Maths HUBS Resources

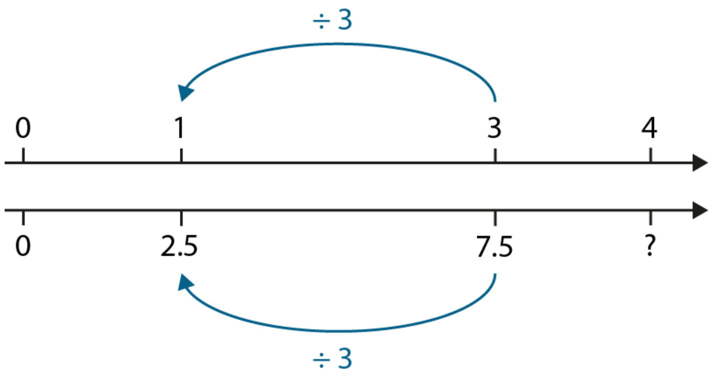
National Centre for Excellence in the Teaching of Mathematics

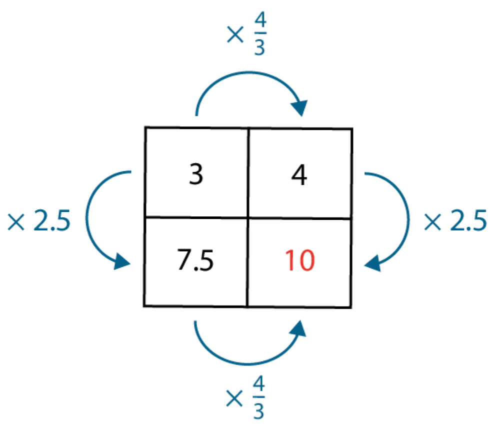
Mastery Professional Development

Mathematical representations

Double number lines (and ratio tables)

Guidance document | Key Stage 3

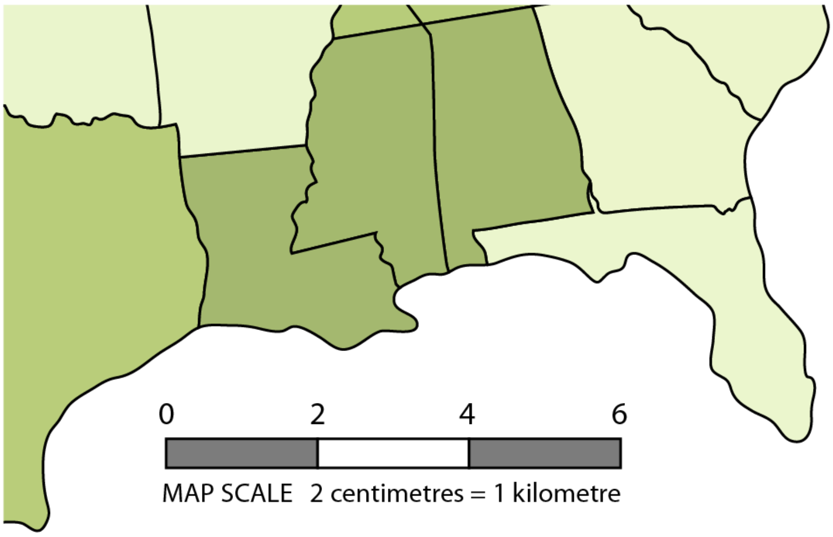


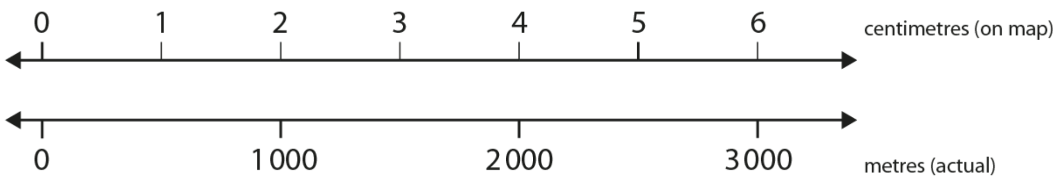


Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

What they are

Double number lines (also known as 'stacked number lines') consist of two single number lines with corresponding pairs of values lined up. A scale on a map, with distances on the map often measured in centimetres and corresponding distances measured in (kilo)metres, is an example of a double number line that students are likely to already be familiar with.



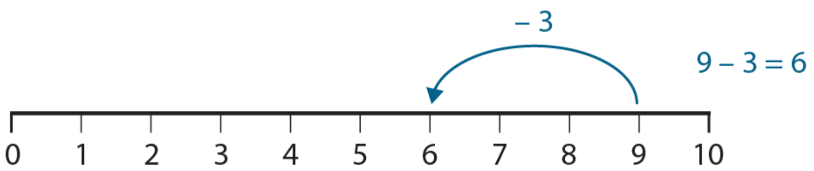


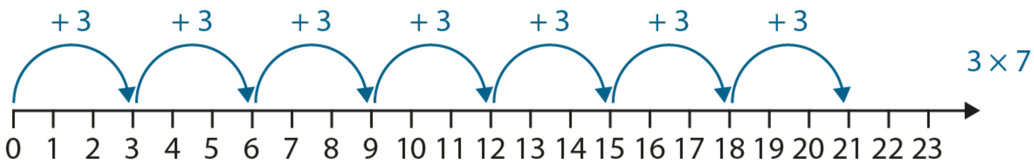
Double number lines provide a way to visually compare two quantities and are a representation that can be applied to many different mathematical situations.

Why they are important

Double number lines are a powerful way of representing multiplicative relationships and ratios, and can help students to visualise equivalent forms of the same ratio. Key to students' success at secondary school is an ability to reason with proportions. Double number lines support such reasoning by offering a strong visual image of how proportional relationships work. When a double number line is used to compare two different measures that are proportional, it provides a model to think with and enables conversion from one measure to the other.

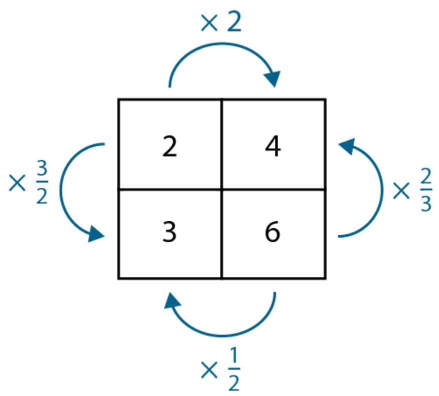
It is likely that students will have used single number lines in Key Stages 1 and 2 to support their understanding of addition, subtraction, multiplication (as repeated addition) and division (as repeated subtraction).

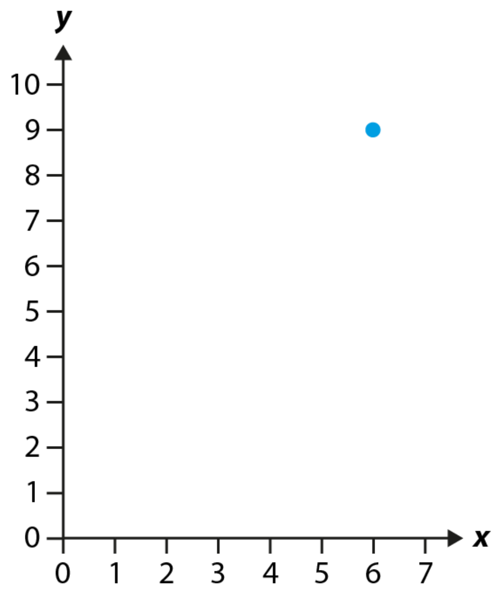




However, learning to construct and use the double number line model is not trivial for students. It is therefore important to provide opportunities for students to both work with and create double number lines in different contexts, as they explore a variety of multiplicative relationships.

The double number line links well to other representations, such as ratio tables and Cartesian graphs. The ratio table displays two particular pairs of values from the double number line; the graph is the result of rotating one of the lines and placing both zeros at the origin. A point on the graph then represents the relationship between two proportional points on the double number line.





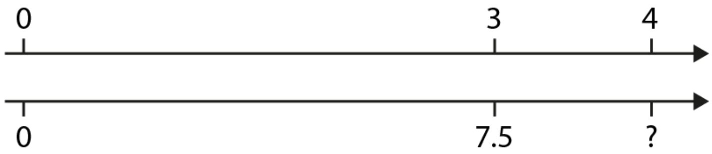
While the ratio table is likely to be more efficient than the double number line, some of the structure may get lost in the compression. Furthermore, the double number line has the advantage of offering a sense of scale. Cartesian graphs provide a familiar representation, with orthogonal rather than parallel lines, but require at least two pairs of corresponding values to be known initially, to be able to plot a straight line. Comparing and contrasting the double number line to these other representations by offering a situation and asking students to represent it in more than one way, can support students in making connections both between the representations themselves and in their understanding of the mathematical structures they represent.

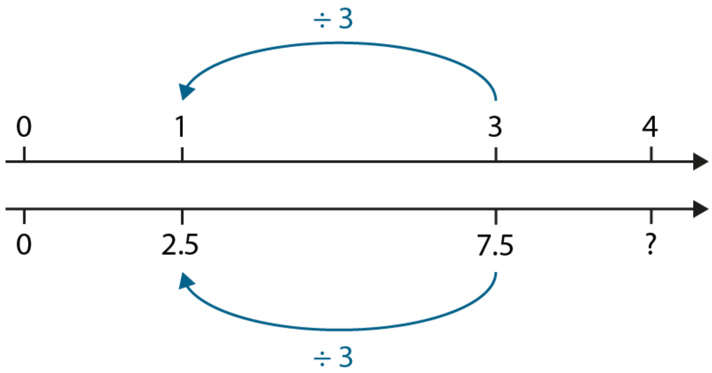
How they might be used

Scalar and functional multipliers

The double number line offers a way of supporting students in exploring multipliers, as they build upon prior strategies involving additive and multiplicative thinking.

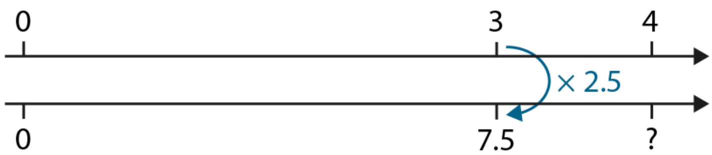
For example, if we have a linear relationship that maps 3 onto 7.5 and we want to find what 4 would map onto, we can do this using two different methods.





While the idea of dividing along the number line may not have been used by students at primary school, it is helpful in exposing a common misconception, that because 3 + 1 = 4, 7.5 + 1 = 8.5, so 4 must map onto 8.5. Identifying that this incorrect addition strategy of adding one should really be adding 'one more 2.5' helps to support students in moving from additive to multiplicative thinking, paving the way for the unitary method, i.e. finding what value one maps onto and then multiplying this by four. Ultimately, this 'along the lines' method, in this example, can be seen as conflating a division by three with a multiplication by four, to give the scalar multiplier of $\frac{4}{3}$.

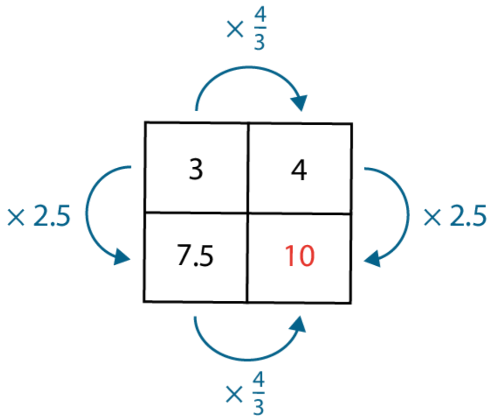
In contrast, the 'between the lines' method focuses on the relationship between the top and bottom number lines to find the functional multiplier.



7.5 is 2.5 times 3, so we can multiply 4 by 2.5 to find the corresponding mapped value of 10. Here, 2.5 is the functional multiplier and can be found by division (7.5 ÷ 3) or rated addition (3 + 3 + half of 3 = 7.5, so 3 × 2.5 = 7.5 or 3 × 2 = 6, 3 × 3 = 9; because 7.5 is halfway between 6 and 9, the multiplier must be halfway between 2 and 3, i.e. 2.5).

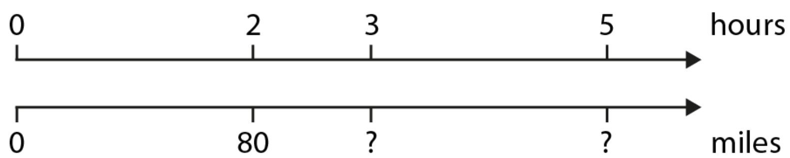
It is likely that students will prefer to work along the lines, but the effectiveness of this approach can be highly dependent on the value of the input and may need to be tailored to the specific number being mapped. Students may, therefore, need greater experience with simple functional multipliers to ensure that they are aware that the 'between the lines' approach can be applied directly to any number on the top number line, and reduces the problem to multiplying by a single number. The nature and effectiveness of both 'along the lines' and 'between the lines' strategies may differ for different contexts, and it is important that students are given the opportunity to explore a variety of scenarios.

Recognising that the numbers originally presented on the double number line provide the values for a ratio table is also an important detail for students to appreciate. Encouraging students to explore multiple representations enables them to find the most effective way of displaying the information given, that fits both the context and their own visualisation needs.



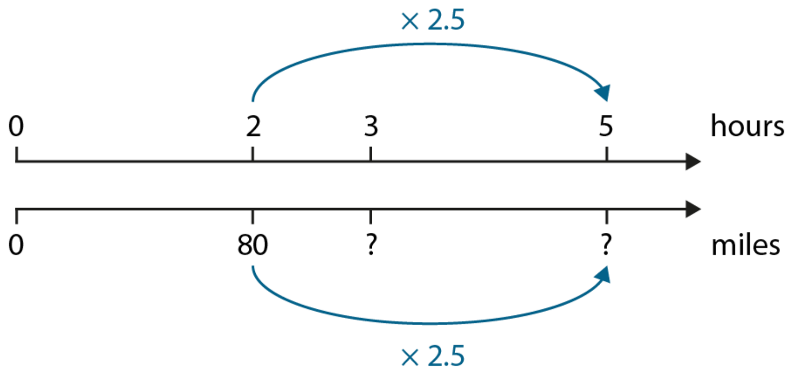
Compound measures

The double number line can help students to think about problems involving compound measures. For example, if a car has travelled 80 miles in two hours, the distance travelled by the car in one hour gives the average speed of the car in miles per hour. Showing the given information on a double number line enables predictions for how far the car could travel, if it maintained the same average speed, to be made for different lengths of time.

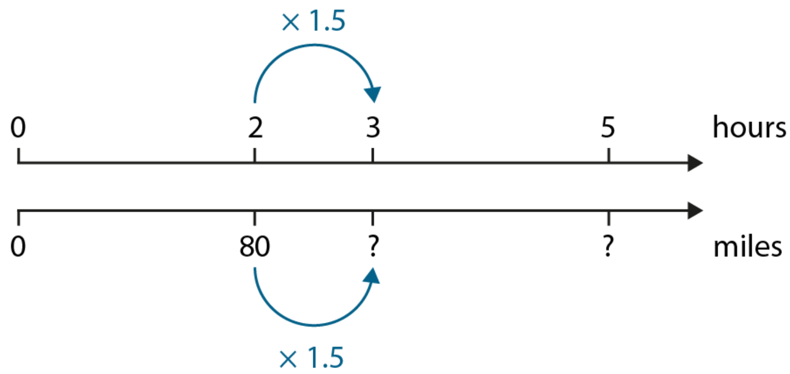


Working along the lines enables students to estimate the distance travelled after three hours and five hours, if the car were to maintain the same average speed. This could be achieved in a variety of ways, and it is important that students are given the opportunity to explore for themselves different methods for thinking about how this can be done.

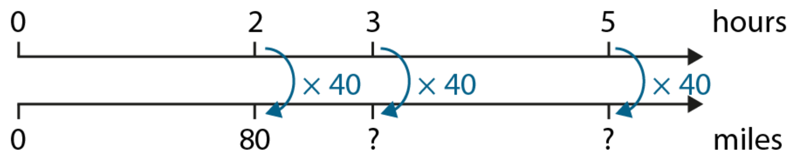
For example, students may recognise that 2 multiplied by 2.5 gives 5, and so 80 multiplied by 2.5 will give the distance travelled after five hours (200 miles).



To find the distance travelled after three hours, students may recognise that 2 + 3 = 5 and so 80 + ? = 200, where the addend is the distance travelled (120 miles). Alternatively, students may look at the relationship between two and three hours and use the fact that 2 multiplied by 1.5 gives 3 and so 80 multiplied by 1.5 will give the distance travelled after three hours.

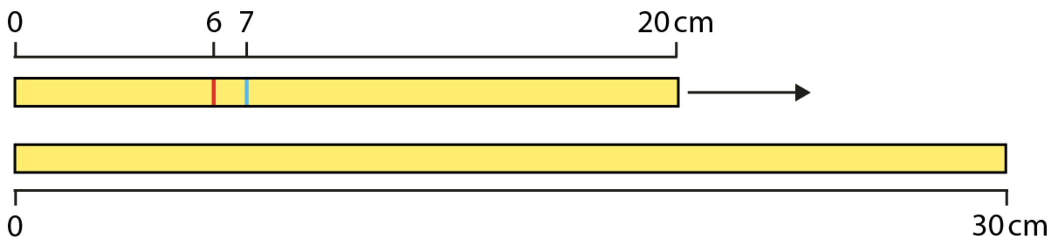


Working between the lines, students may calculate the average speed of the car by finding the (functional) multiplier from 2 to 80 and then use this multiplier to find the distance travelled after three and five hours.



One-way stretch

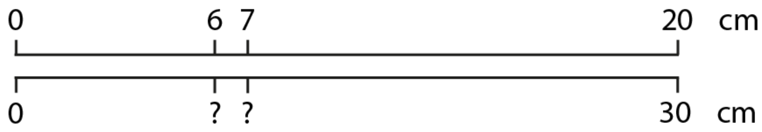
A double number line can be used to represent a one-way stretch, as demonstrated in this ICCAMS elastic strip task†.



† ICCAMS Maths, 2012, *Multiplicative Reasoning: Lesson 3 STARTER (draft trial material)*, iccams-maths.org

In this task, a strip of elastic 20 cm long is stretched so that it is 30 cm long. The strip has two marks on it, a red mark and a blue mark. Before the stretch, the red mark is 6 cm from the left-hand end of the elastic and the blue mark is 7 cm from the left-hand end. The task is to determine how far away from the left-hand end the red mark is, and the gap between the red and blue marks, after the elastic has been stretched from 20 cm to 30 cm.

A double number line can be used to model the situation:



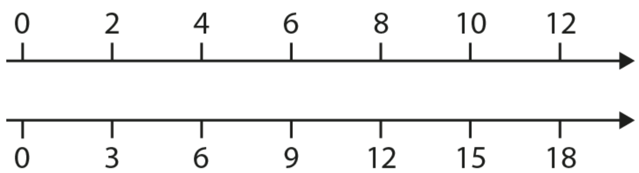
A possible misconception when completing this task might be to model the stretch as an addition (or shift), rather than a multiplication (or scaling) and therefore say that the red mark is now 16 cm away from the left-hand end (adding 10 cm on to 6 cm). One way of addressing this additive misconception may be to ask what happens to the (10 cm) midpoint of the strip or a point very near to the fixed end (the 1 cm or 2 cm mark, for example).

To establish the correct answer of the red mark being 9 cm away from the left-hand end after the stretch, students need to recognise that the elastic has been scaled by a (functional) multiplier of 1.5 or apply a rated addition strategy, identifying that the stretched strip is half as long again;
6 cm + half of 6 cm (3 cm) = 9 cm. Students who are able to identify that because the gap between the red and blue marks is 1 cm prior to the stretch, the gap after the stretch will be equal to the value of the functional multiplier, are demonstrating a deep understanding of the proportional relationship. Looking for such relationships should be encouraged and seen as a more efficient way of solving the task than, for example, determining the gap by finding the distance from the left-hand end of both marks after the stretch and calculating the difference.

Using the double number line as a model for the elastic strip task provides students with the opportunity to both explore multiplication as a one-way stretch and find accessible methods for exposing any misconceptions based on additive relationships when applied to scaling contexts.

Ratio relationships

Double number lines are a powerful way of representing ratios and, once students have used the double number line in practical contexts, it may be appropriate to introduce more abstract ones. A ratio of 2 : 3, for example, modelled using a double number line, would have one number line showing the multiples of two and the other the multiples of three, with respective multiples in line, one underneath the other.

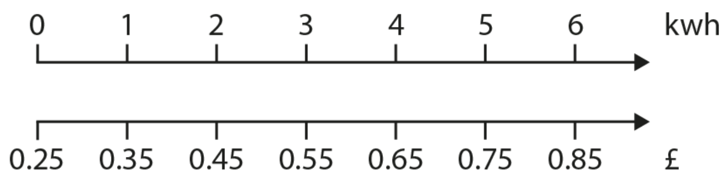


Any number on the top number line is in the ratio 2 : 3 with the number below, making it visible that there are infinitely many pairs of *numbers* in the same ratio and allowing students to develop a greater insight into the nature of ratio relationships. Any two of the numbers from the top number line with the corresponding pair of numbers on the bottom number line can be used to form a ratio table, and this direct link between ratio tables and double number lines can again be emphasised.

Appropriate use of the double number line

Double number lines are most commonly used to represent proportional relationships of the form $f(x) = kx$, where the zeros on the two scales are aligned and the scales themselves are both linear. It is important that students are able to recognise when situations are not proportional and can select an appropriate representation to model these, identifying the limitations of the double number line.

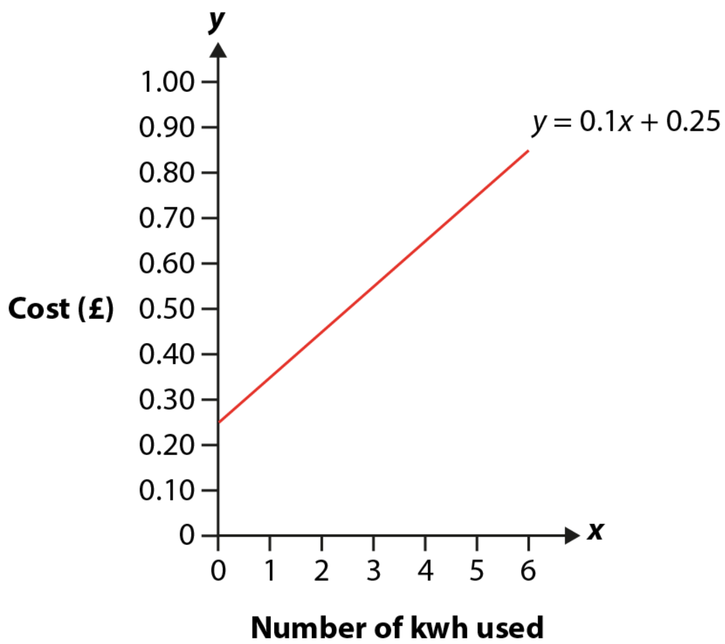
An example of a non-proportional relationship would be to consider the scenario of daily electricity charges, where usage is charged at a 25p per day standing charge plus 10p per kwh of electricity used. Showing this on a double number line highlights the fact that the relationship between the number of kwh used and the amount charged is a non-proportional one.



It is clear from the double number line that the two number lines do not both begin at zero, and although the bottom scale is linear, the relationship is not proportional and so cannot be represented appropriately by the double number line. Other less obvious, but significant, features are that:

- the multiplicative relationship between the lines is not constant (i.e. $0.75 \div 5 = 0.15$, whereas $0.35 \div 1 = 0.35$) so the 'between the lines' strategies will not work
- the multiplicative relationships between the numbers on the top number line are not consistent with those on the bottom number line. For example, 4 is double 2, but 0.65 is not double 0.45, so the 'along the lines' strategies do not work.

Showing the information on a Cartesian graph would give a straight line that does not go through the origin, highlighting again the linear but non-proportional relationship and helping students to identify that only graphs going through the origin represent proportional relationships.



Further resources

Creating number lines can be a time-consuming process, so being able to use an online resource that will generate customisable double number lines allows more time for working with the number line and using it to explore the mathematics of a problem. The ability to stretch and shrink a number line is a desirable additional feature and allows the relationship between the two number lines in a double number line to be investigated.

See, for example:

GeoGebra double number line tool
https://www.geogebra.org/m/YmGMG7Sa
This double number line tool allows the two number lines to be customised in different ways, via sliders and text boxes. The amount the numbers go up in can be varied for each line using the 'jump' (for the bottom number line) and 'topjump' (for the top number line) sliders. Available values range from 0 to 50, including decimals with one decimal place. A 'length' slider can be used to adjust the gap between the numbers on the number line and controls the gaps on both number slides simultaneously. The number lines can be labelled via two text boxes to describe what they each represent.

Interactive double number line tool
https://oercommons.s3.amazonaws.com/media/courseware/relatedresource/file/imth-6-3-5-3-1-double_number_line_tool/index.html
The number lines can be customised via the two settings buttons, with the option to choose 'Whole and decimal numbers' or 'Fractions'. Once selected, the value between the tickmarks can be customised as 0.1, 0.01 or 0.001 and $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$ or $\frac{1}{10}$. There is also the option to label the line via the legend text box. Once generated, the number lines can be locked using the padlock button. When unlocked, the two number lines can be stretched and compressed simultaneously, or there is the option of anchoring one line while the other line is stretched/compressed (via the chain link button). A point on one number line can be linked to a number on the other line, and pressing the 'line' button resets the number lines so that the adjoining line is vertical and the two joined points are directly above/below each other.

When students are using interactive tools, like those described above, for themselves, it is important that they have a clear understanding of the purpose of the tools. It is important that use of the tools is productive in supporting students in deepening their understanding of multiplicative relationships.

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