11 Geometry

Mastery Professional Development

11.4 3D shapes

Guidance document | Key Stage 4

Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The fifth of the Key Stage 4 themes (the eleventh of the themes in the suite of Secondary Mastery Materials) is *Geometry*, which covers the following interconnected core concepts:

11.1 Transformations and relative position

11.2 Reasoning with the properties of a circle

11.3 Trigonometry

11.4 3D shapes

This guidance document breaks down core concept 11.4 3D shapes into two statements of **knowledge, skills and understanding**:

11.4 3D shapes

- 11.4.1 Representations and surface area of 3D shapes
- 11.4.2 Volume of 3D shapes

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

- 11.4.1 Representations and surface area of 3D shapes
  - 11.4.1.1 Understand that 3D shapes can be represented 2 dimensionally (plans and elevations)
  - 11.4.1.2 Understand that 3D shapes can be represented 2 dimensionally (nets)
  - 11.4.1.3 Use 2D representations to quantify the surface area of prisms, cylinders, pyramids and composite shapes
  - 11.4.1.4 Solve problems involving the surface area of cones
  - 11.4.1.5 Solve problems involving the surface area of spheres
- 11.4.2 Volume of 3D shapes
  - 11.4.2.1 Understand, analyse and quantify the volume of prisms and cylinders
  - 11.4.2.2 Solve problems involving the volume of pyramids and cones
  - 11.4.2.3 Solve problems involving the volume of spheres
  - 11.4.2.4 Solve problems involving the volume of composite 3D shapes

Overview

This final core concept brings together and builds on students' understanding of area and volume, to promote a deep, connected understanding of how to quantify the surface area and volume of three-dimensional (3D) shapes. It extends beyond a reliance on formulae recall to a recognition of how calculations are derived from geometrical structures, based on mathematical reasoning.

Opportunities should be provided for students to identify 3D shapes from scale drawings of the plan and elevation views, as well as to construct them for a given shape. Visualising 3D shapes, and interpreting various two dimensional (2D) representations of them, is a key skill that is relevant not only for mathematics, but also for other areas of the curriculum such as design technology, geography and art. Regardless of their chosen career path or future study, students will regularly encounter plans and elevations in everyday life, whether house-hunting, navigating or making purchasing decisions. It is important that students recognise that drawing and interpreting 2D representations of 3D shapes is not a standalone topic from the mathematics classroom, but a vital skill that they will use to throughout their lives.

In the mathematics curriculum, 2D representations of 3D shapes are essential to support students to unpick the properties of 3D shapes and deduce information about them. Nets, for example, are a helpful representation for conceptualising and calculating surface area. Students will have been building up a set of formulae for finding the area of shapes, starting with rectangles, triangles and parallelograms in Key Stage 2, and including trapezia and circles from Key Stage 3. This repertoire of formulae can now be applied to find the surface area of 3D shapes. Particular attention should be paid to identifying where lengths correspond on nets so that, for example, students can understand why the circumference of a cylinder's circular face is equal to the length of the rectangular face that wraps around it.

Using formulae to find the volume of a shape, introduced at Key Stage 2 and built on at Key Stage 3, will so far have involved the volume of cuboids, prisms and cylinders. At Key Stage 4, students' recognition that the volume of a prism can be quantified by multiplying the area of the cross section by the length is extended to a consideration of the volume of pyramids, cones, spheres and composite shapes. It can be tempting to reduce the concept of volume to a series of formulae to be memorised and applied to the relevant shapes. However, this (perhaps counterintuitively) can result in the topic feeling bigger and more unmanageable to students. Understanding how the formulae for the volumes of 3D shapes have been derived supports students to see the connections between the different properties of 3D shapes, and helps to promote a deep and connected understanding of the procedures used to find their volumes. Recognising when a standard formula can be applied, and when a more tailored approach is needed, plays an important part in the development of students' understanding of mathematical structure.

The calculations required for surface area and volume can require students to draw on knowledge from across the mathematics curriculum – including Pythagoras' theorem, proportional reasoning and algebraic manipulation. Many of the problems that they will encounter at Key Stage 4 will be set in context. Students need to be aware that the surface area or volume is not necessarily the 'final answer'; they might need to use the values obtained in further calculations, such as to find the rate or density, or make a decision about other contextual factors.

Prior learning

Students explore a variety of 3D shapes from their earliest childhood experiences. From the start of their time studying mathematics in school, they will have worked with shapes including prisms, pyramids, cones and spheres. They should be able name the shapes, describe their properties, and recognise them in different orientations when represented in two dimensions. At Key Stage 3, students should have applied and consolidated their knowledge of the properties of 3D shapes, using information about faces, surfaces, edges and vertices to solve problems in three dimensions. Given the range of experiences students will bring, there is a possibility that misconceptions have been accrued. Teachers should ensure that students are able to identify precisely the particular properties of each shape.

Students' understanding of nets as 2D representations of 3D shapes, and their ability to identify and create nets for common 3D shapes, began at Key Stage 2. Their understanding should have been maintained at Key Stage 3, so that they are able to utilise their understanding of nets to support calculation of the surface area. They may already have been introduced to the idea of plans and elevations as an alternative way to represent a 3D shape in 2D, but for some this might represent new learning at Key Stage 4. Exploration of these different views helps to develop a deeper understanding of the properties and structure of 3D shapes, and provides support for solving problems involving surface area and volume.

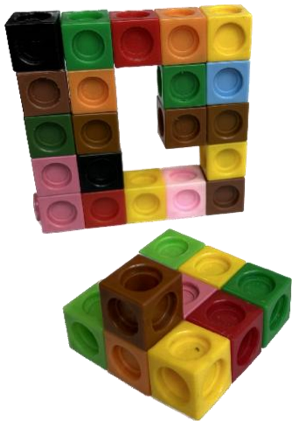
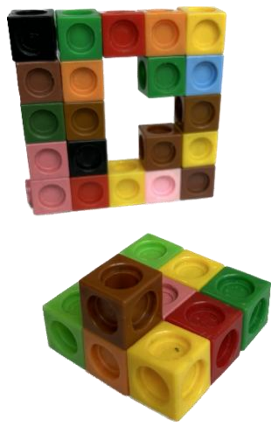
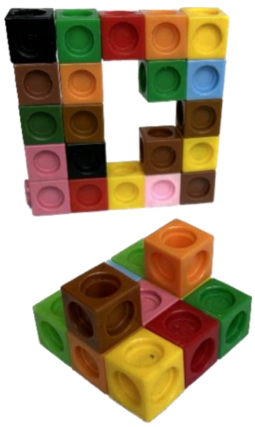
Working on the surface area of 3D shapes at Key Stage 4 develops and builds on the key formulae learnt previously. It is important to check that students recognise and can derive and apply the formulae for the area of a rectangle, parallelogram and triangle from Key Stage 2, and a trapezium and circle from Key Stage 3.

Students will have explored the volume of cubes and cuboids in Key Stage 2 and extended this understanding to include prisms and cylinders at Key Stage 3. Teachers should check students' grasp of the constant cross-sectional area property of prisms and cylinders, to ensure that students are aware of the difference between prisms/cylinders and pyramids/cones, before learning how to derive the formulae for their respective volumes.

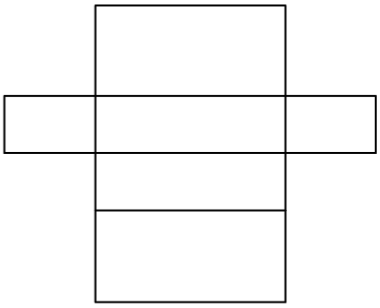
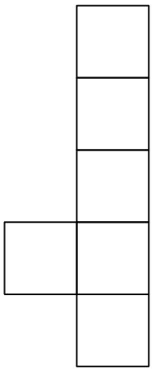
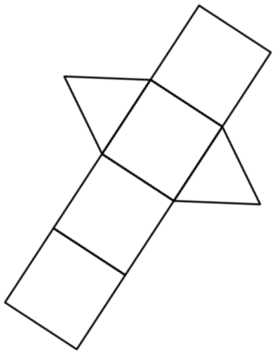
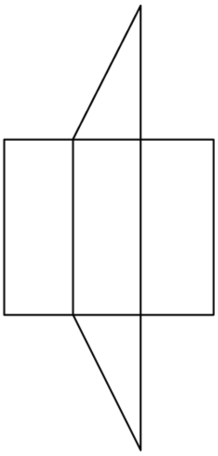
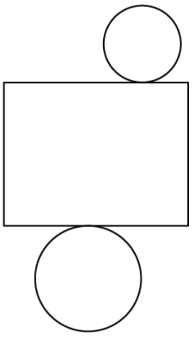
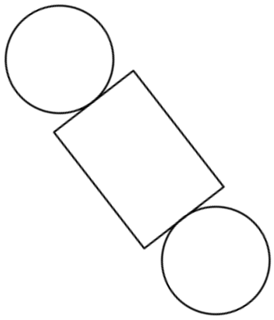
The core concept documents *1.4 Simplifying and manipulating expressions, equations and formulae*, *3.1 Understanding multiplicative relationships*, *3.2 Trigonometry*, *6.1 Geometrical properties* and *6.2 Perimeter, area and volume* from the Key Stage 3 PD materials all explore the prior knowledge required for this core concept in more depth.

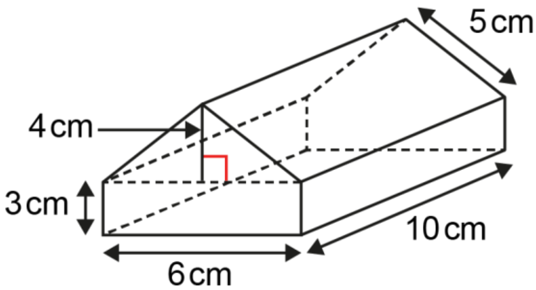
Checking prior learning

The following activities from the NCETM secondary assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

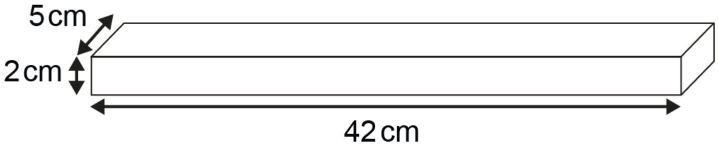
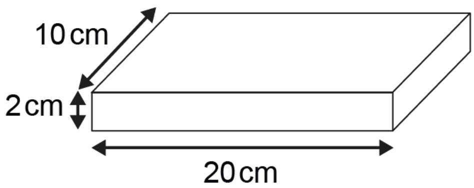
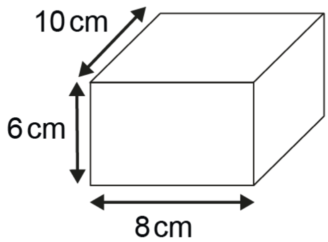
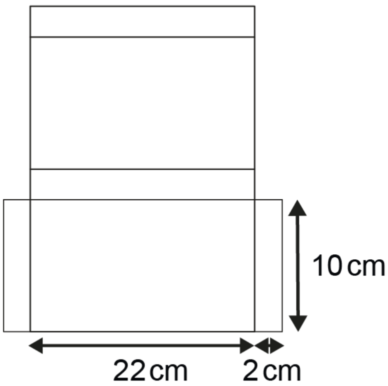
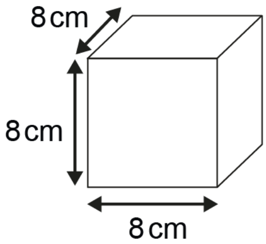
Reference	Activity
Checkpoints '*Perimeter, area and volume 2*', 'Checkpoint 10: Hole in the wall'	a) Will the block fit through the hole in the wall in each of these situations? (i)  (ii)  (iii)  (iv)  (v)  b) Design a different shape that would fit through each hole.

<table>
<tr>
<td>Checkpoints ‘Perimeter, area and volume 2’, ‘Checkpoint 13: Wrong ‘uns’</td>
<td>
a) What is wrong with these nets? How could you correct them?

(i) Cuboid

(ii) Cube

(iii) Triangular prism

(iv) Triangular prism

(v) Cylinder

(vi) Cylinder


b) Draw a different correct net for each 3D shape.
</td>
</tr>
<tr>
<td>Key Stage 3 PD materials document ‘6.2 Perimeter, area and volume’, Key idea 6.2.2.5, Example 6</td>
<td>


John thinks the surface area of the prism is:
Surface area = $6 \times 3 \times 4 \times 10 = 720$ cm^2
Do you agree with John? Explain your answer.
</td>
</tr>
</table>

Checkpoints '*Perimeter, area and volume 2*', '*Checkpoint 25: Marshall's parcel*'	Marshall is sending a parcel and needs a box with a minimum volume of 420 cm^3. a) Which of these boxes could he use? A  B  C  D  E  The item in Marshall's parcel is a book 21 cm long. b) Does this information change your answer to part a? c) Sketch some other boxes that Marshall could use.

Key vocabulary

Key terms used in Key Stage 3 materials

Key Stage 3

- cylinder
- prism
- surface area

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
cone	A cone is a 3D shape consisting of a circular base, a vertex in a different plane, and line segments joining all the points on the circle to the vertex. If the vertex A lies directly above the centre O of the base, then the axis of the cone AO is perpendicular to the base, and the shape is a right circular cone.
cross section	In geometry, a section in which the plane that cuts a figure is at right angles to an axis of the figure. For example, in a cube, a square is revealed when a plane cuts at right angles to a face.
elevation	A 2D diagram of a 3D object, usually the views from the side. The front elevation and side elevation are both horizontal mappings of the 3D object onto a 2D plane, taken from positions at right angles to each other.
plan	A 2D diagram of a 3D object, usually the view from directly above. The plan view is a vertical mapping of the 3D object onto a 2D plane.
pyramid	A solid with a polygon as the base and one other vertex, the apex, in another plane. Each vertex of the base is joined to the apex by an edge. Other faces are triangles that meet at the apex. Pyramids are named according to the base: a triangular pyramid (which is also called a tetrahedron, having four faces), a square pyramid, a pentagonal pyramid, etc. If the vertex A lies directly above the centroid O of the base, then the axis of the pyramid AO is perpendicular to the base, and the shape is a right pyramid.

Knowledge, skills and understanding

Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a 🔍. These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

11.4.1 Representations and surface area of 3D shapes

Visualising the surfaces that make up a 3D shape, and recognising that different views result in different images, underpins students' fluency with 3D shapes. While a net shows what a 3D shape looks like when opened out and laid flat, different projections of the 3D shape (such as the plan and elevation views) map its key points onto a 2D plane. This can provide insight into the structure of a shape and support students in making connections between two and three dimensions.

At Key Stage 4, students should use the different types of 2D representations interchangeably. It is important that they can identify all faces when quantifying surface area, recognising that it is the sum of all the faces and not just the visible ones. Students' experience of calculating the surface area of cubes and cuboids at Key Stage 3 is developed at Key Stage 4 with further exploration of prisms and cylinders. It is important for students to recognise that there is no standard formula for calculating the surface area of a prism. Identifying that prisms can have more than six faces is key to understanding that not all prisms can be quantified by calculating $2(lw + wh + lh)$, where l is the length, w is the width and h is the height. This formula may have been introduced if students calculated the surface area of a cuboid at Key Stage 3.

The introduction of the calculation of the surface area of pyramids at Key Stage 4 builds on students' grasp of the need to sum the areas of all the faces to establish the surface area. Students should recognise that, if the base of a right pyramid is a regular polygon, each of the triangular faces will be identical; and so the total area of the triangular faces can be calculated by finding the area of one face and multiplying by the number of triangular faces. This is key to developing their understanding of mathematical structure when working with 3D shapes.

With the introduction of problems involving the surface area of cones and spheres at Key Stage 4, it is important that students can identify the relationship between the formula for the surface area of a cone with the calculation for the area of a sector of a circle and understand how the formula is derived. Supporting students to make connections between different formulae, rather than viewing them as unconnected processes to memorise, is fundamental to deepening their understanding of representations and surface area of 3D shapes.

- 🔍 11.4.1.1 Understand that 3D shapes can be represented two dimensionally (plans and elevations)
- 11.4.1.2 Understand that 3D shapes can be represented two dimensionally (nets)
- 🔍 11.4.1.3 Use 2D representations to quantify the surface area of prisms, cylinders, pyramids and composite shapes
- 11.4.1.4 Solve problems involving the surface area of cones
- 11.4.1.5 Solve problems involving the surface area of spheres

11.4.2 Volume of 3D shapes

At Key Stage 3, students calculate the volume of cubes and cuboids. When finding the volume of other prisms, they learn to identify and visualise the shape of the constant cross section. At Key Stage 4, when exploring the volume of 3D shapes, it is important to emphasise and contrast the common structure of *area of cross section* × *length* for prisms and cylinders with the common structure of $\frac{1}{3}$ × *area of base* × *height* for pyramids and cones. Draw attention to how these formulae relate to the physical properties of the shapes (i.e. that the cross section is congruent along the length for prisms and cylinders, whereas the cross section is similar but not congruent along the length for pyramids and cones). Students should recognise that, when a prism and pyramid have the same base and height, the volume of the pyramid is one-third of the volume of the prism and, similarly, that three cones have the same volume as a cylinder with the same diameter and height. This is key to developing their understanding of mathematical structure and the way in which the volumes of different 3D shapes relate to one another.

When calculating the volume of composite shapes at Key Stage 4, students have an opportunity to divide a shape into its component parts and identify how the volumes of the different portions can be calculated and then summed to give the total volume. It is important that they recognise the different ways in which a composite shape can be split into less complex shapes to allow them to apply geometric properties and formulae that they are already familiar with. A comparison of different approaches encourages students to consider the most efficient way to determine the volume of a complex shape, while providing opportunities for them to develop their reasoning skills when working with 3D shapes.

11.4.2.1 Understand, analyse and quantify the volume of prisms and cylinders

11.4.2.2 Solve problems involving the volume of pyramids and cones

11.4.2.3 Solve problems involving the volume of spheres

11.4.2.4 Solve problems involving the volume of composite 3D shapes

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

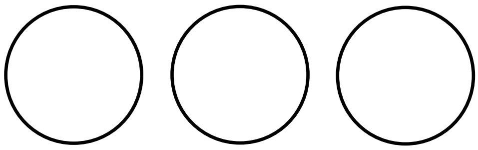
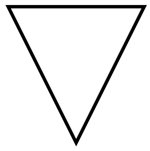
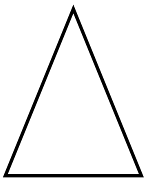
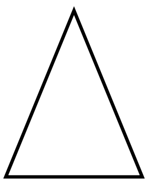
Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

11.4.1.1 Understand that 3D shapes can be represented two dimensionally (plans and elevations)

Common difficulties and misconceptions
Students often struggle to visualise the 3D shape from a 2D representations of it; starting with a 3D shape and representing it in two dimensions can be more accessible for students. Having a physical object to manipulate and view from different angles is essential for students to build practical experience of the different views from different angles. Shadow can be a powerful supportive visual: shining a light onto the board and then placing a 3D shape directly in front of it, results in a 2D shadow projected onto the board. Changing the orientation of the shape allows students to explore plans and elevations in a simple and accessible way. It can also support them to appreciate why slanted faces appear different to the actual face shape when viewed from the side (for example, slanted rectangular faces have reduced height, and equilateral triangles appear isosceles). This can be demonstrated by looking at an equilateral triangle 'face on' and gradually turning it away from the students so that the height reduces. It is important to also provide opportunities that require starting with a 2D representation and using it to determine properties of the 3D shape it represents. This helps with the development of visualisation skills. Recognising that the view from above (the plan view) of a 3D shape can look quite different to what might be expected, is key to understanding a shape's structure and the relationship between 2D and 3D representations. Important links can be made with other representations that rely on plan views, such as maps. Exploring familiar landmarks using interactive map software, freely available online, can help students to appreciate how plan views are constructed. This might need to be revisited regularly to help students build confidence, but does not need to be time consuming or onerous. Working with maps and plans offers the opportunity for potential cross-curricular links, for example with geography, art or design technology, and is also a quick and worthwhile activity in tutor time.

Students need to	Guidance, discussion points and prompts
Understand that different 3D shapes can have the same 2D representation, depending upon the viewpoint *Example 1:* *Below are the plan views of a cylinder, cone and a sphere.* *Shape A Shape B Shape C*  *a) Which shape is which? How do you know?* *Shape A is a cone, shape B is a cylinder and shape C is a sphere.* *b) What will be the same and what will be different about each of their side elevations?* *The diameter of the circle in the plan view is 1.8 cm. All three shapes are the same height.* *c) Sketch the side elevations of each shape, with dimensions included.*	The **variation** in *Example 1* draws attention to the similarities between 2D representations of different 3D shapes. When 3D objects are captured from a single viewpoint, it can be impossible to differentiate between them. By identifying what is shared – that when viewed from above, all three of these shapes will be represented by their largest circular cross section – students can focus on what is not shared – the side elevations. Commonly, a plan-view **representation** of a cone will have a central dot to denote the apex. This has not been included here, as the ambiguity of the three plan views is a key learning point. Teachers could discuss that, ordinarily, they might expect to see a dot on shape A. Alternatively, there is no reason to suppose that the cone is positioned conventionally: if it were balanced on its point, the side view in B would be orientated as on the right.  The final part of this example could be extended through teacher questioning, to help with further **deepening** students' understanding of the properties of 3D shapes. For example, teachers could ask: • *'Which of these three shapes would have the greatest volume? How do you know?'* • *'Which of these three shapes would have the smallest surface area? How might you prove this?'* • *'If the shapes were cut horizontally, what would be the same and different about the cross sections? How would this change as the position of the cut changed?'*
Example 2: *Below is the front elevation of a 3D shape.*  *Niamh says this could be a cone, pyramid or prism.* *a) Is she correct? Why or why not?* *The side elevation of the shape is now shown below.* 	In *Example* 2, students are again focusing on the shared properties of 2D **representations** of 3D shapes. This time, it is the elevations that are the same, so students must imagine the other possible shapes as seen from different viewpoints. They could be encouraged to consider other groups of 3D shapes that would have the same plan or side view – such as a square-based pyramid and cuboid. The **language** of 'prism' is deliberately vague to promote discussion. The term 'prism' is used throughout the full age range of the mathematics curriculum. Consequently, students can come to define it in subtly different ways that are not always explored fully. A key feature of a prism is that it is a solid bounded by two congruent, parallel polygons (the bases). These, therefore, also form a constant cross section throughout the shape, and any prism is named according to the shape of its bases. This shape would only be a prism if the elevation shown was of one of the bases. Students can be prompted to improve on Niamh's conjecture by more accurately defining the possible shape as a 'triangular prism'. It is important also

b) *What shape or shapes have now been ruled out? How do you know?*

c) *What would be the plan views of the remaining possible shapes?*

that students recognise and can explain why a cylinder is not a prism and a cone is not a pyramid.

Example 3:

Cara is a caterer. She serves slices of cakes at an event. She can cut the cakes vertically (so the knife is perpendicular to the plate) or horizontally (so the knife is parallel to the plate).

Below are the different types of slices that she can cut:

- *Identically-sized rectangles*
- *Identically-sized triangles*
- *rectangles of different sizes*
- *squares of different sizes*
- *triangles of different sizes.*

a) *Use the table below to sort the different types of slice shapes according to the whole cakes that they can be cut from:*

Shape of whole cake	***Possible slice shapes***
Triangular prism	
Cuboid (with no square faces)	
Cube	
Cuboid (with two square faces)	

b) *For any cake slices that you could not sort, what shape would the original cake need to be?*

c) *What shapes could Cara cut from a cylindrical cake?*

Example 3 uses a real-life context to consider the different ways that prisms can be cut, and therefore the different 2D shapes that are inherent in their structure. The **language** used to clarify the direction of the cut is essential, to prevent misconceptions forming.

Although this example intentionally uses vertical and horizontal cuts to ensure the learning can be linked directly back to plans and elevations, there is an opportunity for **deepening** students' understanding by considering the planes formed by different cuts – such as different angles and diagonals. Connections can then be made to students' work on applying Pythagoras' theorem and trigonometry in 3D. If students have previously explored the different planes and 2D shapes that it is possible to form in 3D shapes, then there is less 'new' content to be thinking about when calculating missing lengths and angles for the first time.

This example is based on additional activity H 'Slices of cake', from the Checkpoints deck *Perimeter, area and volume 2* (available to download on the NCETM website). Checkpoints have a dual purpose: they are designed to be used as classroom assessment activities, and as professional development tools, with a focus on the prior knowledge needed to access the Key Stage 3 curriculum. A valuable use of departmental time could be to explore *Example 3* alongside the Checkpoint. Discussion points to use with colleagues include:

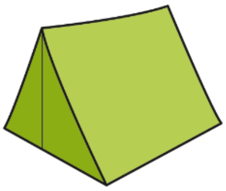
- *'What is the same and what is different about the two different tasks?'*
- *'How does* Example 3 *extend and deepen students' thinking?'*
- *'How might the representations from the Checkpoint be used to support students with this example? How would this change the pitch of the task?'*
- *'What properties of prisms is this task contingent on? What might teachers need to do to ensure students have fully understood these salient features?'*
- *'What misconceptions about prisms might this task uncover? What misconceptions might this task not uncover?'*
- *'How would this task need to be adapted to consider pyramids instead?'*

Connect plans and elevations with the corresponding 3D shape

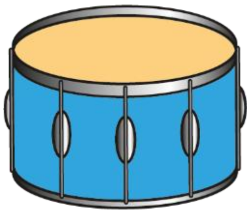
Example 4:

Below are images of four different 3D objects.

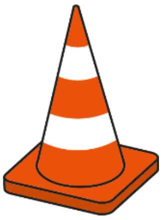
A Tent



B Snare drum



C Traffic cone



D Pyramid of Giza

The table below shows the plan, front and side elevations for the objects above, but some of the representations are missing:

Object	***Plan***	***Front***	***Side***
			
			
			
			

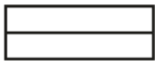
Determine which object is being represented in each row and complete the missing views.

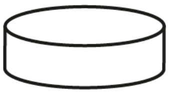
In *Example 4*, students explore 2D representations of real-life 3D objects and consider what they look like from different viewpoints. Discuss students' thought processes when determining which of the four 3D objects is being represented, to gauge their understanding and support them to develop their reasoning skills. The **variation** between the rows of the table and the information provided should encourage students to think deeply about the similarities and differences in the front and side elevations. This should lead them to identifying the corresponding 3D objects and complete the plan views.

When completing the blank cells of the table, emphasise that 2D **representations** of 3D shapes need to be consistent in terms of scale. For example, the diameter of the circle in the plan view of the drum should be approximately equal to the width of the rectangle for the front/side views. Students should also recognise that the front view determines the orientation of the plan view:

- As the front view of the tent is the triangular face, the plan view should be orientated vertically, as shown right.

- If the front view of the tent was looking at the rectangular face, then the plan view would be represented with a horizontal orientation, as shown right.





For the purposes of this example, exclude details such as tension rods and lugs at the side of the snare, as shown to the left.

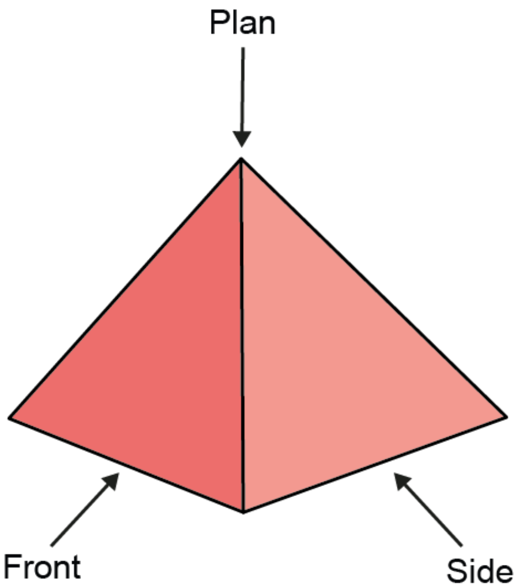
Distinguishing between the front and side views of the drum, and the rectangular front and side elevations, is key to **deepening** students' understanding of plans and elevations. In this example, the front views of the traffic cone, drum and pyramid are the same as each object's respective side view. This helps students to recognise that, for some 3D shapes, one elevation may not provide additional information about the shape, but gives important insight into the shape's structure.

When discussing plans and elevations, emphasise the correct use of **language**, such as vertex, face and apex, as well as classifying the shapes as being a prism, cylinder, cone or pyramid.

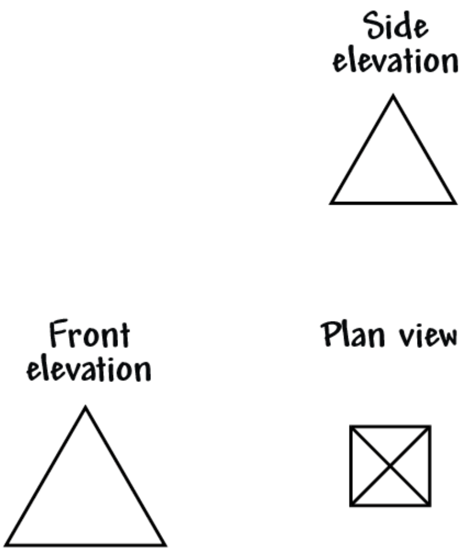
The objects in *Example 4* are recognisable but would not be found in the classroom. This emphasises the mental visualisation of 3D objects, rather than physical manipulation. Discuss alternatives that could be used instead, such as a tin can, rather than a snare drum. When might it be appropriate to have the 3D objects available as scaffolding and how would this change the focus of the task?

Example 5:

Mrs Preece shows her class a square-based pyramid, as shown below. She asks them to construct the elevations marked:



Tilda's work is shown below:



What feedback should Mrs Preece give?

In *Example 5*, students are provided with a **representation** of a 3D shape and asked to consider the plans and elevations that a fictional student has drawn. There is a narrow focus to this task, exposing a common error that often features in students' own work: incorrect dimensions on the plans and elevations. Students can be so focused on identifying the 2D shapes for each viewpoint, that they neglect the detail of which lengths need to correspond with each other. Here, students need to identify not only that the front and side elevations should be identically sized, but that the lengths of the square should be the same as the bases of the triangles. Using light horizontal/vertical construction lines to connect corresponding vertices can help them to conceptualise this.

There are no lengths given in this diagram, but we know from the rubric that the shape is a square-based pyramid. Discuss with students what might be different if this specific **language** had not been used, and it was just described as a shape. Could we still assume that the triangles in the front and side views are the same? Which lengths of the rectangle would need to correspond with which bases?

How often do you use tasks like this, where there is a particular misconception being exposed? Discuss with colleagues how you might use this in your curriculum. Would you want to plan it in so that all students experience it as part of their learning? Or would you want teachers to have access to it, to use if students make this error in class?

Use 2D representations to identify the structure of 3D shapes

Example 6:

Below this example are the nets of five 3D shapes, labelled A to E. The length of every edge in these shapes is 10 cm.

For A, C and E, the square face should be considered as the base. For B and D, the topmost triangular face should be considered the base.

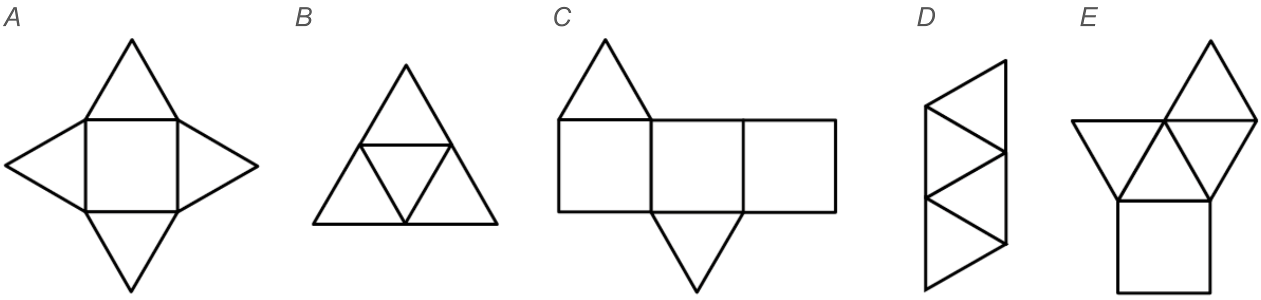
Penny is going to draw accurately the side, front and plan views of each shape.

a) How many different shapes will she draw?

b) How many times will she draw the same shape?

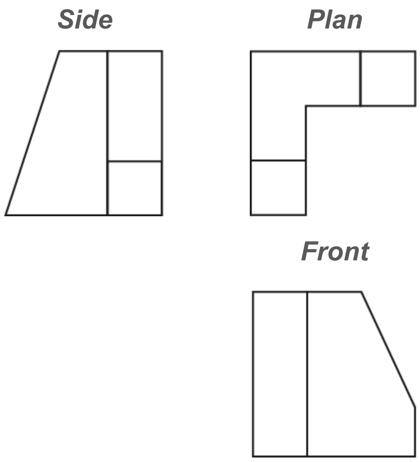
In *Example 6*, students work between different 2D **representations**, strengthening their understanding of how they link to the corresponding 3D shapes. Students may not immediately recognise that there are only three different shapes in the example, as two pairs of nets correspond with each other.

The **variation** in this example is such that there are only two differently-sized faces used throughout: a square and equilateral triangle, each with side lengths of 10 cm. This constant, which students identify in part a, reduces the number of possible shapes for them to use in part b. This allows students to focus on which of the shapes will be visible in different orientations, supporting them to understand more deeply the geometrical properties involved.



Example 7:

The side, plan and front elevations of a 3D shape are given in the table below.

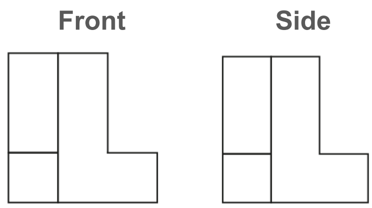


Below are some 3D shapes labelled A, B and C.

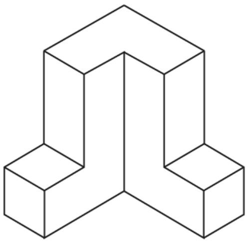
In *Example 7*, students are presented with three 3D shapes that do not correspond with the images given, and asked to explain why they are incorrect. Providing students with non-examples, and asking them to reason about them, focuses their attention on structure and supports them with **deepening** their thinking about the properties of the shape.

The three shapes, A, B and C provide a good starting point for visualising what the 3D shape looks like. **Variation** is used in such a way that each one satisfies one or two of the plan, front or side elevations, but not all three. Once students have identified the ways in which shapes A, B and C do not satisfy the plan, front and/or side elevations given, they should be encouraged to sketch the corresponding views, given below:

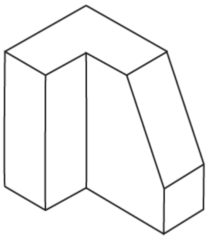
A



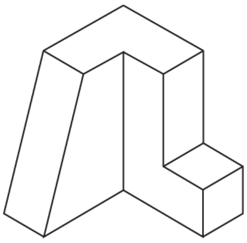
A



B



C

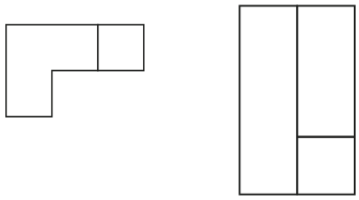


a) *Explain why none of shapes A to C corresponds to the 3D shape represented by the elevations shown in the table.*

b) *Sketch the 3D shape that does correspond to the views in the table.*

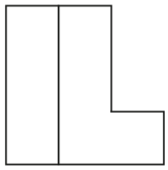
Plan **Side**

B

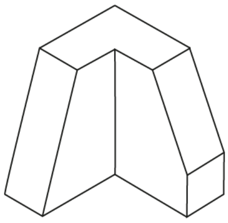


Front

C



Once the correct shape (shown to the right) has been identified, check that students can identify the different 2D shapes that can be seen within the 3D shape, and are able to use the correct **language** when referring to squares, rectangles, trapezia and pentagons.

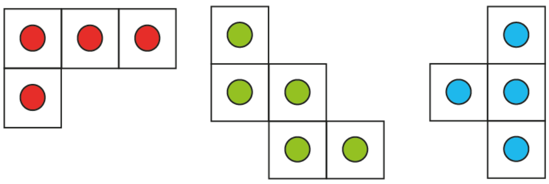


Providing students with non-examples can provide scaffolding and support for students who are struggling to get started. Discuss what other ways teachers might scaffold activities that involve identifying a 3D shape from the plan, front and side elevations.

Example 8:

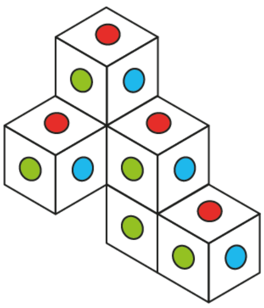
The plan, front and side elevations of a 3D shape made out of connected cubes are given in the table below.

Plan ***Front*** ***Side***



Using the 2D representations given, determine how many cubes the 3D shape has.

In *Example 8*, students are given the plan, front and side elevations to visualise the corresponding 3D object and determine the number of cubes used to create it. Students may assume that the 3D shape is made up of five cubes, as the front elevation consists of five squares and in the **representation** of the shape shown below, for example, only five of the six cubes are visible.



To establish that the total number of cubes is six requires students to consider the structure of the shape and how the cubes join together, **deepening**

	their understanding of what is and is not visible in different views of a shape. Discuss with your colleagues your approaches to this task. Do you all start with the same view? If students are struggling to visualise the 3D shape, which view might provide the best starting point? If they consider the plan/front/side view initially, what 3D object will they identify, and how does this need to be adapted to achieve the correct 3D shape?

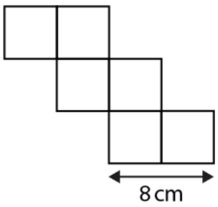
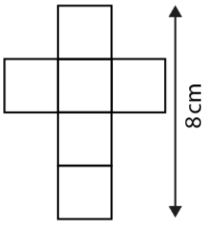
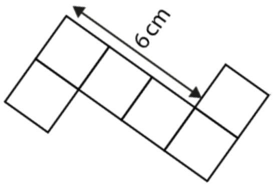
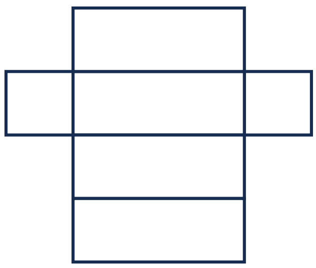
11.4.1.3 Use 2D representations to quantify the surface area of prisms, cylinders, pyramids and composite shapes

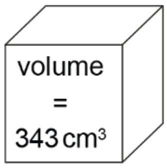
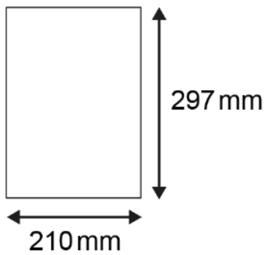
Common difficulties and misconceptions

Having explored the surface area of cubes and cuboids at Key Stage 3, students often assume that the same process (i.e., summing the areas of three pairs of faces) is universal for all prisms. They need to recognise that a general formula cannot be applied when quantifying the surface area of prisms, as the calculation is determined by the number and shape of the faces. To find its surface area, students need a deep understanding of the structure of a given prism, and to be able to connect this understanding with the properties of the shapes that it comprises. The two congruent 'base' faces of a prism have the same area, and the number and structure of the rectangular faces is dependent on the properties of these polygons. Students therefore need to understand how to identify the lengths of the rectangular sides by matching the corresponding lengths on the polygonal bases. This may require the application of other geometrical concepts, such as Pythagoras' theorem.

Students can have difficulty visualising the number of surfaces a prism has, which can result in them making mistakes in surface area calculations. Make links with nets and plans and elevations and provide worked solutions that are incorrect for students to critique. This can focus attention on the structure of a prism and encourage deeper thinking about the surfaces involved.

Even when all the information is provided, finding the surface area can be a lengthy process, and it is important to work with students to generate good habits for recording and checking their work, so that no steps are missed. Annotating and highlighting diagrams can help, as can labelling both the calculation and the relevant shape, to keep track of which faces they have calculated. Encourage students to identify commonalities between the rectangular faces, to promote an appreciation of structure and an efficient calculation approach.

Students need to	Guidance, discussion points and prompts
Use the net of a shape to calculate the surface area *Example 1:* a) *Order the cubes A to C from smallest to greatest surface area.* *A:*  *B:*  *C:*  b) *Sketch the net of a cube with a surface area of:* *(i)* *64 cm^2* *(ii)* *36 cm^2* *(iii)* *24 cm^2*	In order to calculate accurately the surface area of a 3D shape, its net is usually the most helpful **representation**. Teachers should encourage students to annotate lengths and areas, perhaps by highlighting which faces have been included in their calculations, to ensure that no faces are missed. *Example 1* encourages students to engage with three different nets. Rather than simply finding the area of each face, they need to consider how the given lengths relate to the individual faces and then find the surface areas. The intention is that students will be **deepening** their understanding of the structures involved as they identify that two of these cubes have the same surface area. The **variation** in part b is such that the answers include a cube number and a square number, to expose some common assumptions and misconceptions. It is likely that students may confuse the volume and surface area in sub-part (i), and draw a net of a cube with length 4 cm. Instead, they need to divide 64 by six (resulting in a non-integer area of each face: $10\frac{2}{3}$ cm^2). The relevant side length is therefore the square root of this area. Similarly, students may be tempted to aim for an integer answer by square rooting 36 first in sub-part (ii), and then dividing by six, rather than the other way around. It is only in sub-part (iii) that integer answers are achieved, as 24 is a multiple of six and the corresponding factor is square.
Example 2: *The net below is of a cuboid formed from two square and four rectangular faces. It has a surface area of 96 cm^2.*  a) *Suggest dimensions for the cuboid. Is there more than one possible answer?* b) *How would your answers to part a change if none of the faces of the cuboid was square?*	In *Example 2*, students could provide infinite possible answers, so teachers should be alert to students' reasoning and ensure that it is sound. Open questions such as this can support with **deepening** students' understanding, as they need to consider what is the same and what is different about other valid answers. Pay attention to the **language** that students use when explaining their answers. Are they able to work in general terms, rather than just stating the steps for the particular values they have found? If students have found valid lengths for the square faces, then they can double the resultant areas and subtract them from 96. Dividing this remaining area by four, and then by the length of their square, should result in a valid length for the cuboid.

Example 3: *The net of a triangular prism is shown below. All its lengths are integers.* *The triangular faces are isosceles, and each has a base of 6 cm and a height of 4 cm.*  *What is the maximum length of the prism, if the surface area is less than 200 cm²?*	*Example 3* offers another opportunity for **deepening** students' understanding of surface area, requiring them to connect their learning to other previously learnt geometrical concepts such as Pythagoras' theorem. The information that the triangular faces are isosceles should lead students to deduce that they can each be split into two right-angled triangles of height 4 cm and base 3 cm – which forms a Pythagorean triple, therefore giving a hypotenuse of 5 cm. Students should then be able to associate the relevant lengths of the triangle with the appropriate lengths of the rectangular sides. Algebraic **representations** could be employed to find the maximum integer length – labelling the length of the prism as x gives areas of $5x$, $5x$ and $6x$ for the three rectangular faces. Summing these three areas and adding them to the two known areas of 12 cm^2 for the triangular faces results in an expression that can be made into an inequality with 200 cm^2. Forming this equation helps to expose the structure of the surface area of a prism, supporting students to generalise and work with other prisms comprised of different shaped faces.
Appreciate the difference between surface area and volume, and work between them *Example 4:* *A cube has a volume of 343 cm^3.*  a) *Calculate the surface area of the cube.* *An A4 piece of card measures 297 mm by 210 mm.*  b) *Determine whether or not a net of the cube will fit on the piece of card.*	In *Example 4*, students are given the volume of a cube and asked to use this to determine its surface area. Students often confuse volume and surface area when working with 3D shapes. Working with both can help with **deepening** their understanding of how volume and surface area relate to one another. Teachers can assess if students recognise that the volume of a cube is a^3, where a is the length of a side, and understand how to use the structure of the volume calculation to determine a. Once the side length of the cube has been identified, students need to use it to calculate the surface area of one face and then multiply by six to give the total surface area. Students may assume that, because the surface area of the cube is less than the surface area of the A4 card (623.7 cm^3), the net will fit on the piece of card. It is important that they instead consider the structure of the net of a cube to determine that the entire **representation** will indeed fit. While an exploration of all 11 possible cube nets may not be necessary, students need to recognise that any arrangement of the faces that is more three squares wide (3×7) and four squares long (4×7) will not fit. The structure of the calculation for the volume of a cube allows for the length of the side of the cube to be determined when the volume is known. Discuss with your team what additional information would be needed alongside the volume to be able to calculate the surface area of other prisms. How could this be used to develop students' understanding of the structure of surface area calculations for prisms?

Understand the relationship between the plan, front and side elevations and the surface area of a prism

Example 5:

Taj and Quinn are making wooden door stops and door wedges to sell. They plan to varnish the door stops and wedges, so need to work out how many tins of varnish to buy.

Below this example are images of their two different designs, alongside their plan, front and side elevations of each.

Taj does the following calculation for the surface area of the cuboid-shaped door stop:

$$2(12 \times 9) + 2(12 \times 15) + 2(9 \times 15)$$
$$= 216 + 360 + 270$$
$$= 846\ cm^2$$

a) *Is Taj's calculation for the surface area of the door stop correct? Explain how you know.*

Quinn does the following calculation for the surface area of the triangular prism-shaped door wedge.

$$2\left(\frac{1}{2} \times 8 \times 6\right) + 3(10 \times 6)$$
$$= 48 + 180$$
$$= 228\ cm^2$$

Quinn's calculation for the surface area of the door wedge is not correct.

b) *Explain what Quinn has done wrong.*

c) *Show that the correct surface area is 192 cm^2.*

Each tin of varnish covers a surface area of 0.5 m^2.

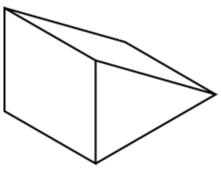
d) *How many door stops and/or wedges could be varnished per tin?*

Students often have difficulty visualising how many surfaces a prism has. In *Example 5*, exploring the plan view and front and side elevations, alongside other images of the shapes, can help to develop their visualisation skills and **deepening** their understanding of the structure of 3D shapes. Ask students to explain the dimensions of the rectangle that forms the side elevation in Taj's workings, to check that they can accurately transfer dimensions when the face in question has not been explicitly labelled.

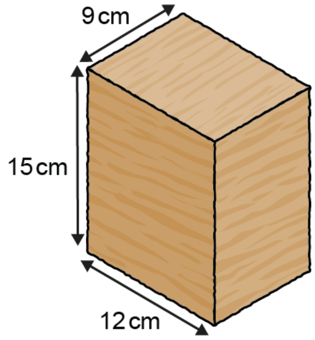
Quinn has correctly identified that the triangular prism consists of five faces, but has assumed that the three rectangular faces all have the same dimensions, possibly because the plan view and side elevations are both rectangles measuring 10 cm by 6 cm. Asking students to identify the view from the back of the prism might be helpful, as well as discussing why the plan view and side elevation are the same rectangle but in different orientations. Discussing other ways of calculating the surface area of the triangular prism, for example using a net, can support students in making connections between different **representations** of 3D shapes.

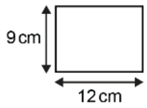
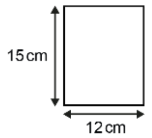
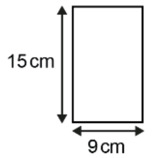
The context offers an additional layer of mathematical **language**, linking back to work on division and upper/lower bounds. When discussing Quinn's calculation, ask students how over-estimating the surface area would impact Quinn's conclusions about the number of door wedges that can be varnished per tin. Is it better for the mistake to result in a larger than actual or smaller than actual surface area?

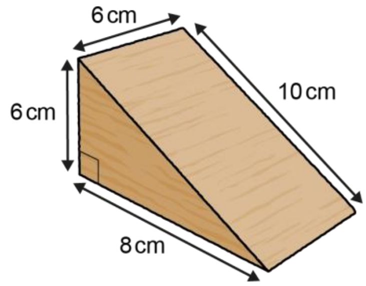
The orientation of 2D representations of prisms is typically such that the base of the prism is horizontal. In this example, the base is not horizontal and only two of the five prism faces are visible. The plan view and front and side elevations are identifiable from the visible faces, without the need for visualisation. Discuss possible **variation** in the orientation of the triangular prism. How might presenting the prism in a different orientation affect students' identification of the dimensions of the five faces. For example, if the triangular prism was presented so that three faces were visible, like the one to the right, what calculation for the surface area are students likely to do? What mistakes or wrong assumptions might they make when identifying the dimensions of the faces of the prism?

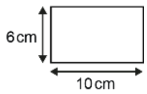
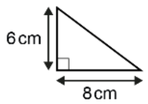
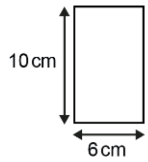


This example uses a worked solution to expose an incorrect assumption and highlight the importance of carefully identifying the dimensions of all the faces. Telling students that a solution is incorrect, and asking them to show that a solution has a given value, is a powerful way of making a task accessible. It can help to increase motivation to have a go at a task, which they may otherwise struggle to get started on. Discuss with your team how often they use deliberate mistakes in this way. What other strategies or classroom 'norms' need to be built alongside for this strategy to be effective?



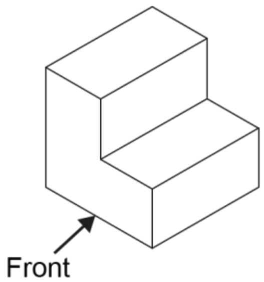
Plan	Front	Side
 9 cm, 12 cm	 15 cm, 12 cm	 15 cm, 9 cm



Plan	Front	Side
 6 cm, 10 cm	 6 cm, 8 cm	 10 cm, 6 cm

Example 6:

a) *Complete the plan view and front and side elevations for this L-shaped object:*



Plan	*Front*	*Side*

b) *Explain why the surface area can be calculated by totalling 2(area of the front elevation), 2(area of the side elevation) and 2(area of the plan view). Will this be true for all prisms?*

c) *Identify the minimum information needed to be able to calculate the surface area of the 3D object.*

Students often assume that the area of an L-shape can be calculated using a standard formula. Using an L-shaped face like the one in *Example 6* provides an opportunity to discuss the different ways of calculating the area of the front elevation of the L-shaped prism, without the need to calculate its area.

Students should readily recognise that the 3D object in this example has two L-shaped faces, although this does rely on well-developed visualisation skills. It is key to **deepening** students' understanding of mathematical structure for them to identify that the area of the base is equal to the sum of the areas of the two rectangles that make up the plan view, and that the 'other side' of the object is a rectangle with an area that is equal to the sum of the area of the two rectangles that make up the side elevation.

Encourage students to be precise in their use of **language** to refer to vertices and faces when identifying the dimensions of the 3D object needed to be able to calculate the surface area.

In this example, students are not required to carry out a calculation of the surface area. Instead, the focus is on developing an understanding of the structure of the 3D shape. What are the benefits of students not needing to carry out the calculation? How does asking students to identify the minimum information needed to be able to calculate the surface area help to develop their understanding of the structure of the 3D shape?

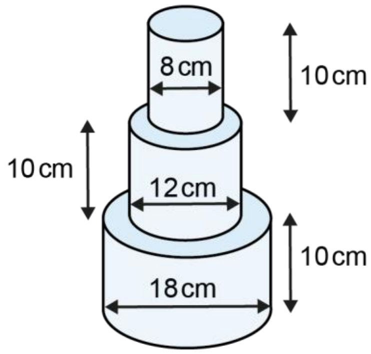
Understand that, for the net of a cylinder, the circumference of the circular face corresponds with the length of the rectangular face

Example 7:

Marlo is going to decorate a three-tiered celebration cake using ready-rolled sheets of fondant icing.

He wants to calculate the surface areas of the three tiers to work out how many sheets of icing he needs to buy.

The three tiers have diameters of 8 cm, 12 cm and 18 cm and are all 10 cm high



a) *Assuming all tiers are completely covered in icing, show that the total surface area is 513π cm^2.*

A pack of ready-roll fondant icing covers an area of 300 cm^2.

b) *How many packs does Marlo need to buy?*

In *Example 7*, students must consider the structure of cylinders and, although this **representation** is not explicitly referenced in the question, use the properties of a cylinder's net to determine surface area. They need to recognise that the net of a cylinder consists of two circles and a rectangle, but that in this context of icing a cake, only one circle and the rectangle need to be considered for each tier.

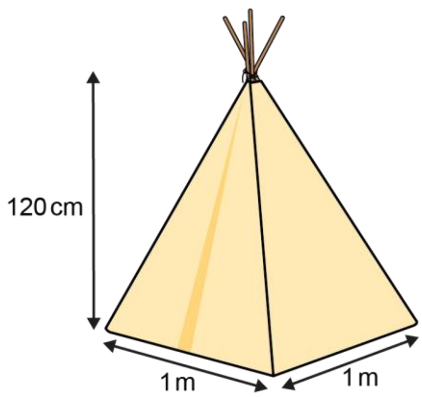
When calculating the surface area of the three tiers, ensure students recognise that the length of the rectangle that forms the sides of each tier is equal to the diameter of the circle that forms the top of that tier, thus **deepening** their understanding of mathematical structure. Asking students to give an answer in terms of π also draws attention to the structure of the calculation, and prevents them from becoming unnecessarily focused on numerical calculations. Students may assume, for part b, that two packs are required, forgetting that part a was expressed in terms of π. Encourage them to estimate rather than calculate that they actually need just over three times more than this.

The **variation** between the diameters of the three cake tiers has been designed so that the diameter of the middle tier is 1.5 times the length of the diameter of the top tier and the diameter of the bottom tier is 1.5 times the length of the diameter of the middle tier. This provides opportunities to explore additive and multiplicative relationships. Teachers could extend students' thinking to consider the effects on the area of a rectangle when one dimension is multiplied by a scale factor of 1.5. How does this compare to what happens to the area of a circle when the diameter/radius is multiplied by a scale factor of 1.5?

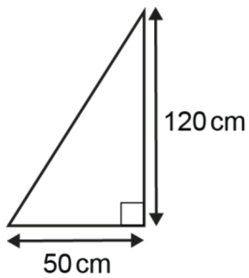
Know the difference between the slant height and vertical height of a pyramid

Example 8:

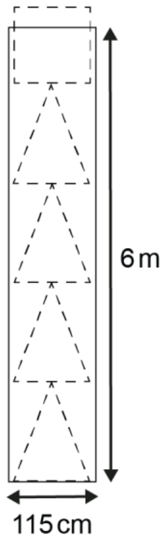
Tia's daughter wants a play teepee and Tia has agreed to make her one.



When finding the surface area of a pyramid, students often confuse the vertical height and slant height in calculating the areas of the triangular sides. In *Example 8*, this difference is explored. It is important to emphasise the correct use of **language**, supporting students to develop a precise understanding of the structure of pyramids – the *vertical* height of the triangular *faces* represents the *slant* height of the *pyramid*. In this example, they can apply their knowledge of the geometry of triangles to a problem involving the surface area of a pyramid, by using Pythagoras' theorem to determine the required measurement. Identifying the existence of a right-angled triangle with base 50 cm and height 120 cm is fundamental to being able to calculate the relevant lengths.



Once students have identified the height of the triangular wall panels (lateral faces)

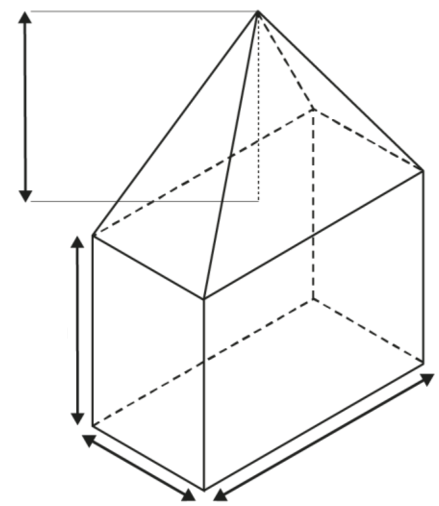
Tia is basing her design on a teepee (above) that she has seen online, which has a 1 m by 1 m square base and a vertical height of 120 cm. *Tia has a 6 m length of fabric that is 115 cm wide. The fabric is patterned, and so all of the triangular faces need to be in the same orientation.* *Show that Tia does not have enough fabric to make the teepee.*	encourage them to sketch a rectangle to show how these might fit onto the 6 m length of fabric. Using a **representation** such as the one on the right supports students to keep the context of the question in mind. In this case, it is not sufficient for the total areas of the faces to be less than the area of the fabric, they also need to fit into the space. They must also think about the positioning of the component parts of the tepee and determine whether a 1 m by 1 m square base and four triangular walls with 1 m base and height 130 cm will fit on the piece of fabric. Establish whether or not students checked to see if there is more than one way of positioning the component pieces of the teepee (whilst maintaining the orientation of the triangular pieces), to be sure that Tia really does not have enough fabric.   When working in contexts such as this, students need to navigate the mathematics and also their understanding and interpretation of the situation. Students bring varying experiences. For some, this context may be a helpful learning aid to conceptualise surface area, while for others it might be so alien that it forms a barrier to learning. Discuss with your team the aspects of this example that might help students understand surface area, and those that might complicate the process for them. How might asking students to draw the net help with finding a solution? Why might the net of the pyramid not be helpful when considering a piece of fabric with a specific width and length?

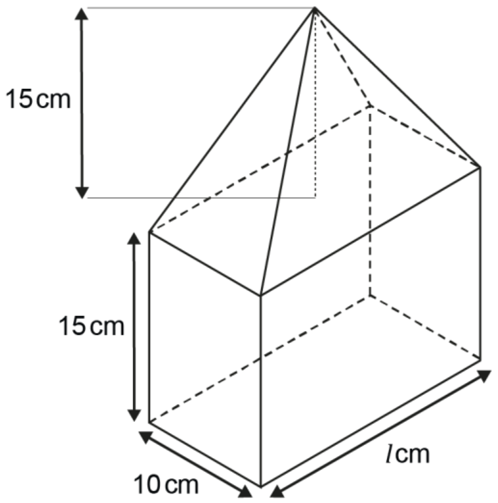
11.4.2.2 Solve problems involving the volume of pyramids and cones

Common difficulties and misconceptions

When calculating the volume of pyramids and cones, students often confuse the volume of a pyramid with that of a prism, and the volume of a cone with that of a cylinder. They may then neglect to multiply the area of the base multiplied by the height of the pyramid/cone by $\frac{1}{3}$ when calculating the volume of a pyramid/cone. Careful use of variation can help, so that students work with pairs of cylinders/cones and pyramids/prisms with the same base and start to appreciate the relationship between them. This can then evolve into problems that involve composite shapes made up of combinations of pyramids/prisms and cones/cylinders with shared dimensions. The intention is to support students to recognise the different structures of the formulae needed for calculating the volume of pyramids and cones, compared with the calculation needed for the volume of prisms and cylinders – rather than memorising a series of separate formulae.

Students can often get confused between the different heights needed for the calculations for surface area and volume of pyramids. The emphasis on the relationship between the volumes of pyramids and prisms can help students to correctly identify the need for vertical height when calculating the volume of pyramids and cones.

Students need to	Guidance, discussion points and prompts
Understand the relationship between the volume of a pyramid and the volume of a prism with the same height and congruent bases *Example 1:* *A 3D shape is made up of a cuboid and a rectangular based pyramid, as shown below.*  *The volume of the cuboid is 480 cm³. The cuboid and pyramid have the same height.* *a) Estimate the volume of the pyramid.* *The total volume is 640 cm³,* *b) Calculate the actual volume of the pyramid.* *c) What do you notice?*	In *Example 1*, students are encouraged to estimate the volume of a pyramid having been provided with the volume of a cuboid with the same base. The volume of 480 cm³ has been selected because it has several factors, and so students can easily calculate different fractions of the total in coming to their estimation. Encourage students to be precise with their **language** when describing how they generated their estimate, so that they are clearly expressing what fraction of the cuboid's volume they believe the pyramid to be. Even if students correctly estimate the volume of the pyramid as one-third of the volume of the cuboid, it is important for them to understand that one particular instance does not constitute mathematical proof of a general relationship. There are several **representations** that can demonstrate this relationship. Some of the most convincing involve pouring liquid between containers – for example, when filling a cuboid from a pyramid-shaped container of the same base, students can easily see that it takes three refills of the pyramid to fill the cuboid. Videos of such demonstrations are readily available online.
Example 2: *A 3D shape, shown below this example, is made up of a cuboid and a rectangular based pyramid. The base of the pyramid and one of the cuboid's faces are congruent.* *a) What fraction of the cuboid's volume is the pyramid?* *b) What fraction of the <u>total</u> volume is taken up by the cuboid?* *The total volume of the composite shape is 4000 cm³.* *c) Show that l = 20 cm.*	In *Example 2*, students work with the same composite shape as *Example 1* but explore the relationships in different ways. Parts a and b are presented without the visual **representation**, so that students must attend to the relevant parts of the question rubric. The question considers the pyramid as a fraction of the cuboid, and then shifts the focus to consider the cuboid as a fraction of the whole. This is designed to support students to connect to prior learning on multiplicative relationships, and to really consider the relationships inherent between the volumes of pyramids and prisms with congruent bases. In part c, students are asked to determine a missing dimension. Providing students with the volume, rather than asking them to calculate it for themselves, encourages them to shift their focus from memory recall of formulae to



a **deepening** appreciation of the mathematical structure of the calculations needed to determine the volume of prisms and pyramids.

The procedural **variation** in this example is such that students are supported by parts a and b to notice a possible approach to part c. However, part c could be offered to students as a standalone task, so that they are not given the suggestion of a fractional approach to determine the solution. Teachers should decide which approach is best for their students.

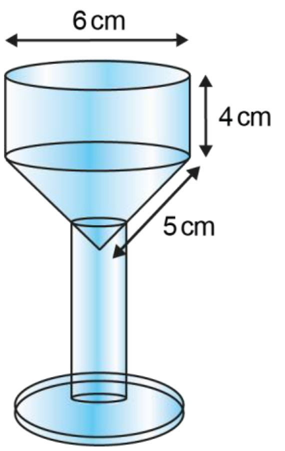
If students' experience of solving problems involving the volume of pyramids is limited to substituting values into a formula, they can often rely on memory recall, without developing an understanding of why the formula works. To what extent do teachers apply a similar approach when posing problems requiring the need to use a mathematical formula? How could teachers promote an understanding of structure for other formulae?

Understand the relationship between the volume of a cone and the volume of a cylinder with the same height and congruent bases

Example 3:

A glass can be represented as a series of cylinders and cones.

There are two solid cylinders that form the base and the stem. On top of these, are an open cone and cylinder that can be filled with liquid.



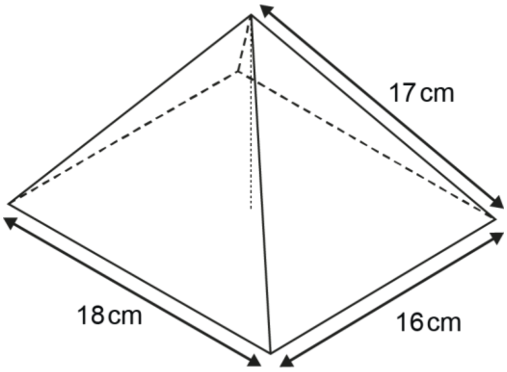
Explain why, when the glass is filled to the base of the cylinder, it is one-quarter full.

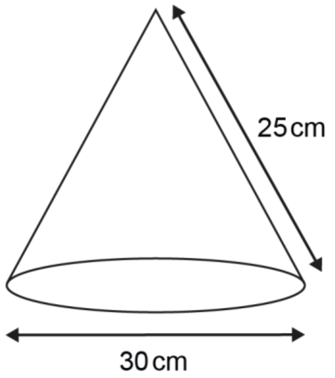
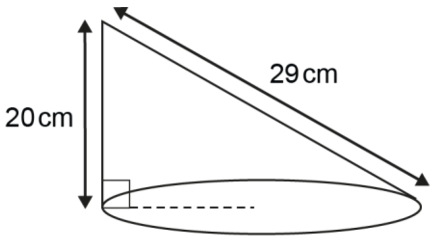
Example 3 makes explicit the relationship between the volume of a cylinder and the volume of a cone with the same vertical height, and whose bases are congruent. It is important that students take time to read the rubric and engage with the **representation**, so they recognise that when the glass is filled to the base of the cylinder, only the cone is filled.

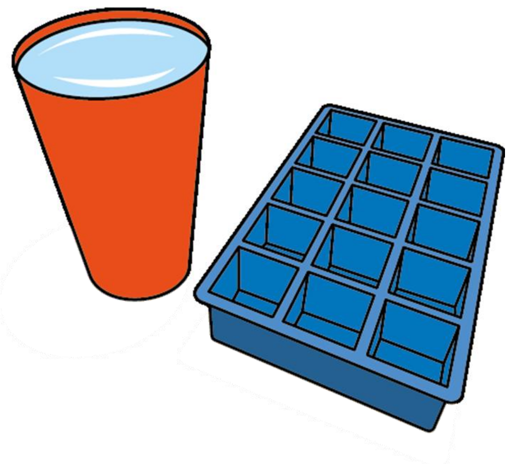
Students need to identify a right-angled triangle with base length equal to the radius of the circle and with a hypotenuse of 5 cm, to establish that the vertical height of the cone and the height of the cylinder are the same. Be precise with **language**, referring to vertical height and slant height rather than just height. When discussing the right-angled triangle, encourage students to use the correct language when referring to its sides. Identifying that the 5 cm is the hypotenuse is key to correctly identifying the vertical height of the cone.

Example 3 focuses on explaining why, rather than showing that, the glass is one-quarter full. Calculations of the volume of the cylinder and cone sections of the glass are not a requirement when answering this question. Exploring without calculations may provide an opportunity for **deepening** students' understanding of the structure of the relationship between cylinders and cones. Some students may nonetheless find the volumes to access this question, those who do should be encouraged to leave their answers in terms of π. Not only does this make the factor of 4 more easily identifiable, but it also ensures that students do not get bogged down by unnecessary calculations.

There are various points in this problem where students may find that they are stuck or make mistakes. For example, the vertical height of the

	cone section of the glass is not given, and the need to find it is not explicitly stated. Students may also assume that they need to show that the volume of the cone is $\frac{1}{4}$ of the volume of the cylinder, rather than $\frac{1}{3}$ of the volume of the cylinder. Teachers will need to judge if and when to intervene with prompts. Discuss with your team: how much ‘productive struggle’ do you all feel comfortable allowing your students? To be confident in approaching unknown problems, students need to be willing to spend time exploring potentially unfruitful or incorrect avenues, but this needs to be balanced carefully so that they do not feel overwhelmed or confused.
Use the vertical height when calculating the volume of a pyramid or cone *Example 4:* *Show that the volume of this rectangular-based pyramid is 1152 cm^3:* 	In *Example 4*, students are given the dimensions of the base of a rectangular-based pyramid and the length of side joining the apex to the corner of the base. While this is enough information for them to be able to calculate the volume of the pyramid, the volume is actually provided. Without this value being given, students may have used the slant height 17 cm as the height of the pyramid and calculate the volume as $\frac{1}{3} \times 18 \times 16 \times 17 = 1632\ cm^3$. Needing to show that the volume is 1152 cm^3 provides a contradiction to this possible solution and encourages students to revisit their method, **deepening** their understanding of what the ‘height’ of a pyramid means. Determining the vertical height of the pyramid requires students to first calculate the slant height, and to understand that this is different to the length of 17 cm provided. Annotating the **representation**, and sketching out the two different right-angled triangles required, will be essential for this. First, a right-angled triangle with base 8 cm and hypotenuse 17 cm can be used to calculate that the slant height is 15 cm. This slant height forms the hypotenuse of a second right-angled triangle with base 9 cm, giving a vertical height of 12 cm. This need to apply Pythagoras’ theorem twice is discussed further in core concept *11.3 Trigonometry*. Students who struggle could be encouraged to think about the structure of the formula for the volume of a pyramid. For example, multiplying the volume of the pyramid by 3 ($1152 \times 3 = 3456\ cm^3$) and dividing by the area of the base ($18 \times 16 = 288\ cm^2$) gives the vertical height (12 cm). This demonstrates a deep understanding of mathematical structure, and how the dimensions of a pyramid relate to its volume. However, it uses the information that students have been asked to show, which should be discouraged in proof questions. Discuss with teachers how they might handle this challenge in the classroom.

Example 5: Find the volume of the cones below. Some of the dimensions will need to be calculated first. a)  b)  	When all the necessary information is given, students can often apply the formula for the volume of the cone without giving thought to its structure. In *Example 5*, they need to first use the information given to calculate an unknown dimension that is required. This will help with **deepening** understanding of the structure of a cone, and the relationship between the properties of a cone and the formula needed to calculate its volume. When practising calculating the volume of a cone, students may become accustomed to being presented with the radius of the base of the cone and its vertical height, and so begin to mechanically substitute these values into the formula $\frac{1}{3}\pi r^2 h$ (where r is the radius of the base and h is the vertical height of the cone). It is important that careful **variation** is employed in the examples that students experience. For example, they should work with problems where they are given both the slant height and the vertical height, and they must select which of the two heights to use. In part a, students need to recognise that the vertical height is needed and determine it, as well as using information about the diameter to calculate the radius. In part b, the information given can be used to calculate the diameter of the base of the cone, which can then be used to determine the radius. When working on part a, students may assume that the only dimension that needs calculating is the radius ($30\ cm\ \div\ 2\ =\ 15\ cm$). They may then use the slant height to incorrectly calculate the volume as $\frac{1}{3} \times \pi \times 15^2 \times 25 = 1875\pi$, instead of using the vertical height (20 cm). Sketching a **representation** of a cylinder with a 30 cm diameter base, that is as tall as the cone, might help them to distinguish between the slant and the vertical heights. Using the **language** of ‘right’ and ‘oblique’ is important when checking that students recognise the difference between the cones presented in parts a and b. It is important to refer to the vertex opposite the base as the apex and the (imaginary) line that connects this to the centre of the base as the axis. Students need to appreciate how they relate to each other differently in the two cones, to establish that for an oblique cone the axis is not perpendicular to the base.
Understand that the volume of a frustum can be found by subtracting the missing cone *Example 6:* *If a cup is filled to the brim and carefully poured into the ice cube tray, every section is filled, with no water left over (see image below).*	In part a of *Example* 6, students are not asked to calculate but to suggest what information they need from an unlabelled diagram. The **language** in the question is left vague to ensure the discussion is as open as possible, supporting teachers to assess students’ understanding. Teacher prompts are essential for making the most of the learning opportunities the question can provide. For example, if students identify that they need to know the ‘volume’ of the cup, teachers should ask what information



a) *What information would you need about the cup to calculate the volume of one of the cube-shaped sections of the tray?*

The cup is a frustum, with vertical height 15 cm. If the cone were extended to its point, its height would be 45 cm.

The circle at the top of the cup has a diameter of 12 cm. The circle at the base has a diameter of 8 cm.

b) *To the nearest centimetre, what are the dimensions of one of the cube-shaped sections in the tray?*

they would need to calculate it. Students may suggest a measuring jug, in which case a useful connection could be made with the fact that 1 cm^3 = 1 ml water.

The intention of part a is to ensure that students are **deepening** their thinking around the mathematical structures involved, rather than immediately launching into calculations. Students should try to connect the information they have with the mathematics that they know, and identify that the volume of the cup can be obtained by imagining the frustum had continued to a complete cone, and subtracting the volume of the imaginary point from the whole.

In the rubric for part b, students are offered lots of information but will need to find their own way to connect this to the **representation**. It is important that students experience situations where the information they need to apply is not readily available, so that they have to annotate or sketch out a situation in order to fully understand it.

Teachers may recognise the image from Checkpoint 21 ‘Cold as ice’ from the *Perimeter, area and volume 2* deck. This task has its genesis in an exploration of how its premise could be extended to include Key Stage 4 knowledge. It is very useful for teachers to adapt and extend existing materials to include new learning or connect to different areas of mathematics. Spend some time with your department using existing Key Stage 2 or 3 question prompts and considering how they could be developed to also include Key Stage 4 content.

Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 11 Geometry* can be found here:

https://www.ncetm.org.uk/media/3xcpkpft/ncetm_ks4_cc_11_solutions.pdf