Planning to teach prime factorisation – Supporting document

This document, which accompanies the 'Planning to teach prime factorisation' video, gives guidance on how one might plan some lessons that allow students to gain understanding of the properties of numbers through exploring their prime factors.

Part 1 – The big idea

Students need to:

- appreciate that every integer greater than 1 is either is a prime number itself or can be represented as the product of prime numbers and, moreover, that this representation is unique (the Fundamental Theorem of Arithmetic)
- understand how to represent numbers as the product of its prime factors and to realise that this is just another way of writing the number
- identify the properties of numbers by examining their prime factors
- identify the Highest Common Factor (HCF) and the Lowest Common Multiple (LCM) by examining their prime factors.

What things typically go wrong?

Common misconceptions include:

- Some students are taught to generate **multiples** through making lists, forming a view that **multiples** are the answers to the times tables. As a result, some students believe they stop at 12 × n.
- That 1 is a prime number. This stems from an erroneous definition that **prime numbers** are numbers that can be divided by one and itself. The correct definition is that prime numbers have exactly two factors.
- It is very common for students to miss out **factors** when thinking of numbers that divide into a number. To address this, encourage students to think of factors as pairs of integers that multiply to make the number and **always start with** 1 and the number itself.
- Students get used to the idea that integers are positive and so therefore forget that **negative multiples exist** (-21 is a multiple of 7 as -21 = -3 × 7)
- An error that seems to continually crop up is that to evaluate an exponent you just multiply the numbers. It is very common for students to think that $2^3 = 6$ (because 2 × 3 = 6).

Part 2 – Pre-requisites

Expectations from KS2

Key stage	Learning outcome
Upper Key Stage 2	• Identify multiples and factors, including finding all factor pairs of a number, and common factors of two numbers • Know and use the vocabulary of prime numbers, prime factors and composite numbers (non-prime, greater than one) • Establish whether a number up to 100 is prime and recall prime numbers up to 19 • Recognise and use square numbers and cube numbers, and the notation for squared (^{2}) and cubed (^{3}) • Solve problems involving multiplication and division, including using their knowledge of factors and multiples, squares and cubes • Identify common factors, common multiples and prime numbers • Use common factors to simplify fractions; use common multiples to express fractions in the same denomination

From the DfE Mathematics programmes of study: key stages 1 and 2 (Sept 2013)

Pre-requisite skills

Factors, multiples and prime numbers all occur in the KS2 maths curriculum so it is likely that there will be some degree of recognition for these concepts on arrival at secondary school. To be able to move forward from firm foundations while still checking that these pre-requisites are understood well, there will need to be some diagnostic assessment of whether understanding has been secured.

Part 3 – Key teaching aspects

Key message

Students need to appreciate a very different way of representing numbers from the additive partitioning they have met before (e.g. 144 = 100 + 40 + 4), i.e. as a product (e.g. $144 = 2^4 \times 3^2$), and that this makes certain properties very apparent (e.g. that 144 is even and a square number).

This is a new and exciting way of looking at numbers and there are some great opportunities for visual representations of prime factors that create a wow factor for students and stimulate thinking.

An important result in maths is 'The Fundamental Theorem of Arithmetic' i.e. that **every integer greater than 1 either is a prime number itself or can be represented as the product of prime numbers** and that, moreover, this representation is unique. A key aim is for students to experience and appreciate the power of this theorem.

KS3 Expectations

Students should be taught to

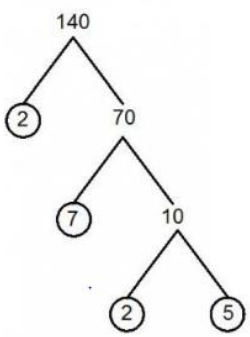
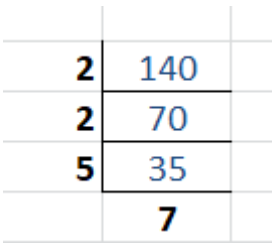
- use the concepts and vocabulary of prime numbers, factors (or divisors), multiples, common factors, common multiples, highest common factor, lowest common multiple, prime factorisation, including using product notation and the unique factorisation property
- use integer powers and associated real roots (square, cube and higher), recognise powers of 2, 3, 4, 5.

From the DfE Mathematics programmes of study: key stage 3 (Sept 2013)

Key skills

Students need to be able to:

- Find all of the factors of a given number. We need to be careful not to demotivate students by offering them tasks which are repetitive and boring to achieve this. Better to present challenges like "There are 10 numbers below 100 that have exactly 6 factors. One of them is 63 (1, 63; 3, 21; and 7, 9). Can you find the other 9 numbers?"
- Identify the prime factors of a given number, recording the product of those factors using index notation. A very common method is to use a factor tree (see below). However, repeated division by the smallest available prime (as shown in the 'repeated division' method below) systematically generates the product of prime factors and more easily transfers to algebraic factorisation later. It is a good idea to check with your head of faculty, head of department or subject leader to see if there is a favoured method within your department.

The 'Factor Tree' method	The 'Repeated Division' method
 Therefore, $140 = 2^2 \times 5 \times 7$	 Therefore, $140 = 2^2 \times 5 \times 7$

- Identify the properties of a number when expressed as a product of primes (***without multiplying all the factors together to obtain one single answer***)
- Be able to generate all of the factors of any number by considering all of the combinations of its prime factors e.g. $2^2 \times 3 \times 7$ has the factors: 1 and 2, 3, 2^2, 2×3, 7, 2×7, 3×7, $2^2 \times 3$, $2^2 \times 7$, $2 \times 3 \times 7$, and $2^2 \times 3 \times 7$
- To examine a set of numbers expressed as the product of primes and deduce the common factors and the highest common factor (HCF) e.g. the two numbers $2 \times 3 \times 5 \times 7$ and $3 \times 5 \times 11$ both contain a factor of 3×5 so both have the factor of 15 which is the HCF.
- Be able to use a Venn diagram (***with understanding***) to determine the highest common factor and the lowest common multiple (LCM) of two numbers expressed as the product of their prime factors [N.B. An outline of how to use a Venn diagram to determine the LCM of two numbers (this was not covered in the recorded slide show) can be found in the appendix at the end of this document].

N.B. Students will need to practise these skills with a range of questions, but this practice needs to be intelligent practice. The careful use of variation in order to draw students' attention to the important structures involved and to guard against unthinking practice is important. A good source of such questions is https://variationtheory.com/category/number/factors-multiples-primes/

Key Terms and Vocabulary

Students should be expected to use correct mathematical language and answer in full sentences.

Key vocabulary includes:

- **Integer -** any whole number such as 1, 2, 3, 4, 5… Integers can be negative and zero is also an integer
- **Sum -** the result when numbers are added, e.g. the sum of 2 and 5 is 7
- **Product -** the result when numbers are multiplied e.g. the product of 2 and 5 is 10
- **Multiple -** multiples of a number are the values which are created when that number is multiplied by any integer e.g. the multiples of 5 are formed from the expression 5n, where n is any integer.
- **Factor -** the factors of a number are any integers (greater than 0) that will divide exactly into that number to leave no remainder e.g. the factors of 12 are 1 and 12, 2 and 6, 3 and 4; 12 has 6 factors.
- **Prime numbers (or primes) -** these are numbers with exactly two factors. Those factors will be 1 and itself. So 13 is prime as 1 and 13 are the only two factors. 1 is not prime as it has only one factor. 2 is the only even number that is prime.
- **Index (or power or exponent) -** the index of a number is written as a superscript and represents the repeated multiplication of that number. n^3 means $n \times n \times n$ and n^4 means $n \times n \times n \times n$. n^4 is read as "n to the **power** of 4" or possibly "the fourth power of n". If there is no index shown on a number, then it is "to the power of 1" and any number to the power of 0 has a value of 1. Exponents can also be fractions or negative integers but that is for much later in the learning process.
- **Co-prime -** Two integers are co-prime if the only positive integer (factor) that divides both of them is 1.

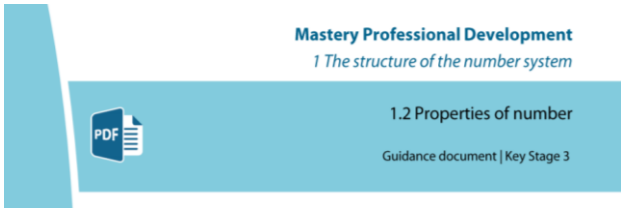
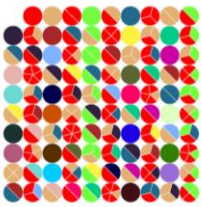
Part 4 – Why this is important

How will this support future learning?

The ideas in this topic helps students develop a deeper understanding of number and number structure, in particular the multiplicative structures, and support future learning in the following areas:

- Simplifying division questions
- Simplifying fractions
- Adding and subtracting fractions
- Solving problems involving common multiples
- A deep understanding of algebraic factorisation

Useful links

	NCETM Mastery Professional Development Materials (1.2. Properties of number) www.ncetm.org.uk/media/dexn0sbp/ncetm_ks3_cc_1_2.pdf
	Variation Theory: Factors, multiples, primes https://variationtheory.com/2020/02/20/product-of-prime-factors-version-2/
NRICH	A collection of short problems related to factors, multiples and prime factorisation include: Counting Factors Factors and multiples Factor Trio
	Make your own 'prime factorisation' diagram – John Graham-Cumming https://blog.jgc.org/2012/04/make-your-own-prime-factorization.html
	'Dancing Primes' www.datapointed.net/visualizations/math/factorization/animated-diagrams/

Appendix – Using a Venn diagram to find the LCM of two numbers

The lowest common multiple (LCM) can be thought of as the lowest integer that each of our numbers will divide into, or alternatively as the lowest integer that has each of the given numbers as a factor.

Multiples of 15: 15, 30, 45, **60**, 75, …

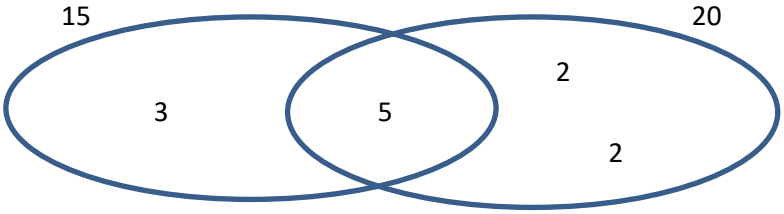
Multiples of 20: 20, 40, **60**, 80, 100, …

60 is the lowest number that is both a multiple of both 15 and 20. 60 is the LCM of 15 and 20.

If we look at this by considering the prime factors, then 15 is 3×5 and 20 is $2^2 \times 5$.

So, $2^2 \times 5 \times 3 \times 5$ is a common multiple, but there is a lower one. As the 5 is a common factor, it is repeated, so we may remove the repeated 5 to give $2^2 \times 5 \times 3$ which is 60.

In a Venn Diagram it is the product of the numbers seen:

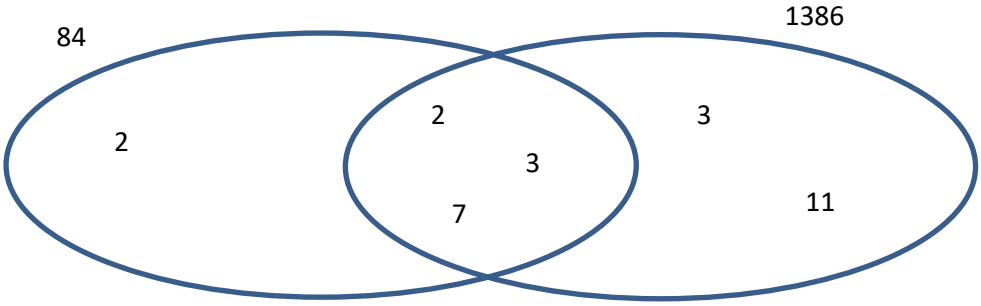


The LCM is $3 \times 5 \times 2 \times 2 = 60$

Another example:

84 is $2^2 \times 3 \times 7$

1,386 is $2 \times 3^2 \times 7 \times 11$



The LCM is $2^2 \times 3^2 \times 7 \times 11 = 2{,}772$