MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Fractions

3.4 Adding and subtracting within one whole

Teacher guide | Year 3

Teaching point 1:

When adding fractions with the same denominators, just add the numerators.

Teaching point 2:

When subtracting fractions with the same denominators, just subtract the numerators.

Teaching point 3:

Addition and subtraction of fractions are the inverse of each other, just as they are for whole numbers.

Teaching point 4:

To subtract from one whole, first convert the whole to a fraction where the denominator and numerator are the same.

Overview of learning

In this segment children will:

- unitise by verbally describing a non-unit fraction as a multiple of its unit fraction; verbalise in both forms, for example *'four-fifths'* and *'four one-fifths'*, and interchange between the two forms
- learn to add and subtract fractions with the same denominator, drawing heavily on unitising and their knowledge that, for example, four-fifths is made up of four one-fifths
- continue to apply fractions through the contexts of area, linear and quantity (sets of objects) models
- continue to develop their understanding that fractions are numbers that can be used in calculations
- meet calculations involving addition and subtraction of fractions with different denominators that can be solved without the use of common denominators.

After completing segment *3.3 Non-unit fractions: identifying, representing and comparing*, children will have a good understanding of non-unit fractions as the repeated addition of unit fractions, for example, five lots of one-eighth make five-eighths. This will provide a solid foundation from which to tackle addition and subtraction of fractions. A deep conceptual understanding of the concept that, for example, five one-eighths make five-eighths, and that this is also the sum of three one-eighths and two one-eighths, will help to minimise the chance of children mistakenly adding or subtracting the denominator. This is a common error that children make when adding and subtracting fractions, but by emphasising the composition of non-unit fractions so heavily, the risk *should* be reduced. Nevertheless, because it is such a common error, these materials encourage you to address it head-on and spend time with children discussing why, for example, $\frac{3}{8}+\frac{2}{8}$ cannot possibly equal $\frac{5}{16}$.

To support the ability to unitise (i.e. to see a non-unit fraction as composed of several unit fractions), teachers should use dual-naming of fractions throughout this segment as they did in the last. For example, *'two one-eighths add three one-eighths equals five one-eighths'* as well as *'two-eighths add three-eighths equals five-eighths'*. Children will need to be able to interchange fluently between these ways of naming fractions when adding and subtraction fractions. They are already very used to adding and subtracting in all sorts of other units, and this approach links addition and subtraction of fractions to their previous learning.

For example:

- 3 + 2 = 5
- 3 cats + 2 cats = 5 cats
- £3 + £2 = £5
- 3 kg + 2 kg = 5 kg
- 3 million + 2 million = 5 million
- 3-eighths + 2-eighths = 5-eighths.

In segment *3.3*, children learnt that as well as operators (for example, $\frac{1}{4}$ *of* something), fractions are also numbers with a position on a number line (for example, the number $\frac{1}{4}$). Throughout this segment, area models (such as bars or circles) are shown alongside number lines. Using number lines to represent calculations reinforces that fractions are numbers but also helps to guide children not to add the denominators. Teachers should take any appropriate opportunities to refer to fractions as numbers in their questioning, for example, rather than saying, *'Now we are going to add a different pair of fractions'*, instead say *'Now we are going to add a different pair of numbers'*.

Within this segment, children are introduced to generalisations for adding and subtracting fractions, such as: ***'When adding (or subtracting) fractions with the same denominators, just add (or subtract) the numerators.'*** Generalisations are extremely useful, and adults often draw on them heavily – for instance 'just knowing' to add only the numerators when adding fractions with the same denominator, saves us from having to stop and think from first principles every time. Recognising mathematical relationships and forming generalisations about how those relationships hold true to relevant contexts is the essence of learning mathematics. However, it is really important that children don't *only* take the generalisation from this segment, but also have an underlying understanding of *why* that generalisation holds true. As children meet different types of fraction calculations (addition and subtraction of fractions with different denominators, and multiplication and division of fractions), they will learn further generalisations. If these are just learnt by rote, with the underpinning rationale cast adrift, children are less likely to be able to connect future learning or develop the necessary depth of understanding. Lots of guidance is provided throughout the segment to ensure you continue to reinforce the *why*, rather than falling back on just the generalisation.

An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast*. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.*

Teaching point 1:

When adding fractions with the same denominators, just add the numerators.

Steps in learning

Guidance

1:1 This segment starts by exploring the addition of fractions within simple story contexts. The introduction of a written addition equation is delayed until after the initial foundations of this teaching point have been introduced, to minimise the possibility of children falling into the 'add the top, then add the bottom' misconception.

The initial focus here is on children conceptually understanding that if we add, for example, three-eighths and two-eighths, the solution must be five-eighths. The significant focus on unitising in earlier segments, combined with an awareness of non-unit fractions as repeated addition of unit fractions, will help children with this.

As a starting point, you might wish to tell children a fraction story, for example:

'An apple is divided into eight equal parts.'

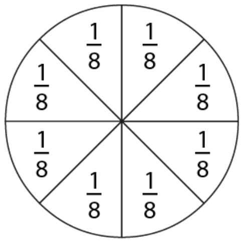
- *'Onithi eats three-eighths of the apple.'*
- *'Joel eats two-eighths of the apple.'*

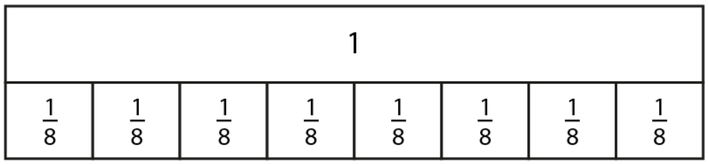
'How much of the apple do they eat altogether?'

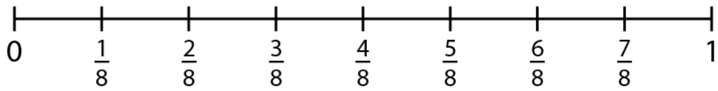
The examples opposite may be useful for prompting discussion around the story, for example:

'I'd like you to discuss this with your partner. There are some images on the board you can use and which you might find helpful. You may be able to think of your own different way to explain how much of the apple Onithi and Joel have eaten altogether.'

Representations





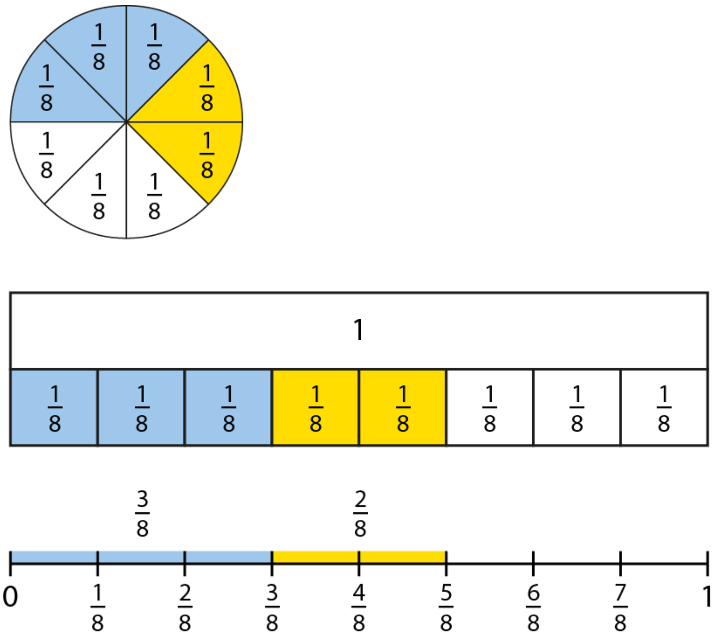


1:2

After children have had an opportunity to discuss and work with their partner, encourage them to explain their ideas to the group.

- Some children may choose to draw on the diagrams on the board (in ways similar to those shown opposite) to support their explanations.
- Some children may not refer to the diagram, preferring to describe their solution verbally.
- Some children may use words like '*plus*' and '*add*'. This is fine, but at this stage the teacher should adhere to 'non-mathematical' words like '*and*'.

As children give their explanations, reinforce or model unitising language in your responses, interchangeably referring to $\frac{3}{8}$ as '*three-eighths*' and '*three one-eighths*'. Encourage the children to do the same.



1:3

Once you have discussed the children's explanations, you may wish to summarise:

'*Onithi eats three one-eighths of the apple and Joel eats two one-eighths of the apple. You have convinced me that altogether they eat five one-eighths of the apple.*'

Show the children how to write an equation to represent the story. You might find it helpful to say the following points and write on the board at the same time, as indicated:

Say	Write
'Onithi eats three-eighths of the apple…'	$\frac{3}{8}$
'…and Joel eats two-eighths of the apple.'	$+\frac{2}{8}$
'Altogether they have eaten five-eighths of the apple.'	$=\frac{5}{8}$

Pose the following questions to probe their understanding:

- *'What does the number* $\frac{3}{8}$ *represent?'*
  (It represents the amount of the apple that Onithi ate.)
- *'What does the number* $\frac{2}{8}$ *represent?'*
  (It represents the amount of the apple that Joel ate.)

- *'What does the number $\frac{5}{8}$ represent?'*

  (It represents how much of the apple they ate altogether.)

  $\frac{3}{8}+\frac{2}{8}=\frac{5}{8}$

1:4 The children have learnt how to write an equation to summarise something they already know. Note that children haven't been asked to *'add fractions'*. Instead, we have encouraged them to think conceptually about what happens if you have three one-eighths and two one-eighths, and we have introduced symbols to represent this thinking.

Now, you might like to read this written equation using the language of addition. Point to the relevant part of the equation as you speak, explaining that you are going to summarise the equation in mathematical language.

Say	Write
'Three one-eighths…'	$\frac{3}{8}$
'…plus two one-eighths…'	$+\frac{2}{8}$
'…is equal to five one-eighths.'	$=\frac{5}{8}$

Repeat this, again pointing out parts of the equation, but this time say: *'Three-eighths plus two-eighths is equal to five-eighths.'*

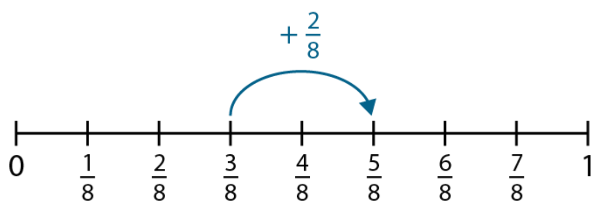
Give children the opportunity to practise reading the equation aloud for themselves.

As we have now introduced the addition symbol, you can also show this calculation on a number line in the same way the children will be familiar with for whole numbers. Show children how to represent the calculation. Working with a number line will help to reinforce that fractions are numbers.

Equation:

$\frac{3}{8}+\frac{2}{8}=\frac{5}{8}$

Number line:



1:5	Probe what the children notice about the numbers in the equation. *'What has stayed the same and what has changed in the sum of the equation?'* They should tell you that the denominator has stayed the same, but the numerator has changed. This is a good point to discuss why this is the case. Refer back to the image from *segment 3:2 Unit fractions: identifying, representing and comparing* (shown opposite). Reiterate that the denominator is the number of equal parts in a whole. The denominator is eight in all the fractions in our apple story because the whole apple is divided into eight equal parts. The number of parts the apple is divided into doesn't change; it isn't cut up into any more pieces. All through the story, the unit we are working in is one-eighth. The numerator is the number of parts of the whole. The numerators *are* different because the number of parts of the whole apple we are talking about changes through the story. Onithi and Joel have a different number of parts of the whole apple; Onithi has three of the eight parts and Joel has two of the eight parts.	**1** ← Numerator (1 for a unit fraction) One of the parts of the whole **2** ← Denominator The number of equal parts in the whole

1:6

In *Spine 1: Number, Addition and Subtraction*, segments *1.5* and *1.6*, children learnt about two different addition structures; aggregation and augmentation. Our apple story was an *aggregation* story – a context involving the bringing together of two parts.

Now we are going to focus on a story involving *augmentation* – where we start with a certain amount and then increase it (also referred to as a '*First…, then…, now…*' context).

Tell a story involving an augmentation context, for example: *'When first measured, the plant is six-tenths of a metre tall. It then grows two-tenths of a metre. How tall is it now?'*

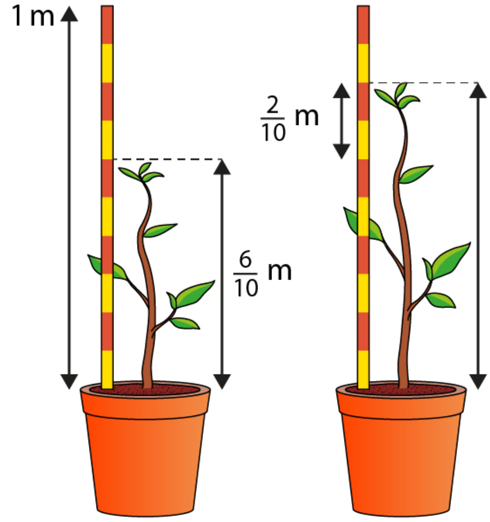
Show the examples opposite and point out the relevant features and information. Explain that the metre stick is one metre tall. There are ten equal parts on the metre stick, so each horizontal line on the metre stick represents one-tenths of a metre. When first measured, the plant is six-tenths of a metre tall. It then grows by two-tenths of a metre.

Recap that in the last story we were working in units of eighths. Ask: *'What is the unit we are working in for this story?'* (The unit is now tenths.)

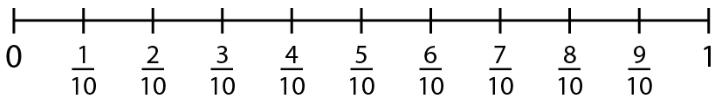
Challenge children to work out how tall the plant is now and to justify their answers. As before, provide a range of models that children may use to aid their explanation.

'When first measured, the plant is six-tenths of a metre tall. It then grows two-tenths of a metre. How tall is it now?'

- *'What unit are we working in for this story?'*



Representations:



1									
$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

1:7

Discuss the children's completed models and responses. These may include explanations such as the following:

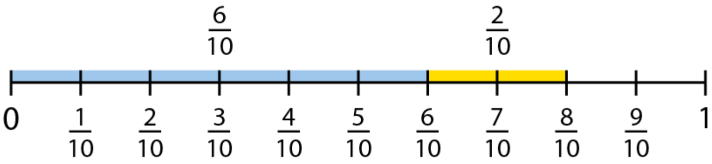
- Shading the number line:
  - *'We need to draw a line that is six divisions long to represent six-tenths of a metre. We add another line, two divisions long to represent two-tenths of a metre.'*
  - *'Altogether our line is eight divisions or eight-tenths of a metre long.'*
- Jumps on the number line:
  - *'When first measured, the plant was six-tenths of a metre tall. It then grew two-tenths of a metre, so I will show this as a jump on the number line. Now it is eight-tenths of a metre tall.'*
- Using the bar model:
  - *'I shaded six one-tenths blue, and then I shaded two more one-tenths yellow. That means that eight one-tenths are shaded altogether.'*

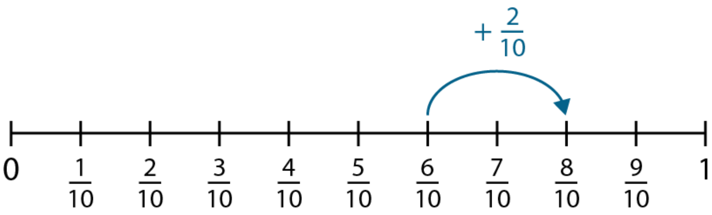
Refer back to the original question and give the solution in context: *'The plant is now eight-tenths of a metre tall.'*

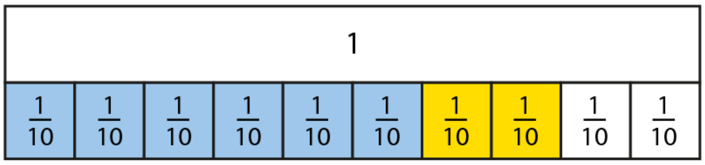
For each of the models, probe their understanding with questions such as:

- *'What does the "$\frac{6}{10}$" represent?'*
- *'And the "$\frac{2}{10}$"?'*
- *'And the "$\frac{8}{10}$"?'*

- *'What does the "$\frac{6}{10}$" represent?'*
- *'And the "$\frac{2}{10}$"?'*
- *'And the "$\frac{8}{10}$"?'*







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<tr><td>1:8</td><td colspan="2">
Explain that you are going to tell the plant story again and you would like each pair to write the equation down as you say the steps. The children have followed this process many times in the additive segment of Spine 1, and so should be familiar with the routine. They have also seen you model it for the previous story. (It is fine if children already offered a written equation when you discussed their models, but still do this now so that all children have the chance to write an addition equation involving fractions.)

Retell the story, writing the equation yourself on the class board as you go:

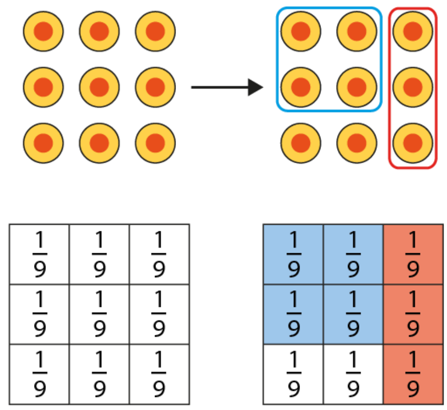
<table>
<tr><th>Say</th><th>Write</th></tr>
<tr><td>'When first measured, the plant was six-tenths of a metre tall.'</td><td>$\frac{6}{10}$</td></tr>
<tr><td>'It then grew another two-tenths of a metre.'</td><td>$+\frac{2}{10}$</td></tr>
<tr><td>'Now the plant is eight-tenths of a metre tall.'</td><td>$=\frac{8}{10}$</td></tr>
</table>
Dictating the whole story at this stage, as we did previously, should minimise the chances of children mistakenly adding the denominators. Note that some children may include the unit 'm' in the equation and some may not; both are fine. However, modelling without the unit draws attention to the numbers in the equation without any potential distractions.

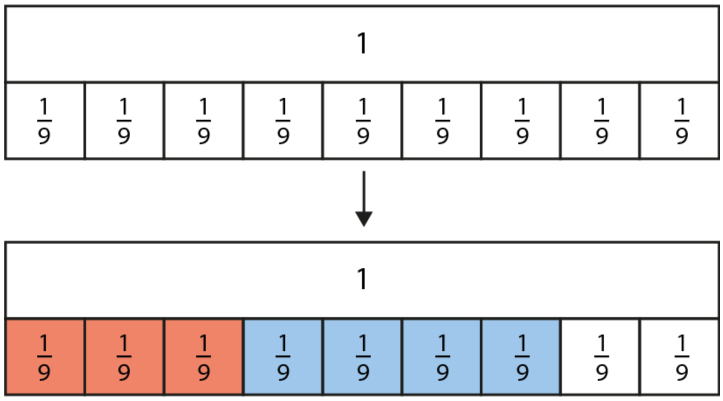
Once the equation is written, you can summarise it as before, pointing at the relevant part of the equation as you say each step and asking the children to join in with you.
• 'Six one-tenths plus two one-tenths is eight one-tenths.'
And
• 'Six-tenths plus two-tenths is eight-tenths.'

You might wish to probe further, for instance: 'What is the same in all the fractions?' Reiterate that the denominator is the same because we are working with units of tenths throughout. We are simply combining parts. The number of equal parts in the whole has not changed.
</td></tr>
<tr><td></td><td colspan="2">
$\frac{6}{10}+\frac{2}{10}=\frac{8}{10}$
or
$\frac{6}{10}\text{m}+\frac{2}{10}\text{m}=\frac{8}{10}\text{m}$
</td></tr>
<tr><td>1:9</td><td>So far, diagrams and number lines have been used to support children as they make sense of the addition of fractions. At this point, introduce some stem sentences to scaffold addition calculations when children don't have accompanying images.</td><td>
Comparing fractions:
• '$\frac{6}{10}$ is six lots of $\frac{1}{10}$.'
• '$\frac{2}{10}$ is two lots of $\frac{1}{10}$.'
• 'I know that six is greater than two '
• '…so, $\frac{6}{10}$ is greater than $\frac{2}{10}$.'
</td></tr>
</table>

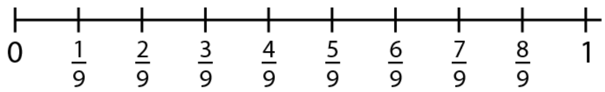
	In segment *3.3 Non-unit fractions: identifying, representing and comparing*, we used the following stem sentences: • ***'*** $\frac{\square}{\square}$ ***is ___ lots of*** $\frac{\square}{\square}$ ***'*** • ***'*** $\frac{\square}{\square}$ ***is ___ lots of*** $\frac{\square}{\square}$ ***'*** • ***'I know that ___ is greater than ___'*** • ***'…so,*** $\frac{\square}{\square}$ ***is greater than*** $\frac{\square}{\square}$ ***'*** We can now adapt these stem sentences to support the addition of fractions: • ***'*** $\frac{\square}{\square}$ ***is ___ lots of*** $\frac{\square}{\square}$ ***'*** • ***'*** $\frac{\square}{\square}$ ***is ___ lots of*** $\frac{\square}{\square}$ ***'*** • ***'I know that ___ + ___ = ___'*** • ***'…so, I know that*** $\frac{\square}{\square} + \frac{\square}{\square} = \frac{\square}{\square}$ ***'***	Adding fractions: • *'* $\frac{6}{10}$ *is six lots of* $\frac{1}{10}$*.'* • *'* $\frac{2}{10}$ *is two lots of* $\frac{1}{10}$*.'* • *'I know that* $6 + 2 = 8$*.'* • *'…so, I know that* $\frac{6}{10} + \frac{2}{10} = \frac{8}{10}$*.'*
1:10	Present another example story for practice. For variation, use a quantity model such as the example opposite. Children now have three familiar methods to justify our answers, just as in segment *3.3 Non-unit fractions: identifying, representing and comparing.* This time, also challenge the children to write an equation to show their thinking.	*'Dad baked a tray of biscuits. Olivia took* $\frac{3}{9}$ *of the biscuits and Dinesh took* $\frac{4}{9}$ *of the biscuits. What fraction of the biscuits did they take altogether?'*

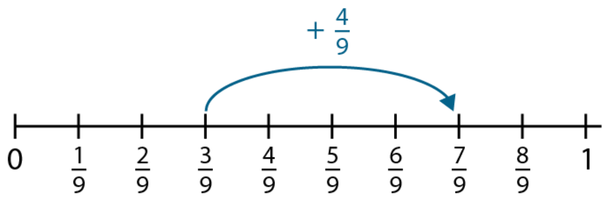
Method 1 – using a diagram:





Method 2 – using a number line:





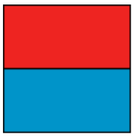
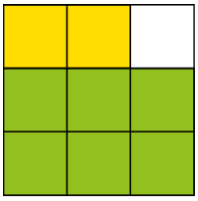
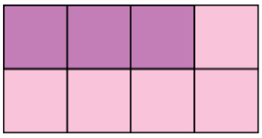
Method 3 – verbal reasoning:

- *'$\frac{3}{9}$ is six lots of $\frac{1}{9}$.'*
- *'$\frac{4}{9}$ is two lots of $\frac{1}{9}$.'*
- *'I know that 3 + 4 = 7.'*
- *'…so, I know that $\frac{3}{9}+\frac{4}{9}=\frac{7}{9}$.'*

Equation:

$\frac{3}{9}+\frac{4}{9}=\frac{7}{9}$

1:11	Present a new calculation, but this time challenge children to solve it without a visual prompt, relying solely on verbal reasoning. Provide the partially completed stem sentences as support: • ***'$\frac{4}{15}$ is ___ lots of $\frac{1}{15}$.'*** • ***'$\frac{2}{15}$ is ___ lots of $\frac{1}{15}$.'*** • ***'I know that ___ + ___ = ___'*** • ***'…so, I know that $\frac{4}{15} + \frac{2}{15} = \frac{\square}{\square}$'*** Discuss explicitly that the denominator stays the same because we are working in units of one-ninth throughout, so we *only add the numerators.*	$\frac{4}{15} + \frac{\square}{\square} =$ • *'$\frac{4}{15}$ is four lots of $\frac{1}{15}$.'* • *'$\frac{2}{15}$ is two lots of $\frac{1}{15}$.'* • *'I know that 4 + 2 = 6.'* • *'…so, I know that $\frac{4}{15} + \frac{2}{15} = \frac{6}{15}$.'*
1:12	Display all the equations you have presented so far. Use them to justify the following generalisation: ***'When adding fractions with the same denominators, just add the numerators.'***	$\frac{3}{8} + \frac{2}{8} = \frac{5}{8}$ $\frac{6}{10} + \frac{2}{10} = \frac{8}{10}$ $\frac{3}{9} + \frac{4}{9} = \frac{7}{9}$ $\frac{4}{15} + \frac{2}{15} = \frac{6}{15}$
1:13	The careful steps already taken should minimise the risk of children incorrectly adding the denominators. However, we will now highlight this possible misconception in order to address it head-on. Present the problem opposite and ask: *'Do you agree with Diego or with Mark? What mistake has been made? Explain your answers clearly.'* By this stage, children are equipped with a range of potential methods in order to validate Diego's answer. They can draw a model or a number line, or use the stem sentences or the generalisation. They should recognise that Mark's answer implies that the whole has been divided into a greater number of parts (24 parts instead of 12 parts). They should also identify that	• *'Diego says that:'* $\frac{3}{12} + \frac{5}{12} = \frac{8}{12}$ • *'Mark says that:'* $\frac{3}{12} + \frac{5}{12} = \frac{8}{24}$

	this isn't the case and that Mark has made an error by adding the denominators. Repeat the generalisation: ***'When adding fractions with the same denominators, just add the numerators.'*** Mark has made the mistake of not leaving the denominators unchanged.	
1:14	Use a true/false style problem, such as the one opposite, and consider some further equations. These types of question that encourage children to prove or disprove a statement are extremely useful in developing reasoning skills. For the final example, explain that: '$\frac{1}{11}+\frac{3}{11}+\frac{5}{11}=\frac{9}{11}$ *because* $1+3+5=9$'	*'True or false?'* $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}$  $\frac{2}{9}+\frac{6}{9}=\frac{8}{18}$  $\frac{3}{8}+\frac{5}{8}=1$  $\frac{1}{11}+\frac{3}{11}+\frac{5}{11}=\frac{9}{11}$ 
1:15	Provide children with some problems that can be used for independent practice, such as those opposite. Include examples where they need to add three fractions as well as addition contextual problems. Finally, children should consolidate and further deepen their understanding through practice with a range of questions, such as the dòng nǎo jīn problems provided.	Missing-number problems: *'Fill in the missing numbers.'* $\frac{5}{9}+\frac{1}{9}=\frac{\square}{\square}$ $\frac{4}{9}+\frac{5}{9}=\frac{\square}{\square}$ $\frac{5}{12}+\frac{3}{12}=\frac{\square}{\square}$ $\frac{5}{10}+\frac{3}{10}=\frac{\square}{\square}$ $\frac{5}{14}+\frac{7}{14}=\frac{\square}{\square}$ $\frac{13}{47}+\frac{23}{47}=\frac{\square}{\square}$

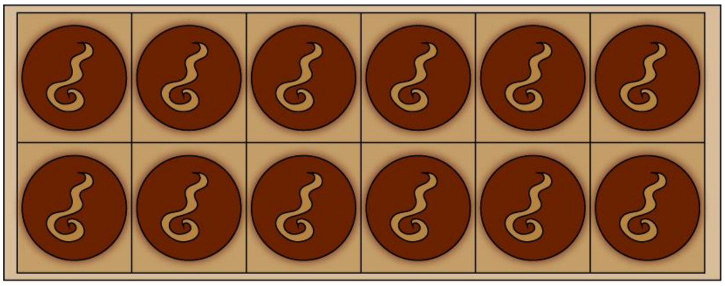
$$\frac{1}{8} + \frac{1}{8} + \frac{3}{8} = \frac{\square}{\square}$$

$$\frac{1}{8} + \frac{3}{8} + \frac{1}{8} = \frac{\square}{\square}$$

$$\frac{3}{16} + \frac{\square}{16} = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

Word problem:

'I was given a box of chocolates as a gift. I ate $\frac{1}{6}$ *of the chocolates myself and gave* $\frac{3}{6}$ *of them to my family.'*



- *'What fraction has been eaten?'*
- *'What fraction is left?'*
- *'How many chocolates are left?'*

Dòng nǎo jīn:

- *'What is the largest possible numerator that can be used to complete this comparison statement?'*

$$\frac{11}{20} + \frac{\square}{20} < \frac{17}{20}$$

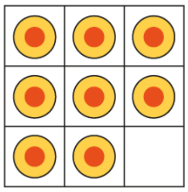
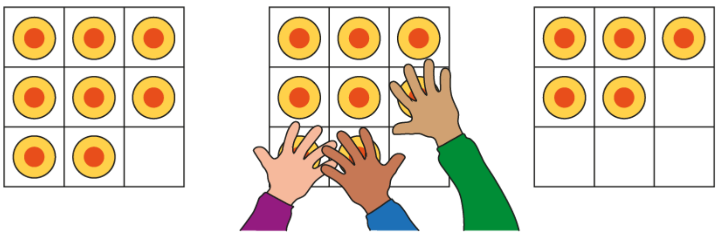
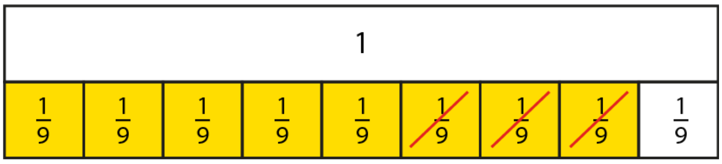
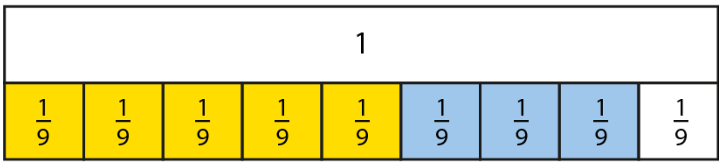
- *'How many different ways can you complete this equation?'*

$$\frac{\square}{9} + \frac{\square}{9} = \frac{1}{9} + \frac{2}{9} + \frac{3}{9} + \frac{2}{9} + \frac{1}{9}$$

Teaching point 2:

When subtracting fractions with the same denominators, just subtract the numerators.

Steps in learning

	Guidance	Representations
2:1	*Teaching point 2* focuses on subtracting fractions. Use a fraction story with images to introduce this point. *'There is $\frac{8}{9}$ of a full box of biscuits. Each biscuit is $\frac{1}{9}$ of a box. If three biscuits are taken, what fraction of the box remains?'* This story uses the *'First…, then…, now…'* structure that children were introduced to in *Spine 1: Number, Addition and Subtraction*, segment *1.6:* • **First** *'What fraction of the box is there to start with?'* (There is eight-ninths of a full box.) • **Then** *'What fraction of the box is taken?'* (Three-ninths of the box is taken.) • **Now** *'What fraction of the box remains?'* (There is five-ninths of the box left.) Invite children to use a variety of strategies to convince you that there are five-ninths left in the box. You might want to remind them of the methods they have used with fractions: • representing it on a diagram • representing it on a number line • using stem sentences to support verbal reasoning. Some example images, which may be useful to stimulate and support discussions, are shown opposite. The children may well suggest other options as well. For example, they may look at the biscuit image and choose to adapt a stem sentence from segment	*'Look at the image. There is $\frac{8}{9}$ of a full box of biscuits. Each biscuit is $\frac{1}{9}$ of a box. If three biscuits are taken, what fraction of the box remains?'*  Method 1 – using a diagram:   or 

3.2 Unit fractions: identifying, representing and comparing: *'The whole is divided into nine equal parts. We have five of the parts, so there is five-ninths of a box of biscuits.'*

When using the stem sentences, continue to use dual language. For example:

- *'Eight one-ninths minus three one-ninths is equal to five one-ninths.'*

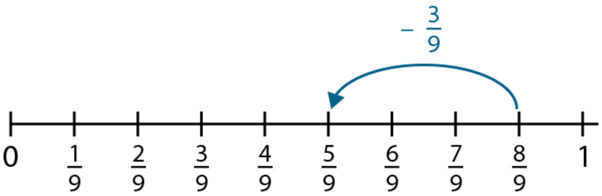
and

- *'Eight-ninths minus three-ninths is equal to five-ninths.'*

Use the stem sentence below and adapt it to suit your chosen examples, such as in the exemplar opposite.

- ***'$\frac{\square}{\square}$ is ___ lots of $\frac{\square}{\square}$'***
- ***'$\frac{\square}{\square}$ is ___ lots of $\frac{\square}{\square}$'***
- ***'I know that ___ – ___ = ___'***
- ***'…so, I know that $\frac{\square}{\square} - \frac{\square}{\square} = \frac{\square}{\square}$'***

Method 2 – using a number line:



Method 3 – verbal reasoning:

- *'$\frac{8}{9}$ is eight lots of $\frac{1}{9}$.'*
- *'$\frac{2}{9}$ is two lots of $\frac{1}{9}$.'*
- *'I know that* $8 - 2 = 6$.'
- *'…so, I know that* $\frac{8}{9} - \frac{2}{9} = \frac{6}{9}$.'

2:2 The next step is to write the equation. Read the biscuit story again and model the construction of the equation.

	Say	**Write**
First	*'There is eight-ninths of a full box…'*	$\frac{8}{9}$
Then	*'Three-ninths of the box is taken…'*	$-\frac{3}{9}$
Now	*'There is five-ninths of a box left.'*	$=\frac{5}{9}$

Summarise, again using dual language:

- *'Eight one-ninths minus three one-ninths is five one-ninths.'*

and

- *'Eight-ninths minus three-ninths is five-ninths.'*

Give children the opportunity to practise reading the equation aloud for themselves.

Focus on the written equation and, as before, discuss what is the same and what is different about the three terms. The denominator hasn't changed because the number of parts the box is divided into hasn't changed. We are still working in the same unit and our unit is *ninths*. We have simply removed some of the biscuits from the box, which has caused the numerators to change.

$\frac{8}{9} - \frac{3}{9} = \frac{5}{9}$

2:3

Now use a linear model to illustrate another fraction story, such as the beetle example opposite.

Confirm with the children that for this scenario, the unit we are working in is tenths. As with previous examples, ask children to repeat the now-familiar methods, and explain their solutions. They may wish to draw their own representation and employ their knowledge of the composition of non-unit fractions. Some of the models they might draw are shown opposite.

Some children may already be using the stem sentences but display the examples opposite for all children to consider. Provide plenty of opportunity to say the stem sentences as a class, so that all children become fluent in them.

When using the stem sentences, continue to summarise using the dual language:

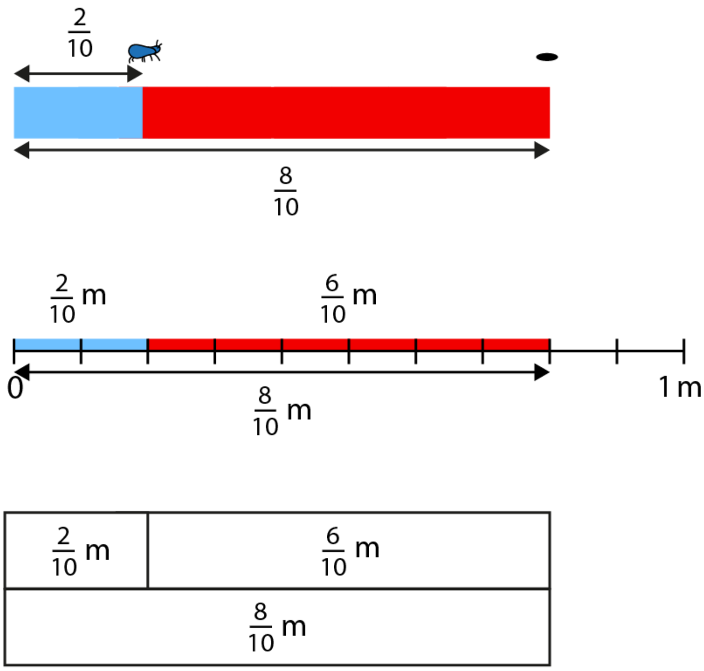
- *'Eight one-tenths minus two one-tenths is equal to six one-tenths.'*

and

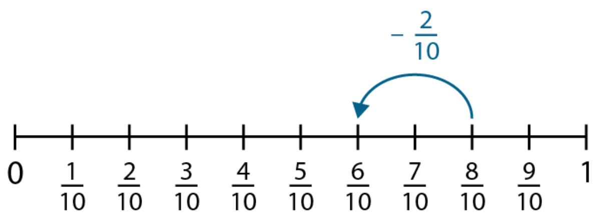
- *'Eight-tenths minus two-tenths is equal to six-tenths.'*

'A beetle was $\frac{8}{10}$ m away from its hole. It walked $\frac{2}{10}$ m towards its hole. How much further is it to the beetle's hole?'

Method 1 – using a diagram:



Method 2 – using a number line:

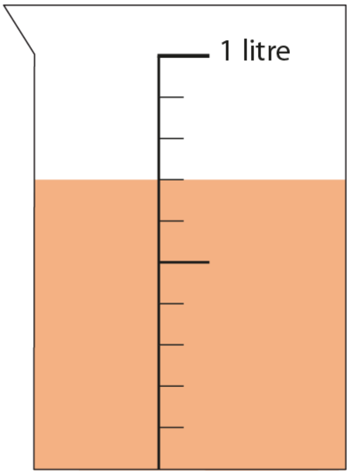


Method 3 – verbal reasoning:

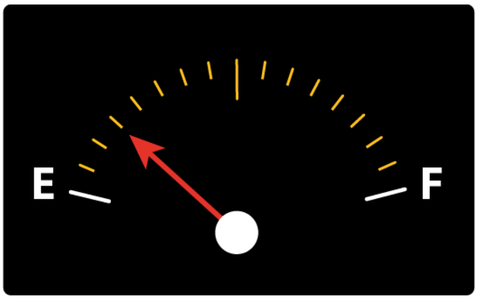
- *'$\frac{8}{10}$ is eight lots of $\frac{1}{10}$.'*
- *'$\frac{2}{10}$ is two lots of $\frac{1}{10}$.'*
- *'I know that 8 – 2 = 6.'*

'…so, I know that $\frac{8}{10} - \frac{2}{10} = \frac{6}{10}$.'

2:4	Challenge the children to write an equation to express the beetle story. Ask them to consider the following: • *'What does the* $\frac{8}{10}$ *represent?'* • *'What does the* $\frac{2}{10}$ *represent?'* • *'What does the* $\frac{6}{10}$ *represent?'* Again, observe what is happening with the numerators and denominators. We are working in the unit of tenths, so the denominator is always ten.	$\frac{8}{10} - \frac{2}{10} = \frac{6}{10}$ or $\frac{8}{10}m - \frac{2}{10}m = \frac{6}{10}m$
2:5	Display both equations opposite and ask children to describe their observations. They should now be in a position to understand and apply the generalisation that: ***'When subtracting fractions with the same denominators just subtract the numerators.'***	$\frac{8}{9} - \frac{3}{9} = \frac{5}{9}$ $\frac{8}{10} - \frac{2}{10} = \frac{6}{10}$
2:6	Consider some equations and explore common misconceptions through a true/false style problem. Children may offer a variety of explanations for their answers, but ensure that amongst these there is plenty of opportunity for all children to apply the generalisation, for example: *'This is false because **when the denominators are the same, you just subtract the numerators**, but here both the denominator and the numerator have been subtracted.'* Encourage children to describe why an equation is true, as well as why some are false. They may apply previous learning and use the format of the stem sentences to prove an equation is true.	*'True or false?'* | | **True (✓) or false (×)?** | |---|---| | $\frac{7}{12} - \frac{2}{12} = \frac{5}{12}$ | | | $\frac{4}{7} - \frac{2}{7} = \frac{2}{0}$ | | | $\frac{8}{10} - \frac{2}{10} - \frac{1}{10} = \frac{3}{10}$ | | | $\frac{7}{9} - \frac{7}{9} = 0$ | | | $\frac{5}{8} - \frac{2}{8} - \frac{2}{8} = \frac{1}{8}$ | |

2:7	Provide varied practice to consolidate the learning in this teaching point, including: • missing-number problems • word problems, with measurement contexts, where applicable.	Missing-number problems: *'Fill in the missing numbers.'* $\frac{6}{8} - \frac{3}{8} = \frac{\square}{\square}$ $\frac{6}{8} - \frac{2}{8} = \frac{\square}{\square}$ $\frac{14}{15} - \frac{3}{15} = \frac{\square}{\square}$ $\frac{9}{11} - \frac{6}{11} = \frac{\square}{\square}$ $\frac{8}{14} - \frac{8}{14} = \frac{\square}{\square}$ $\frac{9}{10} - 0 = \frac{\square}{\square}$ $\frac{15}{20} - \frac{2}{20} - \frac{2}{20} = \frac{\square}{\square}$ $\frac{17}{24} - \frac{7}{24} - \frac{9}{24} = \frac{\square}{\square}$ $\frac{5}{9} - \frac{2}{9} - \frac{3}{9} = \frac{\square}{\square}$ Word problems: *'Sofia had a jug with $\frac{7}{10}$ litre of juice. She drank $\frac{3}{10}$ litre. How much does she have left?'* 

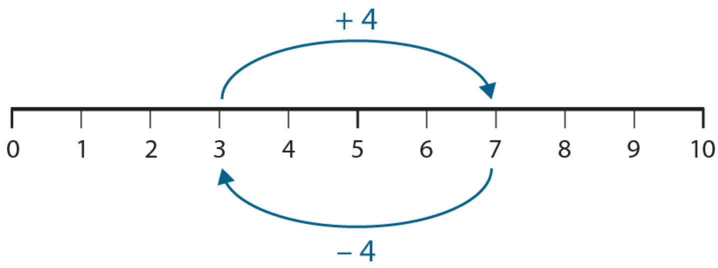
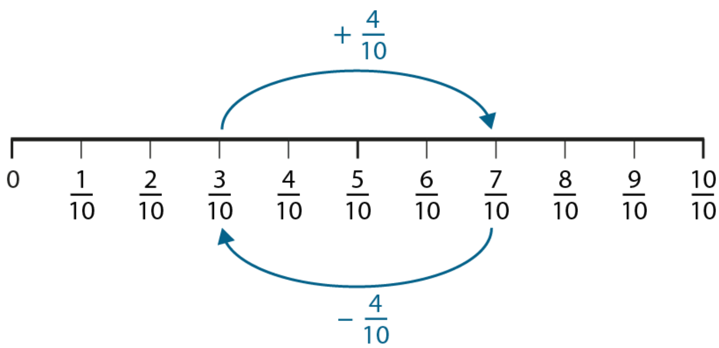
2:8	To promote and assess depth of understanding, use dòng nǎo jīn problems and mixed addition and subtraction problems.	Dòng nǎo jīn:
		• *'What fractions could be placed in this equation to make it correct?'*
		$\frac{2}{8} = \frac{7}{8} - \frac{\square}{8} - \frac{\square}{8}$
		  • *'How many ways are there to complete it?'*
		
		• *'Find all the solutions to this equation.'*
		$\frac{\square}{12} - \frac{2}{12} + \frac{\square}{12} = \frac{7}{12}$
		  • *'Explore this question further by changing the given fractions.'*
		  • *'What do you notice?'*
		
		• *'How many different numbers could be used to complete this comparison statement?'*
		$\frac{3}{12} < \frac{8}{12} - \frac{\square}{12}$
		
		Mixed addition and subtraction calculations:
		'Fill in the missing numbers.'
		$\frac{17}{25} - \frac{5}{25} + \frac{8}{25} = \frac{\square}{\square}$
		$\frac{4}{17} + \frac{9}{17} - \frac{6}{17} = \frac{\square}{\square}$
		$\frac{16}{25} + \frac{5}{25} - \frac{\square}{\square} = \frac{13}{25}$
		$\frac{6}{14} - \frac{\square}{\square} + \frac{5}{14} = \frac{7}{14}$
		$\frac{\square}{\square} + \frac{5}{13} - \frac{8}{13} = \frac{5}{13}$

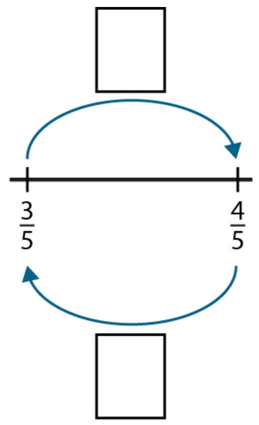
		Word problems: *'Mira's car had $\frac{3}{16}$ of a tank of fuel. She topped it up with $\frac{12}{16}$ of a tank, then went on a journey which used $\frac{10}{16}$ of a tank. What fraction of a tank was left by the end of the journey?'* 

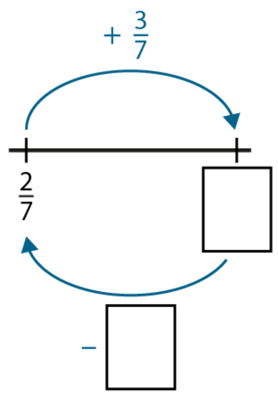
Teaching point 3:

Addition and subtraction of fractions are the inverse of each other, just as they are for whole numbers.

Steps in learning

	Guidance	Representations
3:1	This teaching point links back to *Spine 1: Number, Addition and Subtraction* segment *1:6, Teaching point 4,* which explores addition and subtraction as inverse operations. Remind children of this, using whole-number examples first. In the first example opposite, the number line is used to demonstrate how $3 + 4 = 7$ and $7 - 4 = 3$ are linked. Similarly, the second number line example shows how $\frac{3}{10} + \frac{4}{10} = \frac{7}{10}$ and $\frac{7}{10} - \frac{4}{10} = \frac{3}{10}$ are linked. Using examples like the ones shown opposite, encourage children to discuss what they notice. Ask: • *'What is the same?'* • *'What is different?'* Children could then write for themselves pairs of equations that represent the movements shown on the number lines. Remind children that we say that addition and subtraction are the *inverse* of each other. This applies to addition and subtraction of fractions in the same way it applies to whole number calculations.	• *'What is the same?'* • *'What is different?'*   $3 + 4 = 7$ $7 - 4 = 3$ $\frac{3}{10} + \frac{4}{10} = \frac{7}{10}$ $\frac{7}{10} - \frac{4}{10} = \frac{3}{10}$

<table>
<tr>
<td>3:2</td>
<td>Look at further examples of inverse calculations presented on a number line. Again, ask the children to identify the equations being shown in the two number line examples opposite.</td>
<td>


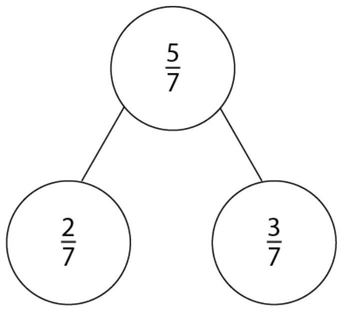


$\frac{3}{5}+\frac{1}{5}=\frac{4}{5}$ $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$
$\frac{4}{5}-\frac{1}{5}=\frac{3}{5}$ $\frac{5}{7}-\frac{3}{7}=\frac{2}{7}$
</td>
</tr>
<tr>
<td>3:3</td>
<td>Provide some more examples using different representations.

Demonstrate how the bar model can be used to show both addition and subtraction calculations in the context of fractions. The children should be very familiar with this from their previous work on whole number composition and calculation. Examine each of the written equations in turn. Discuss each step in each equation in relation to the parts and whole of the bar model.

Next, show how equations can also be represented as a part–part–whole model. Do children notice how this is similar to the approach used in the bar model?</td>
<td>
Bar model:


$\frac{2}{8}+\frac{4}{8}=\frac{6}{8}$ $\frac{6}{8}=\frac{2}{8}+\frac{4}{8}$
$\frac{4}{8}+\frac{2}{8}=\frac{6}{8}$ $\frac{6}{8}=\frac{4}{8}+\frac{2}{8}$
$\frac{6}{8}-\frac{4}{8}=\frac{2}{8}$ $\frac{2}{8}=\frac{6}{8}-\frac{4}{8}$
$\frac{6}{8}-\frac{2}{8}=\frac{4}{8}$ $\frac{4}{8}=\frac{6}{8}-\frac{2}{8}$

Part–part–whole model:


$\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$ $\frac{5}{7}=\frac{2}{7}+\frac{3}{7}$
$\frac{3}{7}+\frac{2}{7}=\frac{5}{7}$ $\frac{5}{7}=\frac{3}{7}+\frac{2}{7}$
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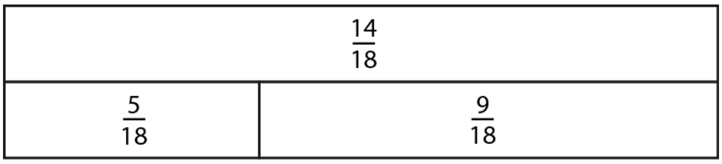
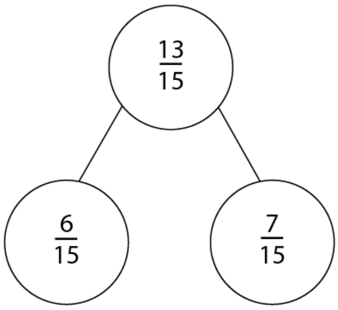
		$\frac{5}{7}-\frac{2}{7}=\frac{3}{7}$ $\frac{3}{7}=\frac{5}{7}-\frac{2}{7}$ $\frac{5}{7}-\frac{3}{7}=\frac{2}{7}$ $\frac{2}{7}=\frac{5}{7}-\frac{3}{7}$
3:4	Give plenty of practice with the concepts covered in this teaching point, including: • missing-number problems • writing equations. In the pairs of linked equations, notice the variation in the position of the '=' sign. Ensure children can still identify what they need to do in order to find the missing number. If they need scaffolding to support them in understanding the relationships represented by these equations, encourage them to draw a bar model or part–part–whole model. Once children have had plenty of opportunity to practise missing number questions, move on to writing equations, setting the following challenge: *'What are the eight additive equations (either addition or subtraction) that are represented in the bar and part–part–whole models? Can you write all eight equations?'*	Missing-number problems: *'Fill in the missing numbers.'* $\frac{4}{11}+\frac{\square}{\square}=\frac{10}{11}$ $\frac{3}{8}+\frac{\square}{\square}=\frac{7}{8}$ $\frac{10}{11}-\frac{\square}{\square}=\frac{4}{11}$ $\frac{\square}{\square}-\frac{\square}{\square}=\frac{3}{8}$ $\frac{15}{16}=\frac{\square}{\square}+\frac{7}{16}$ $\frac{\square}{\square}+\frac{9}{25}=\frac{23}{25}$ $\frac{15}{16}-\frac{\square}{\square}=\frac{7}{16}$ $\frac{23}{25}-\frac{\square}{\square}=\frac{9}{25}$ $\frac{\square}{\square}=\frac{5}{14}+\frac{8}{14}$ $\frac{\square}{\square}=\frac{6}{12}+\frac{3}{12}$ $\frac{\square}{\square}=\frac{13}{14}-\frac{8}{14}$ $\frac{6}{12}=\frac{\square}{\square}-\frac{3}{12}$ $\frac{6}{9}=\frac{2}{9}+\frac{\square}{\square}$ $\frac{\square}{\square}=\frac{7}{20}+\frac{3}{20}$ $\frac{2}{9}=\frac{6}{9}-\frac{\square}{\square}$ $\frac{7}{20}=\frac{\square}{\square}-\frac{3}{20}$

		Writing equations: *'What are the eight additive equations (either addition or subtraction) that are represented in the bar and part–part–whole diagrams? Can you write all eight equations?'*  
3:5	To further deepen understanding of this concept, present dòng nǎo jīn problems like the ones shown opposite.	Dòng nǎo jīn: *'Fill in the missing symbols (+ – or =). Complete each statement in two different ways. You may use each symbol more than once.'* $\frac{10}{15}$ ◯ $\frac{4}{15}$ ◯ $\frac{6}{15}$ $\frac{10}{15}$ ◯ $\frac{4}{15}$ ◯ $\frac{6}{15}$

Teaching point 4:

To subtract from one whole, first convert the whole to a fraction where the denominator and numerator are the same.

Steps in learning

	Guidance	Representations
4:1	In segment *3.3 Non-unit fractions: identifying, representing and comparing*, children learnt that if the numerator is equal to the denominator, then the fraction is equivalent to one. They wrote equations such as $1=\frac{8}{8}$ and $\frac{7}{7}=\frac{4}{4}$. This knowledge is now applied to addition and subtraction of fractions. Follow these steps:	

- You may wish to introduce this through another story, such as the one given below. Through discussion and reference to the models provided, establish that there are three pieces of watermelon left, or three-eighths of a watermelon.
- Challenge children to represent the watermelon story on a part–part–whole model. One of the parts should show the fraction of the watermelon that is eaten, the other part the fraction of the watermelon that is left.
- Display both part–part–whole models below on the board. Discuss them in comparison to the models the children have drawn.
- Next, ask children to write corresponding equations for each of the part–part–whole models. They should write:
  - $1-\frac{5}{8}=\frac{3}{8}$
  - $\frac{8}{8}-\frac{5}{8}=\frac{3}{8}$

Use the *'First…, then…, now…'* structure to read out the equations, pointing to each element as you read it out:

	Say	Write
First	*'First we had one watermelon…'*	1
Then	*'…then we ate five-eighths of the watermelon.'*	$-\frac{5}{8}$
Now	*'Now there is three-eighths of the watermelon left.'*	$=\frac{3}{8}$

Repeat, this time presenting it as:

	Say	Write
First	*'First we had eight-eighths of a watermelon…'*	1
Then	*'…then we ate five-eighths of the watermelon.'*	$-\frac{5}{8}$
Now	*'Now there is three-eighths of the watermelon left.'*	$=\frac{3}{8}$

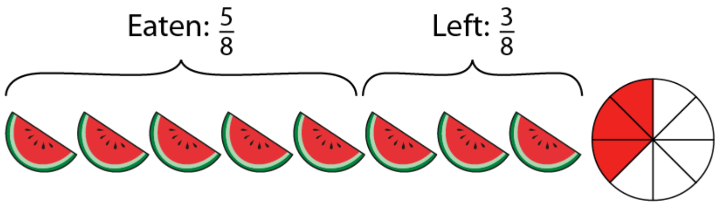
Look at both equations and discuss what each number in the equation represents. Ensure children are comfortable with the fact that both these equations tell the same story, and that the whole watermelon can either be represented by '1' (one watermelon) or by '$\frac{8}{8}$' (eight-eighths of the watermelon).

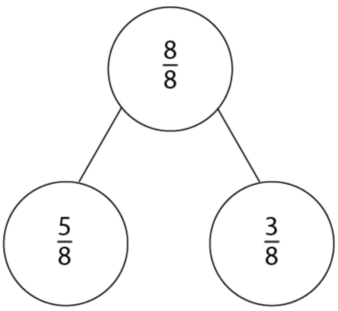
Refer back to the generalisation from segment *3.3* and remind children that: ***'When the numerator and denominator are the same, the fraction is equivalent to one whole.'***

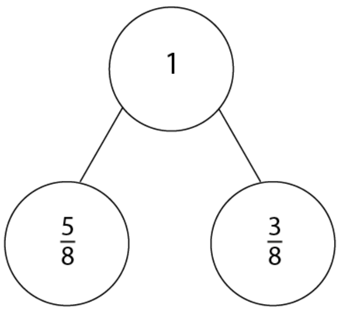
'A watermelon is cut into 8 equal pieces.'

'$\frac{5}{8}$ of the watermelon is eaten.'

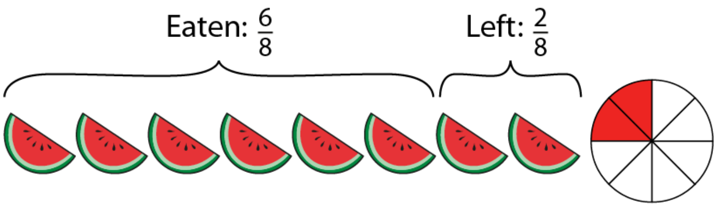
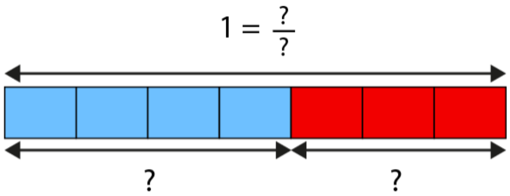
'What fraction of the watermelon is left?'







$\frac{8}{8}-\frac{5}{8}=\frac{3}{8}$ $\quad$ $1-\frac{5}{8}=\frac{3}{8}$

4:2	Look again at the watermelon problem in step *4:1*, and ask the children to consider what the two versions of the equations would be if another piece of melon was eaten. Display and discuss the representations opposite. Ask children to write the matching equations. They should write: • $1-\frac{6}{8}=\frac{2}{8}$ • $\frac{8}{8}-\frac{6}{8}=\frac{2}{8}$	*'A watermelon is cut into 8 equal pieces.'* *'$\frac{6}{8}$ of the watermelon is eaten.'* *'What fraction of the watermelon is left?'* 
4:3	Check that the children are becoming fluent in switching between writing the minuend as '1' and as a fraction where the numerator is the same as the denominator. They should recall from the previous segment that any fraction whose numerator and denominator are the same, has a value of one. Whatever the denominator of the subtrahend is, is both the numerator and denominator in the minuend. Use the bar model opposite to verify children's understanding so far. Do they recognise what each of the question marks represents? Can they write equations to show fractions being subtracted from one *and* from seven-sevenths? Provide some varied practice with examples similar to those opposite.	 • *'$1-\frac{4}{7}=\frac{3}{7}$ can also be written as $\frac{7}{7}-\frac{4}{7}=\frac{3}{7}$'* • *'$1-\frac{3}{7}=\frac{4}{7}$ can also be written as $\frac{7}{7}-\frac{3}{7}=\frac{4}{7}$'* • *'$\frac{4}{7}+\frac{3}{7}=1$ can also be written as $\frac{4}{7}+\frac{3}{7}=\frac{7}{7}$'* • *'$\frac{3}{7}+\frac{4}{7}=1$ can also be written as $\frac{3}{7}+\frac{4}{7}=\frac{7}{7}$'*
4:4	Provide questions that allow children to practise subtracting from '1'. This practice should be delivered through a variety of real-life situations and a range of models (linear, area and quantity), as well as calculations presented without a context. For calculations such as $1-\frac{5}{9}$, some children may prefer to rewrite this as $\frac{9}{9}-\frac{5}{9}$, while others may already be comfortable with thinking of one as	Subtracting from 1 whole: $1-\frac{5}{9}=$ $1-\frac{4}{9}=$ $1-\frac{4}{11}$ $1-\frac{8}{11}$ Linear model: *'If $\frac{2}{9}$ of the number line is red, what fraction is blue?'* 

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<td>nine-ninths and performing this step mentally. Use true/false statements and missing number problems to assess children's understanding. This should offer an opportunity to identity and address any lingering misconceptions.</td>
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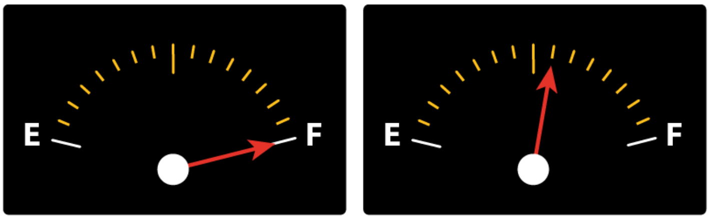
Word problems:

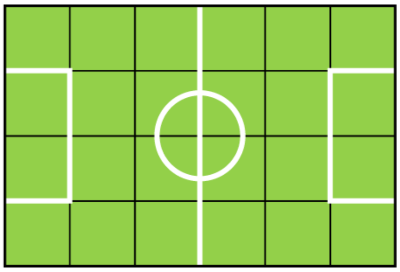
- *'A car set off with a full tank of fuel. At the end of the journey, there was* $\frac{9}{16}$ *left. What fraction of the whole tank of fuel was used?'*



- *'Jamie has read* $\frac{4}{10}$ *of his book. What fraction of the book does he have left to read?'*

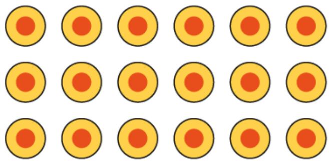
Area model:

'$\frac{7}{24}$ *of a football pitch has been mown. What fraction of the pitch still needs mowing?'*



Quantity model:

- *'*$\frac{11}{18}$ *of a packet of biscuits has been eaten.'*



  - *'What fraction of the packet is left?'*
  - *'How many biscuits were in the packet at the start?'*
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True/false style problems:

'True or false?'

	True (✓) or false (×)?
$1-\frac{5}{12}=\frac{7}{12}$	
$1-\frac{3}{8}=\frac{3}{8}-\frac{8}{8}$	
$1-\frac{3}{7}-\frac{2}{7}=\frac{5}{7}$	
$1-\frac{6}{6}=0$	

Missing number problems:

'Fill in the missing numbers.'

$1-\frac{\square}{\square}=\frac{5}{14}$ $\qquad$ $1-\frac{\square}{\square}-\frac{2}{8}=0$

4:5

Finally, consider how one whole can be written in different ways, noting that some of these ways can actually make a problem appear more difficult than it really is. The aim here is to enable children to confidently and flexibly move between different forms of writing one whole.

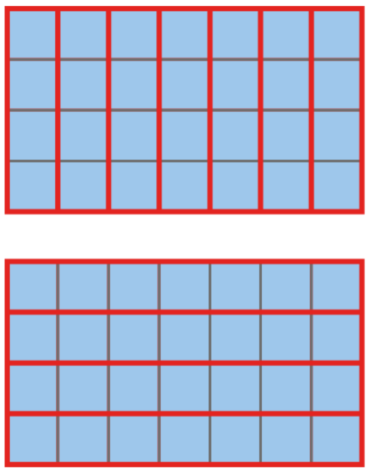
Present an example, such as two rectangles divided into unequal parts and ask:

- *'What is the same?'*
- *'What is different?'*

Elicit that the whole rectangles are the same; they both have 28 small squares. They are different because one rectangle has been divided into sevenths and the other has been divided into quarters. (If using the example opposite, focus in on the red sections.)

This means that we can state that $\frac{28}{28}=\frac{7}{7}=\frac{4}{4}=1$

- *'What is the same?'*
- *'What is different?'*



	Discuss this with the children. Do they understand why this statement is true? If there is any doubt, pull the equation apart and get them to confirm that: • $\frac{28}{28}=1$ • $\frac{7}{7}=1$ • $\frac{4}{4}=1$ And so, it is also true to say that: $\frac{28}{28}=\frac{7}{7}=\frac{4}{4}=1$ Present children with an equation where the denominators are different, e.g. $\frac{7}{7}-\frac{3}{4}$. They will need to recognise seven-sevenths as equivalent to 1. Calculations like this can look really confusing at first glance, so model aloud how you might approach this, for example: • *'What can I do here? I've learnt that we have to have the same denominator to subtract fractions, but these have <u>different denominators</u>. I need to think about this in a different way. The first number is interesting ($\frac{7}{7}$). The numerator and denominator are the same, so I know that it is equal to 1. I will rewrite the equation using 1 instead of $\frac{7}{7}$ and see if that helps me…'* (Write: $1-\frac{3}{4}=$) • *'I can see what to do now. I know that there are four-quarters in one whole, and I am subtracting three-quarters, so there must be one-quarter left.'* Now look at a written chain of equations. Breaking it down like this will help children understand the thought process behind the calculation. Present a different equation, e.g. $\frac{11}{11}-\frac{11}{12}$	Breaking down equations with one whole and different denominators: Example 1 $\frac{7}{7}-\frac{3}{4}=$ $1-\frac{3}{4}=$ $1-\frac{3}{4}=\frac{1}{4}$ Example 2 $\frac{11}{11}-\frac{11}{12}=$ $1-\frac{11}{12}=$ $1-\frac{11}{12}=\frac{1}{12}$

	where children need to think of eleven-elevenths as one in order to subtract eleven-twelfths from one. Again, model your thinking aloud. Write the chain of equations shown opposite on the board so that children can see a visual summary of the thought process. Children haven't yet been introduced to common denominators but even when they are, these calculations are much easier to solve without converting to a common denominator. Provide further examples with different denominators, including some with the same numerators.	