7$ or $x = -7$

11.3.1.3 Use Pythagoras' theorem and trigonometric ratios to solve problems in 3D

Know how to identify right-angled triangles in 3D shapes

Example 1:

a) Responses may vary but should demonstrate an understanding that Simon is assuming the fish can only swim along one edge of the tank. But in a cuboid, the longest straight-line distance is from one vertex of the cuboid to the furthest diagonally opposite vertex.

b) Responses may vary but should demonstrate an understanding that even though the horizontal edge is 5 feet, Flip's diagonal path includes vertical movement too – so the total distance is longer.

c) Responses may vary but should demonstrate an understanding that Flop might have swum from one vertex to the furthest diagonally opposite vertex, or anywhere along the furthest edge.

Example 2:

a) Responses may vary but should demonstrate an understanding that the third value represents the z – coordinate, the depth.

b) 3 units.

c) (0, 0, 3), (3, 0, 3), (0, 3, 3), (3, 3, 3)

d) (3, 3, 3)

Example 3:

a) A cuboid has eight vertices, 12 edges and six faces.

b) A vertex is described by the single letter that labels that point. To describe an edge, you would need the two letters at either end (for example, edge AB is between vertex A and vertex B). To describe a face, you would need to give the four letters at the four vertices (for example, face ABCD is the face between these four vertices).

c)

15 cm	8 cm	6 cm
DE	DA	DC
AH	EH	GH
BG	BC	BA
CF	GF	EF

d) There will be four different lengths of diagonal. (With four examples of each diagonal, so 16 diagonals in total.)

e)

$\sqrt{15^2+8^2}$	$\sqrt{15^2+6^2}$	$\sqrt{6^2+8^2}$
DH	DF	DB
AE	AG	AC
BF	BH	EG
CG	CE	FH

f) The four longest diagonals are missing: AF, BE, DG and CH. The length of these diagonals can also be calculated using Pythagoras' theorem, using one of the diagonals calculated in part e. Any of the following calculations are valid: $\sqrt{AE^2+6^2}$ or $\sqrt{BF^2+6^2}$ or $\sqrt{CG^2+6^2}$ or $\sqrt{DH^2+6^2}$ or $\sqrt{BH^2+8^2}$ or $\sqrt{CE^2+8^2}$ or $\sqrt{AG^2+8^2}$ or $\sqrt{DF^2+8^2}$.

Example 4:

For each of the triangles, students should consider the plane in which the triangle sits, and use this to form the basis of their explanations. For example:

- AFC sits on the plane through AFHC. AFHC is a rectangle and FC is a diagonal of that rectangle. AFC is a right angle.
- BCM sits on the plane through BCEF. BCEF is a rectangle, and M is the midpoint of side EF. There are right angles at the vertices of the rectangle at FBC, BCE, CEF and EFB. Fixing the end of the line at B and moving the other end from F to M reduces the size of the angle, so it will be less than 90°. Similarly for CE to CM.
- Angle CMB is not necessarily a right angle, but, dependent on the ratio of the side lengths, could be, i.e., when BF is half of BC.
- BEH sits on the plane through ABHE, and ABHE is a rectangle with BE a diagonal, therefore BHE is a right angle.

Apply Pythagoras' theorem to determine lengths in three-dimensional contexts

Example 5:

Diagrams may vary but should show a right-angled triangle with one side length of 22 cm (the length of the pencil case) and one side length of 9 cm (the diameter of the circular end of the pencil case). The longest possible distance inside the pencil case is represented by the hypotenuse and can therefore be found using Pythagoras' theorem:

$a^2 + b^2 = c^2$

$22^2 + 9^2 = c^2$

$484 + 81 = c^2$

$\sqrt{565} = c$

$c =$ 23.8 (to 1 d.p.)

The pencil is 23 cm long, and since 23 < 23.8 the pencil will fit in the pencil case.

Example 6:

a) To support teachers, the answers are given as both surds and decimals rounded to two decimal places. However, the emphasis should be on identifying the correct lengths and substituting appropriately into Pythagoras' theorem.

(i) $\sqrt{3^2 + 3^2}$ $(= \sqrt{18} = 3\sqrt{2} \approx 4.24)$

(ii) $\sqrt{3^2 + 3^2}$ $(= \sqrt{18} = 3\sqrt{2} \approx 4.24)$

(iii) $\sqrt{3^2 + 3^2}$ $(= \sqrt{18} = 3\sqrt{2} \approx 4.24)$

(iv) $\sqrt{3^2 + (3\sqrt{2})^2}$ $(= \sqrt{9 + 18} = \sqrt{27} = 3\sqrt{3} \approx 5.20)$

b) For parts (i), (ii) (iii), accept any coordinate pairs from the following set, where the general rule is that one of the coordinates remains the same, and the other two differ by three units each:

(0, 0, 3) to (3, 3, 3)	(0, 3, 0) to (3, 0, 3)	(3, 0, 0) to (0, 3, 0)	(3, 0, 3) to (0, 3, 3)
(0, 3, 0) to (3, 3, 3)	(3, 3, 0) to (3, 0, 3)	(0, 3, 0) to (0, 0, 3)	(0, 0, 0) to (0, 3, 3)
(3, 0, 0) to (3, 3, 3)			

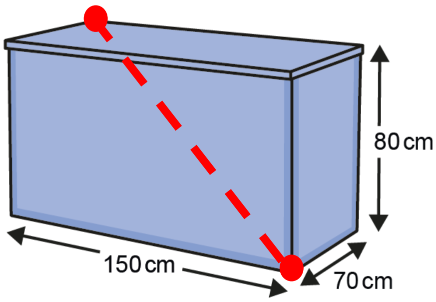
For part (iv), accept (0, 3, 3) to (3, 0, 0).

Example 7:

Responses may vary but should demonstrate an understanding that:

a) Max was likely considering the length of the **face** diagonal, so that the prop was parallel with one of the faces. This is not the greatest length inside a cuboid. The largest face diagonal would be: $\sqrt{150^2 + 80^2} \approx$ 170.0 cm.

b) Nell made it fit by using the longest diagonals of the cuboid, not a face diagonal. This means that the prop is not parallel to any of the faces; instead, it fits in the distance from one vertex and the diagonally opposite vertex on the opposite face. An example is annotated on the diagram to the right. There are four possible diagonals of the same length.



$a^2 + b^2 = c^2$

$80^2 + \left(10\sqrt{274}\right)^2 = c^2$

$\sqrt{33800} = c$

c =183.8 cm (to 1 d.p.)

183.8 > 175 and therefore, the line prop fits within this longest diagonal.

Example 8:

Responses may vary but should demonstrate an understanding that:

a) 201 cm may be the height of the cupboard, but when rotating it to stand it upright, the cupboard needs more vertical space than just its height. During rotation, the top corner traces an arc, and the diagonal across the cupboard becomes relevant. This diagonal (the hypotenuse of a right-angled triangle formed by the base and height) may be longer than the shed’s internal height of 205 cm, so it might not fit during the rotation. Leo may need to build the cupboard in an upright position.

b) Sabrina could calculate the diagonals to demonstrate that there are dimensions within the cupboard that are greater than 205 cm:

Longest face diagonal:

$a^2 + b^2 = c^2$

$100^2 + 201^2 = c^2$

$\sqrt{50\,401} = c$

c =224.5 cm (to 1 d.p.)

Longest diagonal:

$a^2 + b^2 = c^2$

$58^2 + (\sqrt{50\,401})^2 = c^2$

$\sqrt{53\,765} = c$

c = 231.9 cm (to 1 d.p.)

c) As shown in Sabrina’s workings for part b above, the longest diagonal is 231.9 cm (to 1 d.p.). Since 231.9 < 226, the fishing rod will fit inside the cupboard.

Estimate angles in the right-angled triangles that can be created within 3D shapes

Example 9:

a)

< 45°	= 45°	> 45°
GEC	DBC CDB HED EDH	ECG

b) Students should demonstrate an understanding that angle DBC will now be greater than 45° and angle CDB would now be less than 45°. The rest will remain in the same columns.

c) There are a number of different angles that students could identify, but they should accord to the following principles:

< 45°: any angle formed by the cube diagonal and a face diagonal or edge.

= **45°**: any non-right angle in a triangle formed using two edges of a face and the diagonal across that face.

> 45°: any angle formed by the cube diagonal and a single edge of a face.

Use trigonometric ratios to determine lengths and angles in three-dimensional contexts

Example 10:

a) 13 m

b) 67.4° (to 1 d.p.)

Example 11:

a) 41

b) 16° (to the nearest degree)

Example 12:

a) 15.2° (to 1 d.p.) b) 15.2° (to 1 d.p.) c) 74.8° (to 1 d.p.)

11.3.2.5 Use chains of reasoning to derive the cosine rule

Identify relevant features of triangles

Example 1:

a) Some possible responses include:

Same	Different
• Each student has labelled three elements of the triangle: two angles and one side. • All students have used capital letters for angles and lower-case letters for sides, following standard convention.	• Beth has assigned different angles to be A and B compared to the other three students. • Anwar has not followed the convention of the side opposite the angle being labelled with the lower-case version of the same letter. • Daisy has labelled side c, so she does not have a complete pair of opposite sides and angles.

b) Students' responses should demonstrate an understanding of the standard convention for labelling sides and angles. For example, Mr Cooper should advise Anwar to re-label his triangle so that side a is opposite angle A.

c) Students should explain that the specific choice of letters is flexible, provided the standard convention is followed – typically using capital letters for angles and matching lower-case letters for the sides opposite those angles.

d) Responses may vary but should demonstrate an understanding that Callum has the correct information to find the perpendicular height, as has Anwar (despite the unconventional labelling). Beth could also find the height once she has used the two angles given to calculate the third. (Note that Daisy could also find the perpendicular height if she calculated the missing angle and used the sine rule; students' prior knowledge of this rule will depend on your school's scheme of work.)

e) Anwar, Beth and Callum (and Daisy if using sine rule).

Use the fact that all triangles can be divided into two right-angled triangles

Example 2:

a) Nic (and Annabelle if using algebraic manipulation as demonstrated in *Example 9*; teachers will need to use professional judgement as to whether to explore this now.)

b) Shane could have given side b or side c, or Chinraj could have given angle C.

c) Chinraj (by finding the height perpendicular to side b) and Nic. (Note that Annabelle could if she used the information given along with the sine rule to first calculate h).

d) Shane could be given two of the sides.

Example 3:

a) Either side c and angle A, or side a and angle C. (Note that it is also possible to use all three sides, although this requires substantial algebraic manipulation which may not be suitable at this stage in students' learning.)

b) Students' responses may vary in detail but should demonstrate an understanding of the relationship between the perpendicular height and the angle and side required to find it. That is, regardless of which perpendicular height is specified, what is the same is that we need a complete angle (i.e., not split by the perpendicular height) and the side that represents the hypotenuse formed between that angle and the base. What is different is the particular angle and side, and the labels that are used.

Example 4:

a) $l_1 = 8.5\, sin\, 42$

b) $l_1 = c\, sin\, A$

c) Not enough information provided.

d) Not enough information provided.

Example 5:

a) $l_1 = c\, sin(A)$ or $l_1 = a\, sin(C)$ b) $l_2 = c\, sin(B)$ or $l_2 = b\, sin(C)$ c) $l_3 = b\, sin(A)$ or $l_3 = a\, sin(B)$

Example 6:

a) $b - x$

b) Triangle 1: $y^2 = c^2 - x^2$ Triangle 2: $y^2 = a^2 - (b - x)^2$

c) Responses may vary but should demonstrate an understanding that they are both valid equations for y^2 and therefore the relationships between the variables within the shape are the same. However, different sides/angles are used to create the variables in each equation.

d) $c^2 - x^2 = a^2 - (b - x)^2$

Example 7: $cos\ A = \frac{x}{c}$ and $cos\ C = \frac{b-x}{a}$
Example 8: a) $x = \sqrt{(2v)^2 + (w - v)^2}$ or $x = \sqrt{5v^2 - 2vw + w^2}$ b) $x = \sqrt{(g + 2h)^2 - (h + 3f)^2}$ or $x = \sqrt{g^2 + 4gh + 3h^2 - 6fh - 9f^2}$ c) $x = \sqrt{k^2 - (k\ cos\ P)^2}$. Note this could be rearranged and factorised. While not needed for GCSE, the trigonometric identity $1 - cos^2\ P \equiv sin^2\ P$ can be used, resulting in a final equation of $x = k\ sin\ P$. d) $x^2 = x^2(sin^2\ M + cos^2\ M)$ which leads to the identity $1 \equiv sin^2\ M + cos^2\ M$ $x^2 = x^2$ $x = x$ e) $x = \sqrt{(y\ cos\ P)^2 - (y - 1)^2}$
Follow the chains of reasoning involved in deriving the cosine rule *Example 9:* In order: D, B, A, C The detail of students' explanations may may vary, but should capture the following ideas: 4) Simplifying step 3. 5) Using triangle ABD to find x. 6) Substituting $x = c\ cos(A)$ in to step 4. 7) Rearranging to make a^2 the subject.
Appreciate Pythagoras' theorem as a special case of the cosine rule *Example 10:* Responses may vary but should demonstrate an understanding that Kit is correct; the cosine rule is a generalisation of Pythagoras' theorem. $cos(90) = 0$ $a^2 = b^2 + c^2 - 2bccos(A)$ $a^2 = b^2 + c^2 - 2bccos(90)$ $a^2 = b^2 + c^2 - 0$ $a^2 = b^2 + c^2$

11.4.1.1 Understand that 3D shapes can be represented two dimensionally (plans and elevations)

Understand that different 3D shapes can have the same 2D representation, depending upon the viewpoint

Example 1:

a) Responses may vary but should demonstrate an understanding that it is not possible to determine which shape is which, as the largest circular cross section is the same for all of them.

b) Possible answers include:

Same	Different
• They are all the same width at the widest point, as all three solids share congruent circular plans. • All will be symmetrical about a vertical axis (provided the cone is a right cone).	• They will all have different 2D shapes as their side elevations: a triangle (for the cone), a rectangle (for the cylinder), and a circle (for the sphere).

c) Shape A: an isosceles triangle with perpendicular height 1.8 cm and base 1.8 cm

Shape B: a square with sides 1.8 cm

Shape C: a circle with diameter 1.8 cm

Example 2:

a) Niamh is correct in saying a triangle could represent the front elevation of a cone or a pyramid. It could also represent a triangular prism, depending on the orientation of the shape. Students' explanations of why may vary, but should show that they understand that the front elevation is 2D view of the shape, and so a cone or pyramid will appear triangular when viewed from the side, as will the cross-sectional face of a triangular prism.

b) A prism has been ruled out as the side elevation would be a rectangle.

c) Cone: circle (with or without a 'dot' at the centre to represent the peak).

Square-based pyramid: square (with or without diagonals drawn in to represent the edges of the faces that meet at the peak).

Example 3:

a)

Shape of whole cake	Possible slice shapes
Triangular prism	Rectangles of different sizes Identically-sized triangles
Cuboid (with no square faces)	Rectangles of different sizes Identically-sized rectangles
Cube	Identically-sized squares
Cuboid (with two square faces)	Identically-sized rectangles Identically-sized squares

b) Squares of different sizes: square-based pyramid

Triangles of different sizes: tetrahedron (triangular-based pyramid)

c) Cara could cut identically-sized circles or differently-sized rectangles.

Connect plans and elevations with the corresponding 3D shape

Example 4:

Object	Plan	Front	Side
Snare drum	Circle with same diameter as the width of the rectangle given for front elevation.		Identical to front elevation
Tent	Rectangle with the same width as the base of the front elevation, and the same length as the longest dimension in the side elevation (with or without a straight line bisecting the shape parallel to the longest side, representing the ridge of the tent).		
Pyramid of Giza	Square with the same side lengths as the base of the front/side elevation triangles (with or without the diagonals, representing the edges of the triangular faces meeting at the apex).		
Traffic cone	Square with same side lengths as the longest dimension of the rectangle at the bottom of the front elevation. Centred within this square is a circle with the same diameter as the base of the triangle in the side elevation (with or without a dot in the centre of the circle, representing the apex of the cone).		Identical to front elevation

Example 5:

Responses may vary but should demonstrate an understanding that, although the shapes are correct, the dimensions are not. As this is a square-based pyramid, the front and side elevations should be identically sized. The lengths of the square must match the bases of the triangular elevations.

Use 2D representations to identify the structure of 3D shapes

Example 6:

a) Penny will draw four different shapes: a square, an equilateral triangle, an isosceles triangle (front and side view of shapes A, B, D, E) and a rectangle (side view of shape C).

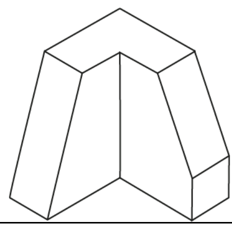
b) She will draw three squares; three equilateral triangles; eight isosceles triangles (although these will not be the same for each shape); and one rectangle.

Example 7:

a) Responses may vary but should demonstrate an understanding that:

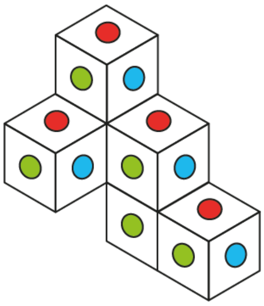
- Shape A does not correspond to the 3D shape because it lacks the slanted sides shown in the front and side elevations.
- Shape B does not correspond because the slanted sides in the elevations indicate two slanted faces in the correct 3D shape, but shape B only has one. It also only shows a difference in height on one of the sides.
- Shape C does not correspond because it also has only one slanted face, whereas the correct 3D shape requires two.

b) The correct 3D shape is a combination of the left-hand sloping side of C and the right-hand truncated sloping side of B, as shown here:



Example 8:

Six cubes as shown here:



11.4.1.3 Use 2D representations to quantify the surface area of prisms, cylinders, pyramids and composite shapes

Use the net of a shape to calculate the surface area

Example 1:

a) Responses may vary but should demonstrate an understanding that cubes B and C have the same (and smallest) surface area. Shape A has the greatest surface area.

b) Accept any net of a cube with side lengths of:

(i) $\sqrt{10.\dot{6}}$ cm (ii) $\sqrt{6}$ cm (iii) 2 cm

Example 2:

a) Responses may vary but should demonstrate an understanding that, if students have found valid lengths for the square faces, then they can double the resultant areas and subtract them from 96. Dividing this remaining area by four, and then by the length of their square, should result in a valid length for the cuboid (i.e., $96 = 2b^2 + 4ab$ where b is the side length of the square and a is the length of the prism).

For example, 1 cm × 6 cm × 6 cm assuming the net wasn't drawn to scale.

There is an infinite number of solutions involving non-integers.

b) Responses may vary but should demonstrate an understanding that there are still infinitely many non-integer solutions with all faces rectangular. However, there are now no integer solutions.

Example 3:

10 cm

Appreciate the difference between surface area and volume, and work between them

Example 4:

a) 294 cm^2

b) Responses may vary but should demonstrate an understanding that the majority of the distinct nets for a cube will fit. However, a net that is more than three squares wide (3 × 7) or four squares long (4 × 7) will not fit.

Understand the relationship between the plan, front and side elevations and the surface area of a prism

Example 5:

a) Responses may vary but should demonstrate an understanding that Taj is correct because there are three differently-sized rectangles, each representing two of the faces of the cuboid.

b) Responses may vary but should demonstrate an understanding that Quinn has assumed that all three rectangular faces are the same, when in fact each is different.

c) $(6 \times 8) + (6 \times 6) + (10 \times 6) + 2\left(\frac{1}{2} \times 6 \times 8\right) = 192$ cm²

d) Responses may vary but should demonstrate an understanding that five cuboid-shaped door stops can be painted with one tin (with enough left to paint four triangular prism-shaped door wedges).

If painting the triangular prism-shaped door wedges only, one tin would cover 26 of these.

Example 6:

a) Students' diagrams should correspond to the following descriptions:

Plan	Front	Side
Rectangle bisected by a vertical line.	L-shape where: • the overall height is the same as the vertical height of the side elevation • the overall width is the same as the horizontal width of the plan view • the shorter vertical length is half of the height of the side view • the shorter horizontal length is half of the width of the plan view.	Rectangle bisected by a horizontal line.

b) Responses may vary but should demonstrate an understanding that the area of the front elevation is equal to the area of the back elevation. The area of the two rectangles that make up the side elevation is equal to the other side. The area of the two rectangles that make up the plan view is equal to the area of the base. This is only true for right prisms – those with 'slanted' sides.

c) Responses may vary but should show that the minimum information needed is the base length, height, depth, and the dimensions of the cut-away section from the front elevation.

Understand that, for the net of a cylinder, the circumference of the circular face corresponds with the length of the rectangular face

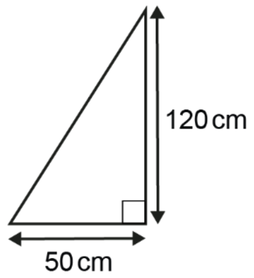
Example 7:

a) $\left(16\pi + 10(8\pi)\right) + \left(36\pi + 10(12\pi)\right) + \left(81\pi + 10(18\pi)\right) = 513\pi$ cm²

b) 6

Know the difference between the slant height and vertical height of a pyramid

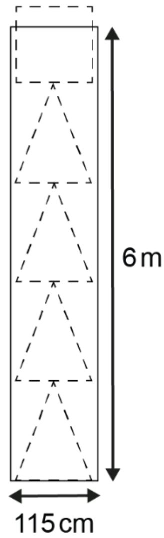
Example 8:



To find the vertical height of the faces:

$h = \sqrt{50^2 + 120^2}$

$h = 130$ cm



Since the fabric pattern means the triangles need to stay in the same orientation:

4×130 cm $= 520$ cm

600 cm $- 520$cm $= 80$ cm

$80 < 100$

Therefore, the 1 m × 1 m square base cannot be cut.

11.4.2.2 Solve problems involving the volume of pyramids and cones

Understand the relationship between the volume of a pyramid and the volume of a prism with the same height and congruent bases

Example 1:

a) Responses may vary but should involve a reasonable estimate with students able to use precise mathematical language if asked to explain their answer.

b) 160 cm^3

c) Responses may vary but should demonstrate an understanding that the volume of the pyramid is one-third of the volume of the cuboid. Students may also choose to comment on the accuracy of their estimation.

Example 2:

a) One-third

b) Three-quarters

c) Responses may vary but should demonstrate an understanding that:

Volume of cuboid = $15 \times 10 \times l = 150l$

Volume of pyramid = $\frac{1}{3} \times 15 \times 10 \times l = 50l$

Total volume of shape = $150l + 50l = 200l$

$200\ l = 4000$ cm^3

$l = 20$ cm

Understand the relationship between the volume of a cone and the volume of a cylinder with the same height and congruent bases

Example 3:

Responses may vary but should demonstrate an understanding that when filled to the base of the cylinder, the cone is entirely filled.

Explanation through reasoning:

- The vertical height of the cone is 4 cm ($3^2 + 4^2 = 5^2$).
- As the cone and the cylinder have the same vertical height and the same 'base', the volume of the cone is one-third of the volume of the cylinder. Therefore, the volume of the cone is x cm^3, the volume of the cylinder is $3x$ cm^3.
- Total volume $4x$ cm^3
- $\frac{1}{4}$ of $4x = x$. Therefore, when the glass is filled to the base of the cylinder, it is one-quarter full.

Demonstration through calculation:

Volume of cylinder section of glass = $\pi \times 3^2 \times 4 = 36\pi$

Volume of cone section of glass = $\frac{1}{3} \times \pi \times 3^2 \times 4 = 12\pi$

Total volume = $36\pi + 12\pi = 48\pi$

$\frac{1}{4}$ of total volume = $\frac{1}{4} \times 48\pi = 12\pi$ = volume of cone section of glass

Use the vertical height when calculating the volume of a pyramid or cone

Example 4:

Note that there are several ways to find the vertical height of the pyramid to demonstrate the volume. One example is given below:

- Let the vertical height of the smaller triangular face be a, creating a right-angled triangle with the height (a), a base which is half of the base of the smaller triangular face (16 ÷ 2) and a hypotenuse which is the given slanted edge length (17):

  $a^2 + 8^2 = 17^2$

  $a = \sqrt{17^2 - 8^2}$

  $a = 15$ cm
- Let the vertical height of the actual pyramid be b, creating a right-angled triangle with the height (b), a base which is half of the base of the larger triangular face (18 ÷ 2) and a hypotenuse which is the vertical height of the smaller face (15):

  $b + 9^2 = 15^2$

  $b = \sqrt{15^2 - 9^2}$

  $b = 12$ cm
- Substituting $b = 12$ into the formulae for the volume of a pyramid gives:

  Volume = $\frac{1}{3} \times 18 \times 16 \times 12 = 1152$ cm^3

Example 5:

a) 1500π cm^3

b) 735π cm^3

Understand that the volume of a frustum can be found by subtracting the missing cone

Example 6:

a) Responses may vary but should demonstrate an understanding that you would need to know if the cup is a frustum. If so, the minimum information needed is the radii of the top and bottom circles and the perpendicular height of the cup.

b) Responses may vary but should demonstrate an understanding that:

Volume of frustum: $\pi \times 6^2 \times 45 - \pi \times 4^2 \times 30 = 380\pi$ cm^3

Volume of one cube: $380\pi \div 15 = \frac{76}{3}\pi$ cm^3

$\sqrt[3]{\frac{76}{3}\pi} = 4.3$ (to 1 d.p.)

Dimensions of each section, to the nearest centimetre, $4 \times 4 \times 4$