MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Number, Addition and Subtraction

1.31 Problems with two unknowns

Teacher guide | Year 6

Teaching point 1:

Problems with two unknowns can have one solution or more than one solution (or no solution). A relationship between the two unknowns can be described in different ways, including additively and multiplicatively.

Teaching point 2:

Model drawing can be used to expose the structure of problems with two unknowns.

Teaching point 3:

A problem with two unknowns has only one solution if the sum of the two unknowns and the difference between them is given (‘*sum-and-difference problems*’) or if the sum of the two unknowns and a multiplicative relationship between them is given (‘*sum-and-multiple problems*’).

Teaching point 4:

Other problems with two unknowns have only one solution.

Teaching point 5:

Some problems with two unknowns can’t easily be solved using model drawing but can be solved by a ‘trial-and-improvement’ approach; these problems may have one solution, several solutions or an infinite number of solutions.

Autumn 2018 pilot

Overview of learning

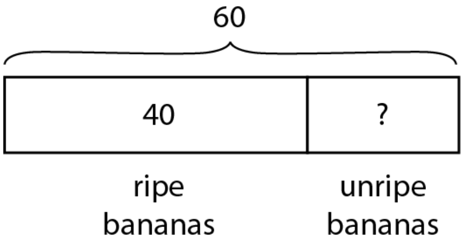
In this segment children will:

- explore, using Cuisenaire® rods, how problems with two unknowns can have one solution or several solutions (*Teaching point 1*)
- use bar-type models to expose the structure of problems with two unknowns that have only one solution, including:
  - sum-and-difference problems, in which both the sum of the two unknowns and the difference between them is given (e.g. $a + b = 27$ and $a - b = 7$)
  - sum-and-multiple problems, in which both the sum of the two unknowns and a multiplicative relationship between them is given (e.g. $a + b = 20$ and $a = 4b$, or $a + b = 48$ and $a = \frac{1}{5}b$)
  - more complex problems (e.g. $4p + 5l = 3.35$ and $4p + 2l = 2.30$).
- work systematically using trial-and-improvement methods to solve problems with more than one solution
- explore problems with an infinite number of solutions (e.g. $x + 50 = y + 20$), applying their knowledge of balancing equations to generalise about the relationship between the unknowns.

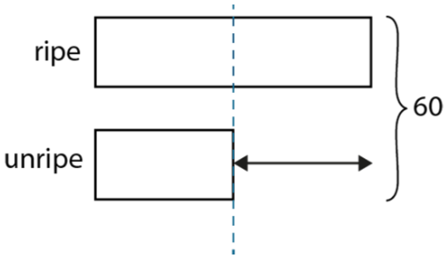
For several years, children have been solving missing-part problems of the following type:

- *'There are 60 bananas in the school kitchen. 40 are ripe. How many are unripe?'*

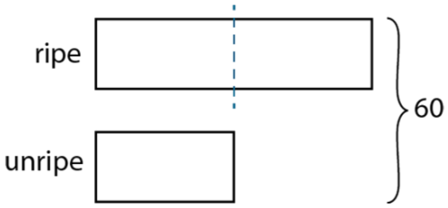
Children should be able to identify that there are 20 bananas that are not ripe. The context can be represented using a bar model:



However, most children will be able to understand the simple structure of this problem well enough to solve it without jottings or diagrams (*40 + number of unripe bananas = 60*). This is a problem with one unknown (the number of unripe bananas) and one solution. This segment begins to look at problems with two unknowns and one solution by considering problems with a very similar structure: the sum is given (for example '*There are 60 bananas.*') and a relationship between the two parts is given; this relationship can be either additive (for example *'There are 20 more ripe bananas than unripe bananas.'*) or multiplicative (for example '*There are twice as many ripe bananas as unripe bananas.*'). In either case, there are now two unknowns (the number of ripe bananas *and* the number of unripe bananas). However, crucially, a qualitative additional piece of information is given that links the numbers of ripe and unripe bananas. *Teaching points 1–3* work towards children being able to confidently model and solve problems with both of these structures (sum-and-difference and sum-and-multiple problems); an approach to solving both can be revealed by drawing a bar-type model for each:



- sum-and-difference problem
  bar-model reveals:
  60 – 40 = 2 × number of unripe bananas



- sum-and-multiple problem
  bar model reveals:
  60 = 3 × number of unripe bananas

Although the first three teaching points focus on problems with this type of structure, a critical aspect of the work is building children's confidence in interpreting and modelling problems for themselves, so that they are able to approach problems with different structures (*Teaching point 4*). This is highlighted towards the end of *Teaching point 3* (step *3:6*), when children are presented with a difference-and-multiple problem (of the form $a – b = 30$ and $a = 3b$) and asked to model it for themselves. The aim is not for children to try to learn an 'off-the-shelf' solution for each type of problem. Indeed, this isn't possible or desirable, as the variety of problems in *Teaching point 4* illustrate.

Bar-type models, such as the ones illustrated above, are very powerful for solving problems with two unknowns. However, encourage children to think of this kind of model drawing as part of a wider family of model drawing, rather than 'bar modelling' being a standalone entity. Looking at any collection of rich mathematical problems, both routine and non-routine, will demonstrate that all sorts of other diagrams and drawings also have their role to play in solving mathematical problems (such as the spatial example in step *4:3*).

Of course, for some mathematical problems drawing isn't very useful and, for others, it isn't at all relevant. In particular, problems that are better solved using a trial-and-improvement, or reasoning, approach are considered in *Teaching point 5*. These may have one solution, several solutions, or an infinite number of solutions. Here the focus is on working systematically by tabulating possible solutions and then, in the case of several solutions, reasoning about whether all solutions have been found, and, in the case of an infinite number of solutions, reasoning towards a generalisation that provides a simple relationship between the unknowns.

Throughout the entire segment, teachers should encourage and guide children to:

- work in small, logical steps
- look for connections and patterns within the solutions
- check their solutions
- reason about whether they have found all possible solutions.

A good practice to foster in children is questioning the number of possible solutions a problem might have. It is suggested that, for every problem, once children have found one solution, teachers ask '*Are there any more solutions? Can you explain why/why not?*' It is important to note that there are no hard-and-fast rules (at least at this level of mathematics) when deciding whether a problem will have one solution or multiple solutions. Children might reason in the following ways:

- For one-solution problems, solved by modelling (such as sum-and-difference problems):
  'As I worked through the problem, I didn't have to make a decision about a value, so I could never have chosen a different value.' (all steps follow logically one after the other)

- For one-solution problems, solved either with modelling or trial-and-improvement: '*Everyone got the same single solution, so there probably is only one.'*
- For trial-and-improvement problems with one or multiple solutions*: 'I've tried all possible combinations "within range" so there can't be any more.'* (a more detailed discussion is included in *Teaching point 5* alongside the different types of problem).

An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Teaching point 1:

Problems with two unknowns can have one solution or more than one solution (or no solution). A relationship between the two unknowns can be described in different ways, including additively and multiplicatively.

Steps in learning

1:1

Guidance

In this teaching point, children will use Cuisenaire® rods to model and compare the structure of problems with one or two unknowns. The problems considered will have either one solution or several solutions (note that some problems with two unknowns have no solutions, but these are not considered here).

As with segment *1.28 Common structures and the part–part–whole relationship*, this allows children to develop a deep understanding of the structures without reference to specific numerical problems and paves the way for the use of bar-like models. In segment *1.28*, *Teaching point 1*, children explored various part–part–whole structures, learning that if one rod is 'unknown', there is only one possible solution (e.g. $B = r + ?$). They also briefly explored the idea that if there are two unknown rods and no further information is provided, there is more than one possible solution (e.g. $B = ? + ?$).

Begin this teaching point by presenting a problem with one unknown, for example:

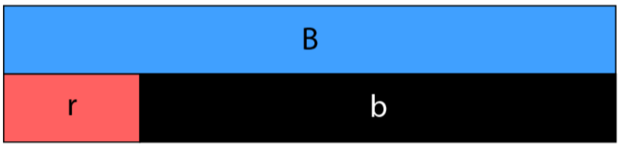
- *'I am thinking of two rods that are equivalent to blue. One of them is red. What is the other one?'*

Working with the manipulatives, children should come to the conclusion that the only rod that 'fits' is the black one. Building on their work in segment *1.29 Using equivalence and the*

Representations

One unknown; one solution:

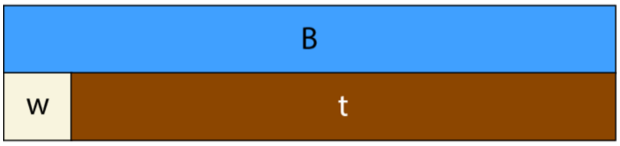
'I am thinking of two rods that are equivalent to blue. One of them is red. What is the other one?'



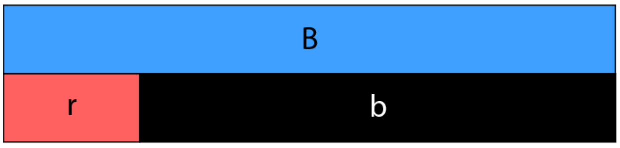
$B = r + b$

Two unknowns; several solutions:

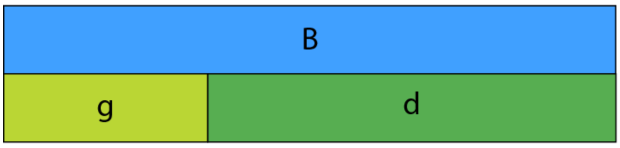
'I am thinking of two rods that together are equivalent to blue. What are they?'



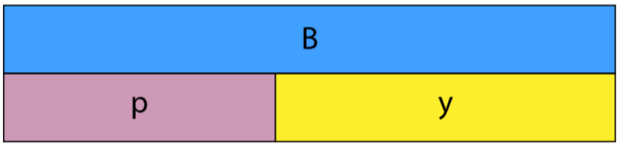
$B = w + t$



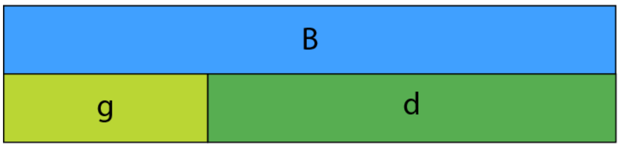
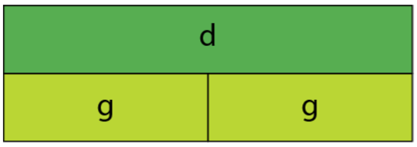
$B = r + b$

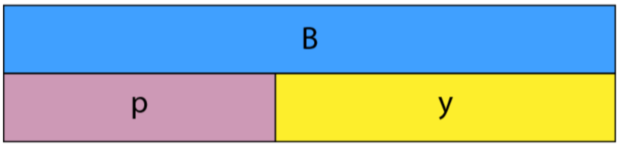
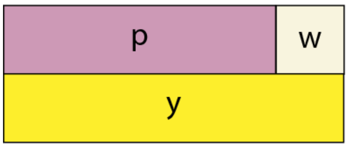


$B = g + d$



$B = p + y$

	compensation property to calculate, they can then write an equation to represent this ($B = r + b$). Then present a problem with two unknowns, for example: • *'I am thinking of two rods that together are equivalent to blue. What are they?'* Now, as children explore the problem using the manipulatives, they will find that there is more than one solution (in this case, four). Again, children should write an equation to represent each solution. Emphasise that when there is one unknown there is only one solution to the problem (none of the other rods 'fit'), while in the case of two unknowns (with no further information) there is more than one solution. Note that, strictly speaking, the correct use of language to describe the relationships would be, for example: • *'The combined length of the two rods is equal to the length of the blue rod.'* rather than: • *'The two rods are equivalent to blue.'* For now, it is fine to use the latter to keep the focus on the structures.	
1:2	Now present the problem with two unknowns from step *1:1*, but provide an additional piece of information relating one unknown to the other *multiplicatively*, for example: • *'I am thinking of two rods that together are equivalent to blue. One is twice as long as the other. What are they?'* Ask children to experiment with the four possible solutions from step *1:1*. By comparing the two rods that are equivalent to blue, children will find that only the light green/dark green combination meets the new criterion. They will find it easier to show this by positioning the rods alongside each	Two unknowns; one solution (multiplicative relationship between the unknowns): *'I am thinking of two rods that together are equivalent to blue. Once is twice as long as the other. What are they?'*  $B = g + d$  $d = 2 \times g$

	other as shown in the second arrangement on the previous page. Later in the segment, children will need to become proficient at drawing models to reveal the structure of problems and will need to be flexible about how they place the bars to best enable problem-solving, so practise this now: - *'Show that light green plus dark green is equal to blue.'* - *'Show that that dark green is twice as long as light green.'*	
1:3	Finally, present the problem with two unknowns from step *1:1*, but now include an additional piece of information relating one unknown to the other *additively*, for example, - *'I am thinking of two rods that together are equivalent to blue. There is a difference of white between the two rods. What are they?'* Again, allow children to experiment with the rods until they find that only the pink and yellow combination satisfies both criteria. As before, encourage children to use different rod arrangements to confirm each relationship: - *'Show that pink plus yellow is equal to blue.'* (first arrangement opposite) - *'Show that pink and yellow have a difference of white.'* (second arrangement opposite)	Two unknowns; one solution (additive relationship between the unknowns): *'I am thinking of two rods that together are equivalent to blue. There is a difference of white between the two rods. What are they?'*  $B = p + y$  $y - p = w$ or $p + w = y$

1:4

Complete this teaching point by providing children with practice finding pairs of rods where both are unknown and only one criterion is provided (several solutions):

- *'Together, two rods are equivalent to dark green (/black/tan…).'* (part–part–whole relationship)
- *'One rod is twice (/three times/four times…) the length of the other.'* (multiplicative relationship between the unknowns)
- *'One rod is white (/red/light green/pink…) longer than the other.'* (additive relationship between the unknowns)

And then provide practice finding pairs of rods where both are unknown and two criteria are provided (one part–part–whole criterion and one relationship between the two unknowns) (only one solution), for example:

- *'Two rods together are equivalent to black and have a difference of light green.'* (additive relationship between the unknowns)
- *'Two rods together are equivalent to dark green and have a difference of red.'* (additive relationship between the unknowns)
- *'Two rods together are equivalent to yellow, and one is four times as long as the other.'* (multiplicative relationship between the unknowns)

Dòng nǎo jīn:

- *'Sort different pairs of rods into this table.'*

sum = orange	
sum = orange ***and*** **difference = red**	
difference = red	

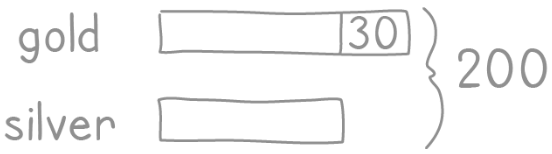
'Convince me that you have found all possible solutions.'

- *'Two rods have a sum equal to the length of orange, and one is two-thirds the length of the other. What are they?'*
- *'Two rods have a sum equal to the length of blue, and one is three-and-a-half times as long as the other. What are they?'*

Teaching point 2:

Model drawing can be used to expose the structure of problems with two unknowns.

Steps in learning

	Guidance	Representations
2:1	This teaching point shows how model drawing can be used to expose the structure of contextual problems with two unknowns in a similar way to modelling with Cuisenaire® rods. Begin by presenting a context with an additive relationship between the two unknowns: • *'Year 6 have earnt 200 stars; the stars are either gold or silver. They have 30 more gold stars than silver. How many are gold?'* First, discuss the problem and identify that there are two unknowns: the number of gold stars is unknown and the number of silver stars is unknown. However, because we know how many gold stars there are relative to the number of silver stars, there is only going to be one possible solution. Now, rather than prescribing a method for solving the problem, present a range of sample approaches that children might use, as shown opposite; some of these solutions are complete, some are incomplete and some are incorrect. Common errors and omissions have been built in to the examples. It is recommended that you provide children with a handout of all five solutions so they can examine and compare them more easily (see slides 7–12 of the accompanying representations presentation). Taking each child, A to E, in turn, ask the class what each child has done to solve the problem and why. For example:	Child A: $\frac{1}{2}$ of 200 = 100 100 + 30 = 130 130 stars are gold Child B: g = 130 s = 70 x g = 100 s = 100 x g = 105 s = 95 x g = 110 s = 90 x g = 111 s = 89 x g = 120 s = 75 x Child C:  200 − 30 = 170 170 ÷ 2 = 85

- *'Child A has split the 200 into two parts; one part is the number of gold stars and one part is the number of silver stars. That gives them 100 of each. Then they've added on 30 because we know that there are 30 more gold stars.'*

At this stage, it is not necessary to ask whether each approach has arrived at the correct answer. This will be revealed as discussions continue. The aim, for now, is to promote and exemplify reflection on the choice of strategy.

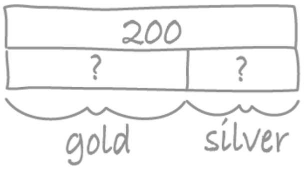
It is important to use mathematically precise language here. For example, the letter *'g'* in Child B's approach, and the top bar in Child C's approach, each represent 'the number of *gold stars*' and not *'gold'* or *'gold stars'*. Ensure that children accurately describe the working. Also make sure that you model the correct language throughout this teaching point (and, indeed, the rest of the segment).

Child D:

gold	silver	difference
150	50	100
140	60	80
130	70	60
120	80	40
110	90	20
115	85	30

115 stars are gold

Child E:



You don't have enough information to solve it

2:2

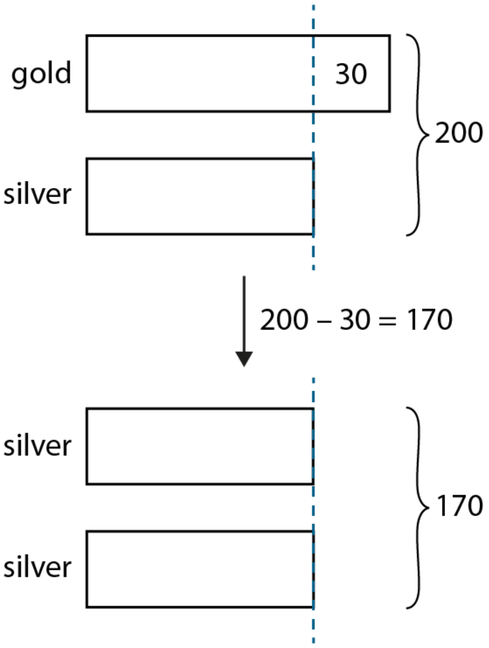
Once all of the example answers have been reviewed, it will probably be clear that the only child who has reached the correct answer is Child D. Discuss each of the other children's approaches, exploring what they have missed out or what errors have they made, and *why* the approach has not resulted in the correct final solution. Recognise that even within the solutions that have not reached the correct answer, there are some correct steps.

There is an enormous amount to discuss; some key points for you to note are as follows:

- Child A has taken a common, but incorrect, approach. Adding 30 onto the 100 to give 130 gold stars means there are only 70 silver stars left (out of the 200), so the difference

	between the numbers of gold and silver stars is now 60 rather than the desired 30. • Child B has tried various values for the numbers of gold and silver stars, making sure that each time they sum to 200. However, the particular values they tried look fairly random and a solution hasn't been found; the child appears to have given up. • Child C has represented the structure of this situation correctly by drawing a model. This has revealed an approach to solving the problem using calculation rather than trial-and-improvement.* Child C has carried out the correct calculations, but has stopped short of providing a final correct answer to the question: they have provided the number of silver stars (85) rather than the number of gold stars (*85 + 30*). • Like Child B, Child D has used a trial-and-improvement approach, though with a little more systematic improvement than Child B. Laying out the options in a table, with explicit recording of the difference between the numbers of gold and silver stars has helped the child to adjust more systematically and to reach the correct final solution (the only child to do so). • Child E has drawn a bar model that correctly represents one of the criteria (the numbers of gold and silver stars sum to 200). However, they haven't shown the second criterion (*'They have 30 more gold stars than silver.'*) on the diagram and therefore haven't been able to link their model to calculations needed to solve the problem. *Note that this model combines both criteria into one diagram by providing the information about the sum to one	

side. This allows us to identify that the number of silver stars is equal to *170 ÷ 2*. This model will be explored in more detail in step *2:5*:



2:3

Summarise some of the previous discussions by asking children if they agree or disagree with statements such as the following:

1. *'Child D has used the best strategy because they are the only one to get the right answer.'*
2. *'Child A is the only one who has used an incorrect approach.'*
3. *'If Child B hadn't given up they would have got the right answer.'*
4. *'The bar models by Child C and Child E show the same information.'*
5. *'Child B and Child D have used the same strategy.'*
6. *'Child D is the only one who has shown the difference between the number of gold stars and the number of silver stars.'*

Try to shift away from using words like '*best*' or '*worst*' when discussing the statements, encouraging children to think carefully about the relative merits and drawbacks of the different approaches, for example:

- *'We might disagree with statement 1, because Child D's strategy is not efficient.'*
- *'We might agree with statement 2, although if Child A had halved the 30 (as well as the 200) and then added, the approach would have worked.'*
- *'We might agree with statement 3, although Child B's approach, while similar to Child D's approach, is not as organised or systematic.'*

2:4

Now highlight the benefits of drawing a model to reveal the underlying algebraic structure of the problem. Compare a version of Child C's approach (adjusted to show the final correct solution) with Child D's approach. Now that they are both complete, which do the children prefer?

Introduce the idea that mathematicians tend to prefer the more efficient and 'elegant' approach, and that they generally will try to find a structural (or algebraic) solution to problems (like Child C), rather than use a trial-and-improvement approach (like Child D) when possible. As such, children will be using the bar-type model extensively through the rest of this segment.

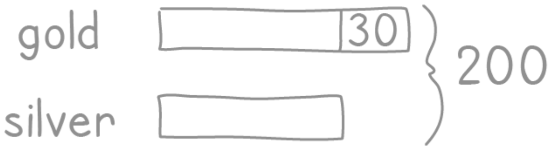
Note that, at secondary school, children will learn how to use algebraic notation to solve problems like this, namely simultaneous equations:

$g + s = 200$ ①

$s + 30 = g$ ②

$(s + 30) + s = 200$ *substitute equation ② into equation ①*

Child C (completed):



200 − 30 = 170

170 ÷ 2 = 85

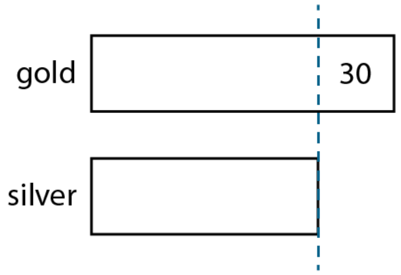
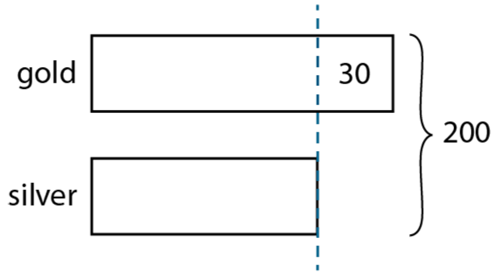
85 + 30 = 115

There are 115 gold stars.

Child D:

gold	silver	difference
150	50	100
140	60	80
130	70	60
120	80	40
110	90	20
115	85	30

115 stars are gold

	$2s + 30 = 200$ $2s = 200 - 30$ $= 170$ $s = \frac{170}{2} = 85$ $g = s + 30$ $= 85 + 30 = 115$ It is useful for teachers to understand the link to the secondary curriculum, but this method is beyond the primary programme of study.	
2:5	Now that the class has identified the bar-type model as an efficient and elegant way to solve problems with two unknowns, explore some different options for representing the given context with bars. Present the two (now complete) model-drawing approaches for Child C and Child E, as shown opposite. For each, work through the creation of the models step-by-step, demonstrating how the information in the question is used to construct the models. Ensure that children understand how each of the models represents the context, by asking questions such as: - *'What does the "200" represent?'* - *'What does the "30" represent?'* - *'Where are the number of gold stars/number of silver stars/all two hundred stars represented?'* - *'Where can we see the difference between the number of gold stars and number of silver stars?'* As described in step *2:1*, make sure that children use language correctly and that they understand that a bar labelled '*silver*' or '*silver stars*' does not represent silver stars, but rather represents 'the	Child C (completed) – step-by-step construction and solution: - *'I know that there are thirty more gold stars than silver stars.'*  - *'I also know that there are two hundred stars altogether.'* 

number of *silver stars*'. Explain that just '*gold*' and '*silver*' have been used so the models can be drawn quickly.

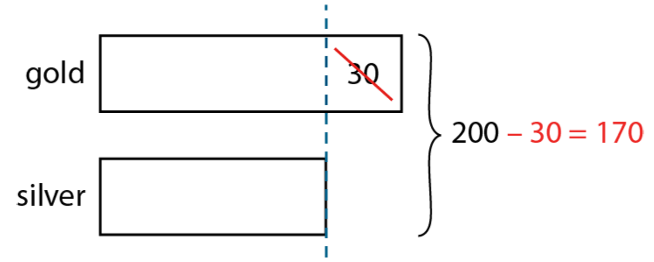
Then compare the two models. Compare how they have been laid out and refer back to the work with Cuisenaire® rods in *Teaching point 1*, where children moved between an 'end-to-end' rod layout and a 'side-by side' rod layout. As a class, discuss the advantages and disadvantages of each model; you may note that:

- Child C's model clearly shows the comparison between the number of gold stars and the number of silver stars, and the difference of 30. Because of this arrangement, the total of 200 is recorded to the side.
- Child E's model uses the end-to-end arrangement, which has been used throughout the spine to show additive relationships, so will probably be more familiar to the children. However, it doesn't show as clearly the equivalence of the remaining two bars once the '30' has been partitioned off.

Note that either approach can be used. The aim of this discussion is to support children to reflect on how they can represent the structure of a situation in different ways. As children encounter each new problem, they can choose a modelling approach as they wish based on how they choose to think about the particular context; avoid teaching a set of bar modelling 'rules' in an attempt to provide an off-the-shelf solution, since this can get in the way of children engaging fully with the problems they encounter. The aim is to develop flexible mathematicians, who can confidently model contexts in an intelligent and useful way.

Teaching point 3 uses the 'side-by-side' arrangement (Child C's approach), but

- *'I subtract the "30".'*



- *'So now I know that the number of silver stars is half of one hundred and seventy.'*

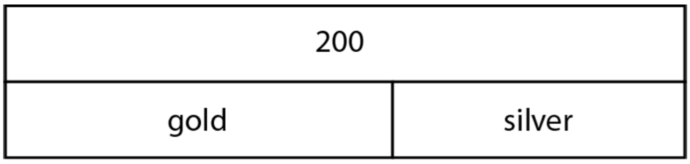
  170 ÷ 2 = 85

  number of silver stars = 85

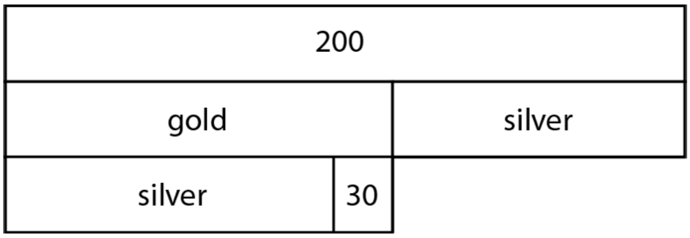
  number of gold stars = 85 + 30 = 115

Child E (completed) – step-by-step construction and solution:

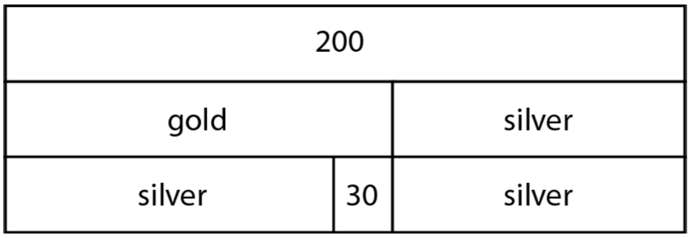
- *'I know that the number of gold stars plus the number of silver stars is equal to two hundred.'*



- *'I also know that there are thirty more gold stars than silver stars.'*



- *'I can fill in the missing part on the bottom row.'*

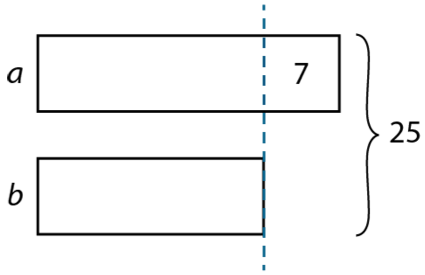
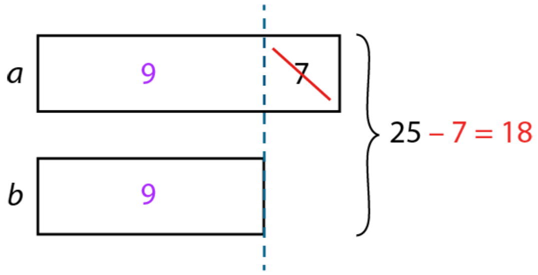


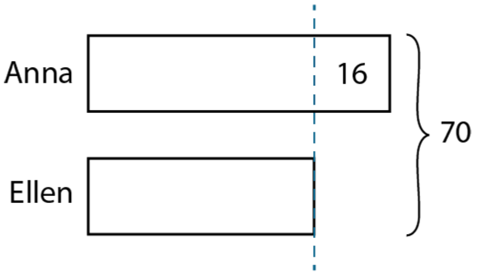
	all of the problems could equally be modelled using the end-to-end arrangement (Child E's approach). Before you move on, double check that all children are comfortable with the arrangement of bars used by Child C, and in particular the new element method of recording the total (here 200) to the side, using a bracket.	- 'So now I can see that: $s + 30 + s = 200$ or $2s = 200 - 30 = 170$ $170 \div 2 = 85$ number of silver stars = 85 number of gold stars = 85 + 30 = 115

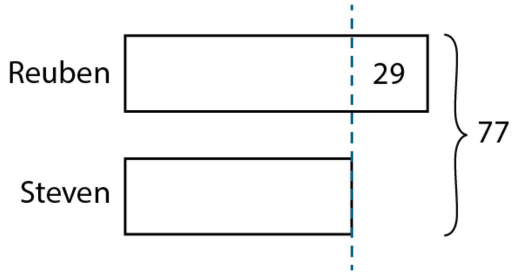
Teaching point 3:

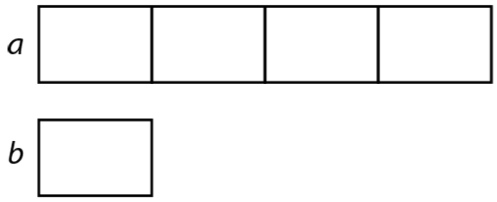
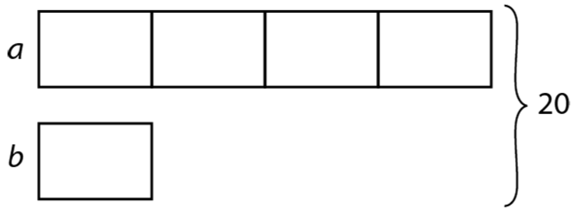
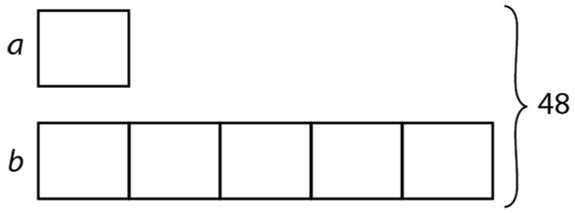
A problem with two unknowns has only one solution if the sum of the two unknowns and the difference between them is given (‘*sum-and-difference problems*’) or if the sum of the two unknowns and a multiplicative relationship between them is given (‘*sum-and-multiple problems*’).

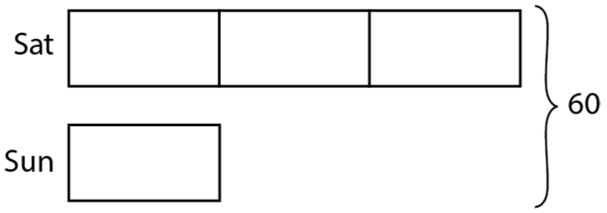
Steps in learning

	Guidance	**Representations**
3:1	In *Teaching point 1*, children learnt to model problems with two unknowns, with the following structures: • only one criterion given, for example: *‘The combined length of the two rods is equal to the length of the blue rod.’* (sum only; several possible solutions) • two criteria given, the second of which is a *multiplicative* relationship between the two unknowns, for example: *‘The combined length of the two rods is equal to the length of the blue rod.’* and *‘One of the rods is twice as long as the other.’* (sum-and-multiple problem; only one possible solution) • two criteria given, the second of which is an *additive* relationship between the two unknowns, for example: *‘The combined length of the two rods is equal to the length of the blue rod.’* and *‘The two rods have a length difference equal to that of the white rod.’* (sum-and-difference problem; only one possible solution) In *Teaching point 2*, children learnt how models can be drawn to help them solve problems that have two unknowns but only one solution. The scenario used included an additive relationship between the two unknowns (a sum-and-difference problem). In this teaching point,	*‘The sum of two numbers,* a *and* b*, is 25, and the difference between them is 7. What are the two numbers?* Step 1 – representing the information given:  Step 2 – deducing the value of *b*:  This can also be expressed as: $25 - 7 = 18$ this is double the smaller amount (b) $b = 18 \div 2 = 9$ $a = 9 + 7 = 16$ *‘The two numbers are “9” and “16”.’* Step 3 – check: $9 + 16 = 25$

	children will look at a range of problems of that type (steps *3:1*–*3:2*), before applying the modelling approach to sum-and-multiple problems (steps *3:3*–*3:5*). Begin by presenting a problem such as: - *‘The sum of two numbers,* a *and* b*, is 25, and the difference between them is 7. What are the two numbers?’* Draw children’s attention to the similarity of this problem to the gold/silver stars problem they have just explored. As a class, construct a model and then ask children to describe what each part of the diagram represents. Then use the model to find the solution. A common omission that children make is to stop once they have found the value of one of the unknowns. Get children into the habit of checking what the question is asking for and making sure they provide that information. Once both values have been found, ask children what they could do to check the solution. In this case, the difference has already been used to calculate the value of the second unknown, but we can check that the two numbers sum to 25.	
3:2	Work through some more examples as a class, including measures contexts. Continue to encourage children to check their answers using the sum. Then compare all of the sum-and-difference problems you have explored and ask children to identify similarities and differences between them. Emphasise that even though the contexts and numbers are different, all of the problems have the same structure.	Example 1: - *‘Anna and Ellen have £70 in total. Anna has £16 more than Ellen. How much money do they each have?’* 

		70 – 16 = 54 this is double Ellen's amount of money Ellen's amount = 54 ÷ 2 = 27 Anna's amount = 27 + 16 = 43 - *'Anna has £43 and Ellen has £27.'* - Check: £43 + £27 = £70 Example 2: - *'Steven is 29 years younger than Reuben. The sum of their ages is 77 years. How old is each person?'*  77 – 29 = 48 this is double Steven's age Steven's age = 48 ÷ 2 = 24 Reuben's age = 24 + 29 = 53 - *'Steven is 24 years old and Reuben is 53 years old.'* - Check: 24 + 53 = 77
3:3	Now move on to sum-and-multiple problems – i.e. those in which the sum of the two unknowns is given, along with a multiplicative relationship between them. Before continuing, you may wish to briefly recap this structure with Cuisenaire® rods, as described in step *1:2*. Begin by presenting a problem in which one of the unknowns is	

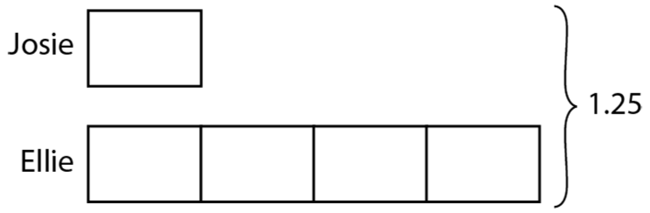
	expressed as a multiple of the other, for example: • *'The sum of two numbers is 20. One number is four times the other number. What are the two numbers?'* Work to construct a model, step-by-step, as a class, building on what children have learnt about sum-and-difference problems. The critical 'realisation' that children need to come to is that the total is made up of *five equal parts* (four parts in one number (*a*) and one part in the other number (*b*)). As before, remind children to check the solution using the sum.	*'The sum of two numbers is 20. One number is four times the other number. What are the two numbers?'* • *'I know that one number (a) is four times the other number (b).'*  • *'I also know that the two numbers sum to twenty.'*  • *'There are five equal parts that sum to twenty.'* one part $= 20 \div 5 = 4$ $b = 4$ $a = 4 \times 4 = 16$ • *'The two numbers are "16" and "4".'* • Check: $16 + 4 = 20$
3:4	Now present a problem in which one of the unknowns is expressed as a fraction of the other: • *'The sum of two numbers is 48. One number is one-fifth of the other number. What are the two numbers?'* This has exactly the same structure as the problem in step *3:3*, but here the multiplicative relationship is described using a fraction (*a* is $\frac{1}{5}$ of *b*), while in the previous problem, the relationship	*'The sum of two numbers is 48. One number is one-fifth of the other number. What are the two numbers?'* • *'There are six equal parts that sum to forty-eight.'* 

	was described using a multiplier (a was equal to $4b$). Encourage children to see that if a is one-fifth of b, we need to iterate a five times to get b ($b = 5a$). Once the model is drawn, the crucial step is then to see that there are six equal parts that sum to 48.	one part $= 48 \div 6 = 8$ $a = 8$ $b = 5 \times 8 = 40$ - *'The two numbers are "8" and "40".'* - Check: $8 + 40 = 48$
3:5	As for sum-and-difference problems, work through some contextual sum-and-multiple problems. Continue to encourage children to check their answers using the sum. Then compare all of the sum-and-multiple problems you have explored, and ask children to identify similarities and differences between them. Emphasise that even though the contexts and numbers are different, all of the problems have the same structure.	Example 1: *'Bill earnt £60 doing odd-jobs one weekend. He earnt three times as much on Saturday as he did on Sunday. How much did Bill earn each day?'* - *'There are four equal parts that sum to sixty.'*  $60 \div 4 = 15$ this is the amount earnt on Sunday amount earnt on Saturday $= 15 \times 3 = 45$ - *'Bill earnt £45 on Saturday and £15 on Sunday.'* - Check: £45 + £15 = £60

Example 2:

'Between them, Josie and Ellie swam 1.25 km during swimming training. Josie swam $\frac{1}{4}$ of the distance that Ellie swam. How far did each of them swim?'

- *'There are five equal parts that sum to 1.25.'*



1.25 ÷ 5 = 0.25 this is the distance Josie swam

distance Ellie swam = 0.25 × 4 = 1

- *'Josie swam 0.25 km and Ellie swam 1 km.'*
- Check: 0.25 + 1 = 1.25

3:6

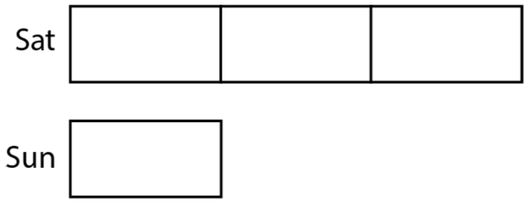
If we take the underlying values from example 1 in step *3:5*, we can see that the problem can be posed in different ways (with different structures).

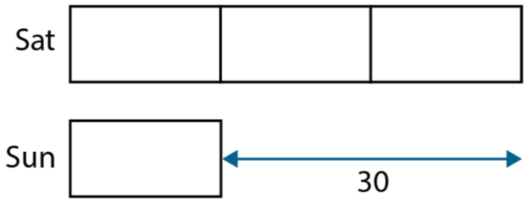
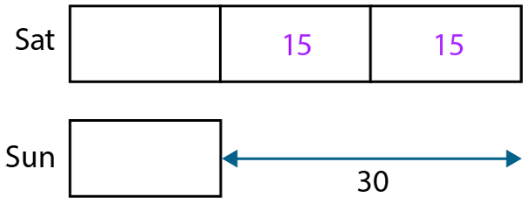
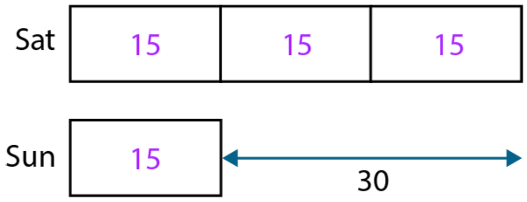
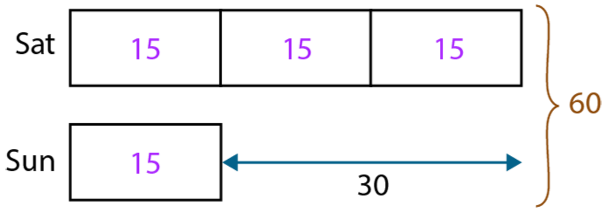
- The underlying values are:
  - *'Bill earnt £45 on Saturday.'*
  - *'Bill earnt £15 on Sunday.'*
- We can pose the problem with a multiplicative relationship between the uknowns:
  - *'Bill earnt three times as much on Saturday as on Sunday.'*
  - *'On Sunday, Bill earnt a third of the amount of money that he earnt on Saturday.'*
- We can pose the problem with an additive relationship between the unknowns:
  - *'Bill earnt £60 in total. He earnt £30 more on Saturday than on Sunday.'*

Modelling your thinking:

'Bill earnt some money doing odd-jobs one weekend. He earnt three times as much on Saturday as he did on Sunday. He earnt £30 more on Saturday than on Sunday. How much did Bill earn in total?'

- *'I'll start by representing the information in the question:*
  - *The question compares the amounts earnt on Saturday and Sunday, so I'll draw two bars side by side.*
  - *Bill earnt three times as much on Saturday as on Sunday, so the bar for Saturday must be made up of three parts each equal in size to the "Sunday" bar.'*



	- *'Bill earnt £60 in total. Bill earnt £30 less on Sunday than he did on Saturday.'* - We can pose the problem using a part-to-whole relationship: - *'Bill earnt £60 in total. He earnt three-fourths of the total on Saturday and one-fourth of the total on Sunday.'* As discussed earlier, we want to avoid children learning 'off-the-shelf' models for every conceivable situation; this is neither a realistic nor a desirable aim, since we want children to develop into confident mathematicians who can engage fully with the problems they encounter. Consider this reframed version of the odd-jobs problem: - *'Bill earnt some money doing odd-jobs one weekend. He earnt three times as much on Saturday as he did on Sunday. He earnt £30 more on Saturday than on Sunday. How much did Bill earn in total?'* (This could be described as a difference-and-multiple problem with the terminology we have used so far.) As you meet new situations like this, model your thinking for the class, as exemplified opposite.	- *'The question also tells us that Bill earns £30 more on Saturday than on Sunday, so I'll add this difference to my model.'*  - *'I can see that the difference of 30 corresponds to two equal parts of the "Saturday" bar. So each of those two parts must be 15.'*  - *'Because all of the parts in the model are equal, I know that the other parts must be 15 as well.'*  - *'So the total amount earnt is four multiplied by £15; that's £60.'* 
3:7	To complete this teaching point, provide children with a range of practice problems as shown below. Avoid being too formulaic in the wording and structure of the problems, encouraging children to fully engage	

with and explore each one they encounter.

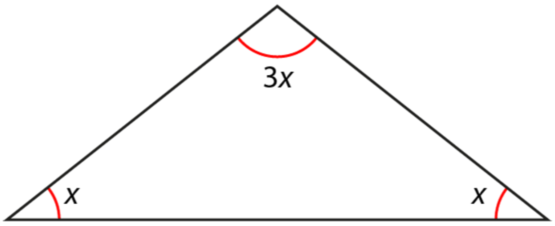
- *'Two numbers have a total of 360. One is three times the other. What are the two numbers?'*
- *'Two numbers have a difference of 5.6 and a sum of 8. What are the two numbers?'*
- *'My dad and I have a combined mass of 96 kg. My dad's mass is three times as much as my mass. How much is each of our masses?'*
- *'It costs £2.65 to buy a watermelon and a pineapple. The pineapple costs 85 p less than the watermelon. What is the cost of each?'*
- *'My garden has an area of 78 m^2. The patio takes up one-third of the area of the garden. The rest is grass. What is the area of the grass?'*
- *'Work out the values of a and b, if* $a + b = 1{,}000$ *and* $a - b = 100$*'*

Dòng nǎo jīn:

- *'Work out the values of p and q, if* $p + p + p = q$ *and* $q - p = 28$*'*
- *'Each side of an isosceles triangle is a whole number of centimetres. The perimeter is 18 cm. I measure two of the sides and find one is 3 cm longer than the other. What are the possible lengths of all three sides?'*
- *'The ages of Anna, Bella and Cara total 18 years. Anna is two years younger than Bella. Bella is two years younger than Cara.'*
  - *'How old is each of them?'*
  - *'How many years is it until their ages total 50?'*
- *'Together Jess, Safa and Amy ran 25 km between them. Jess ran three times as far as Amy and 3 km further than Safa. How far did Jess run?'*
- *'I am thinking of a number. The difference between one-third of this*

Dòng nǎo jīn:

- *'Work out the value of each angle in this triangle.'*



	number and one-fourth of this number is three. What is the number I am thinking of?' Note that the dòng nǎo jīn problem above (regarding the isosceles triangle) actually has *two* solutions, in contrast to each of the other problems in this teaching point which only have *one* solution. However, it is structurally similar to the other problems. The two solutions arise from the uncertainty in the question as to which measured side, the shorter or longer one, is the same length as the other unmeasured side; i.e. the solutions correspond to two possible situations: $2a + b = 18$ and *either* $a - b = 3$ or $b - a = 3$. This is a good example of a problem where children can reason about how many solutions there are, for example '*As I worked through the problem, I had to make a decision about a value/relationship. What would happen if I made a different decision?'*	

Teaching point 4:

Other problems with two unknowns have only one solution.

Steps in learning

Guidance

4:1 Up to this point, we have considered problems with two unknowns in which 'one of each' of the unknowns was related to the other. All of the problems considered so far can be summarised by the following equations, where *a* and *b* are unknowns and *c*, *d* and *n* are given values:

- sum-and-difference
  $a + b = c$ and $a - b = d$
  (e.g. $a + b = 25$ and $a - b = 7$; step *3:1*)
- sum-and-multiple
  $a + b = c$ and $a = nb$
  (e.g. $a + b = 20$ and $a = 4b$; step *3:3*)
  here *n* could be a fraction as in step *3:4*
- difference-and-multiple
  $a - b = c$ and $a = nb$
  (e.g. $a - b = 30$ and $a = 3b$; step *3:6*)

We now broaden the discussion to consider slightly more complex problems, where several of each unknown are considered together, with information such as the following given:

$$4p + 5l = 3.35$$

$$4p + 2l = 2.30$$

(More generally the known information is of the form:

$$na \pm pb = c \quad \text{and} \quad qa \pm rb = d$$

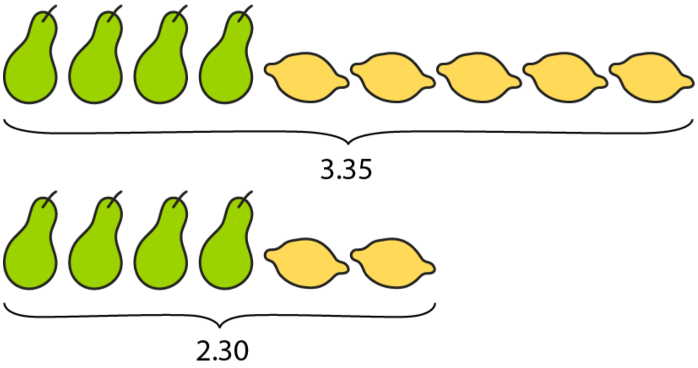
where, again, *a* and *b* are the unknowns and the rest of the letters represent given values. Problems could include situations where one of the coefficients in an equation (for example *n* or *p*) is zero, e.g. $6a = 15$ and $2a + 6b = 32$. However, this would not have the same structure as the other problems in this

Representations

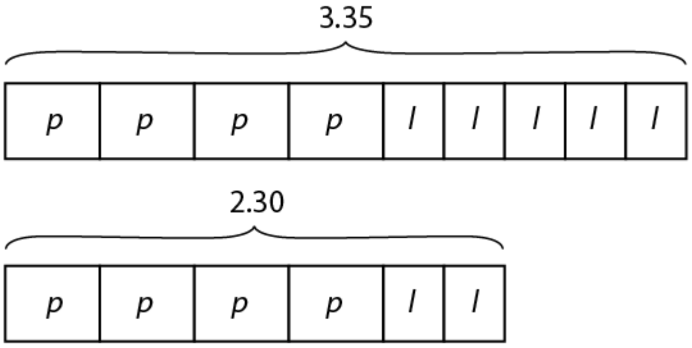
Representing the original information provided in the problem:

'4 pears and 5 lemons cost £3.35.
4 pears and 2 lemons cost £2.30.'

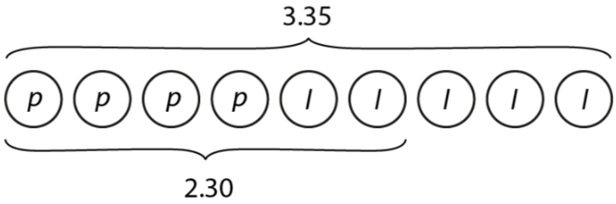
- Pictorial – one diagram for each piece of information



- Bar-type model – one diagram for each piece of information



- Circles/shapes – one diagram for both pieces of information



teaching point, since one equation alone contains enough information to find the value of one of the unknowns (which can then be used, along with the other equation, to find the value of the other unknown); such problems are not considered here.

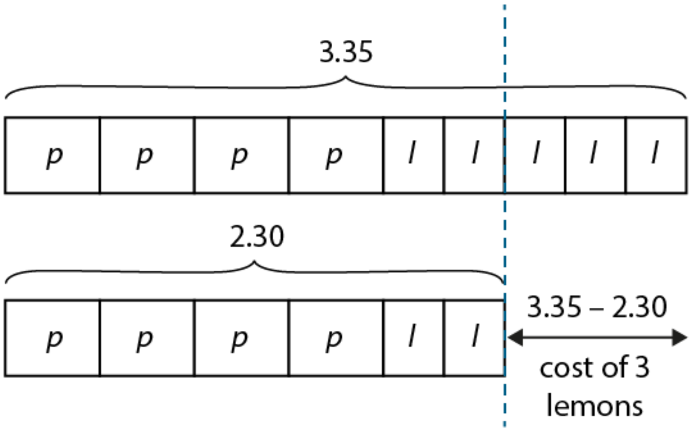
Needless to say, this algebraic discussion is aimed at teachers only, to outline the scope of this teaching point and the common features of the problems considered. A good understanding of this scope will help you to construct your own problems.)

Begin this teaching point by presenting a problem for which one of the unknowns has the same coefficient in each equation (i.e. there is a common quantity in both equations), for example:

- *'**4 pears** and 5 lemons cost £3.35.*
  ***4 pears** and 2 lemons cost £2.30.*
  How much does one pear cost?
  How much does one lemon cost?'

To solve a problem such as this, we essentially need to subtract one equation from the other to find the value of one of the unknowns, then use one of the original equations to find the other. As a first step, ask children to represent the given information. As already mentioned, there is no 'perfect' model that all children should be steered towards. Instead ask them to think about how they can represent the two pieces of information given in the problem, then, as a class, look at some of their suggested diagrams:

- Some children may choose to draw two separate diagrams, one for each piece of information given.
- Some may choose to draw one diagram that combines both pieces of information.

	- Based on their previous experience, some may choose to draw bars. - Others may choose to draw circles (or similar shapes) as they think of the individual pieces of fruit. You may wish to show a pictorial version on the whiteboard, but encourage children to draw simplified diagrams.	
4:2	Whatever the type of diagram, the key feature that is needed is a layout that clearly exposes the difference between the two scenarios, since we are looking to relate one piece of information to the other to find a solution. In this example, the diagram must clearly expose the fact that the difference between the two purchasing situations is the cost of three lemons. To draw attention to this, look again at the two pieces of information given, and ask children: - *'What's the same?'* (the number of pears) - *'What's different?'* (the number of lemons and the total cost) Choose *one* diagram that already clearly shows the given information (or your pictorial representation) and add this comparison information to it. Then ask children to describe what each part of the diagram represents, for example, for the pictorial representation: - *'What does each pear represent?'* (each pear represents the *cost* of one pear) - *'What does the "£1.05" represent?'* (the cost of three lemons) etc. Then work through the rest of the solution, beginning with the fact that we know that the difference between the two costs is equal to the cost of	Comparing the information and finding the solution: *'4 pears and 5 lemons cost £3.35.* *4 pears and 2 lemons cost £2.30.* *How much does one pear cost?* *How much does one lemon cost?'* - Step 1 – finding the cost of a lemon  cost of 3 lemons = 3.35 – 2.30 = 1.05 so cost of 1 lemon = 1.05 ÷ 3 = 0.35 *'One lemon costs 35 p.'*

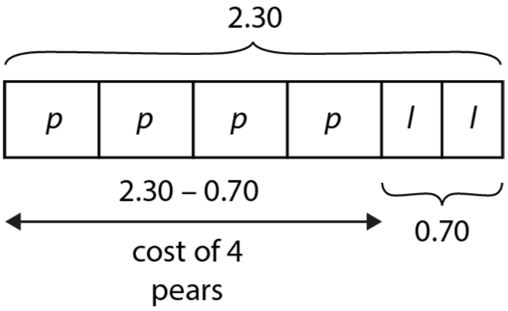
three lemons and so can calculate the cost of *one* lemon. Then demonstrate how we can calculate the cost of a pear; opposite we have substituted the cost of one lemon into the second equation/second part of the diagram (there are fewer lemons in the second one so the calculation is slightly simpler).

As emphasised throughout the segment, children must always be encouraged to check their solutions. Here we have already used the second equation/piece of information to find the cost of a pear, so now we check the solution is consistent with the first equation/piece of information.

Work through some more problems of this form, for example:

- *'A school sells two types of tickets for the school concert: adult tickets and child tickets. For the Thursday night performance, they sell 72 adult tickets and 11 child tickets, making £387.50 in total. For the Friday night performance, they sell 67 adult tickets and 11 child tickets, making £362.50. How much does each type of ticket cost?'*

- Step 2 – finding the cost of a pear



cost of 2 lemons = 70 p (or £0.70)

cost of 4 pears = 2.30 – 0.70 = 1.60

so

cost of 1 pear = 1.60 ÷ 4 = 0.40

'One pear costs 40 p.'

- Check

'4 pears and 5 lemons cost £3.35.'

cost of 4 pears = 4 × 40p = £1.60

cost of 5 lemons = 5 × 35p = £1.75

£1.60 + £1.75 = £3.35

'The solution must be correct.'

4:3

Now explore a spatial problem for which two pieces of information about grouped shapes are given, as in the example below. At first glance, this problem does not seem related to those in the previous steps, but the underlying structure is the same (we essentially have two equations, $s + b = 34$ and $2s + b = 42$, where s is the side-length of one small square and b is the side length of the big/large square).

To start, encourage children to describe what the 34 cm and 42 cm each represent:

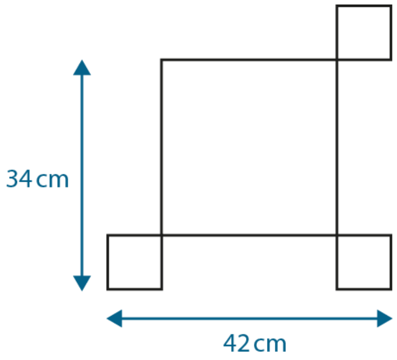
- 34 cm = length of small + length of big
- 42 cm = length of two smalls + length of big

(Here we abbreviate, for example, '*side-length of a small square*' to '*length of small*'; avoid using language such as *'small plus big equals thirty-four'*, instead being precise, i.e. *'side-length of a small square plus side-length of a big square is equal to thirty-four centimetres.'*)

Then ask children to compare the two pieces of information, asking *'What's the same?'* and *'What's different?'* This should draw out the fact that the difference between the 34 cm height and the 42 cm width is equal to the side-length of one small square.

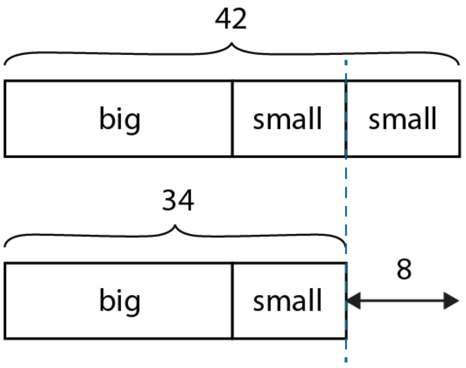
For spatial problems, children may feel comfortable annotating the diagram provided or they may prefer to draw separate models.

'This pattern is made from two different sized squares. What is the side-length of each type of square?'

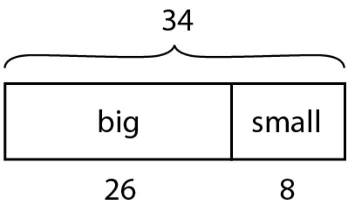


Bar-type model:

- Step 1 – finding the side-length of the small square

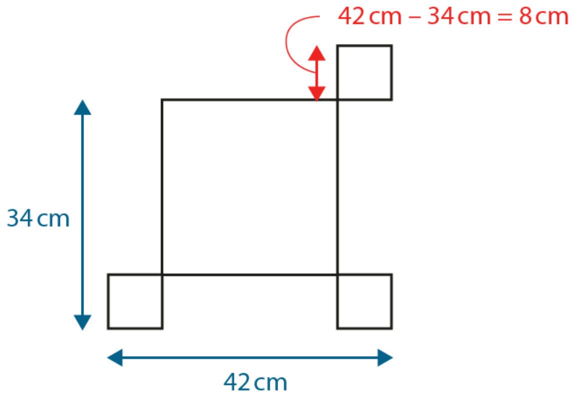


- Step 2 – finding the side-length of the big square

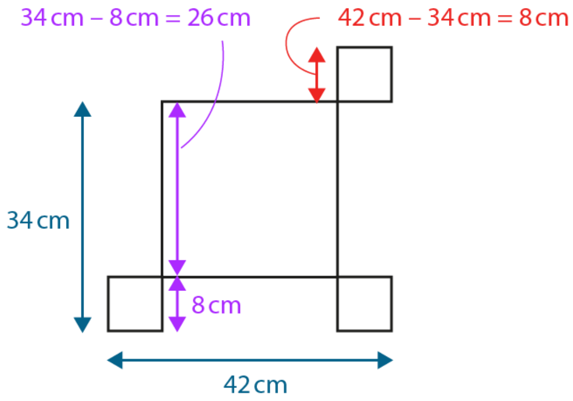


Annotating the spatial diagram:

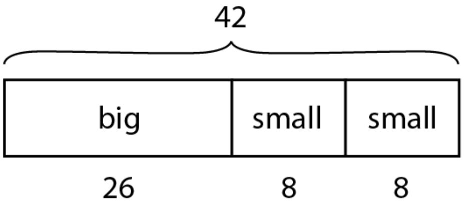
- Step 1 – finding the side-length of the small square



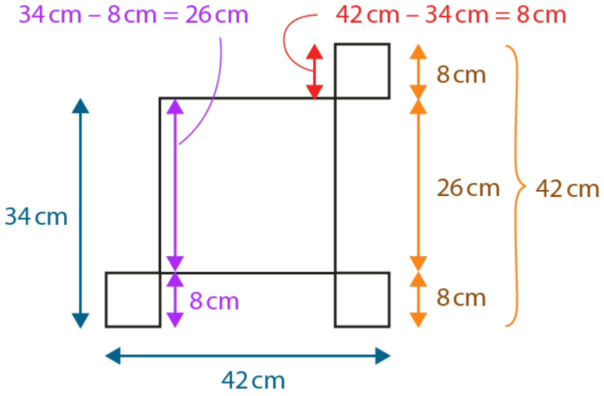
- Step 2 – finding the side-length of the big square



- Step 3 – check



- Step 3 – check



4:4

Now explore the following problem:

- *'An apple costs 15 p more than a banana. 2 apples and 3 bananas cost £1.30. How much do apples and bananas cost each?'*

This fits into the same general structure described in step *4:1*:

$na \pm pb = c$ and $qa \pm rb = d$

but here we recognise that one of the equations is the difference:

$\boldsymbol{a - b = 15\text{ p}}$ and $a + 3b = \pounds 1.30$

As mentioned before, when drawing models you are likely to abbreviate the notation, writing for example, '*3 bananas*' instead of '*cost of 3 bananas*'. This is a practical approach to annotating models, but ensure that you use the correct language when describing them; for example, *'The top bar represents the cost of three bananas.'*

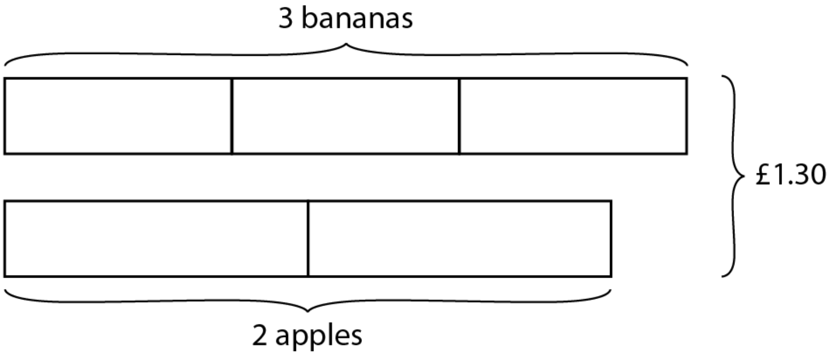
Note that at the start of *step 2* below, you could include an interim visual step in which the bar representing the cost of each apple is split into a part representing the cost of a banana and 15 p; the parts on the bottom row can then be rearranged into the format shown for *step 2*.

Some children may prefer the 'end-to-end' arrangement of bars as summarised below.

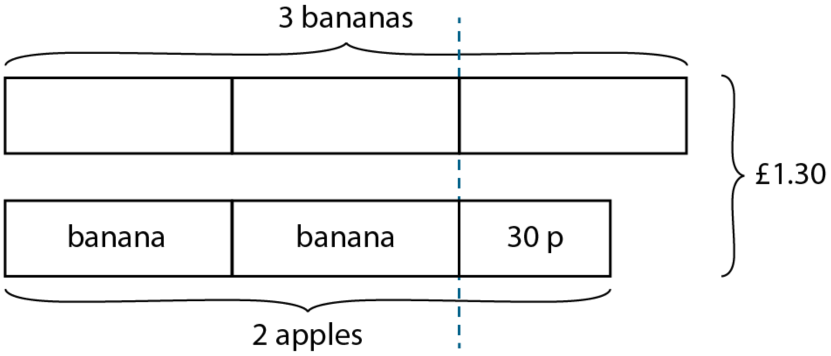
Side-by-side layout of bars:

'An apple costs 15 p more than a banana. 2 apples and 3 bananas cost £1.30. How much do apples and bananas cost each?'

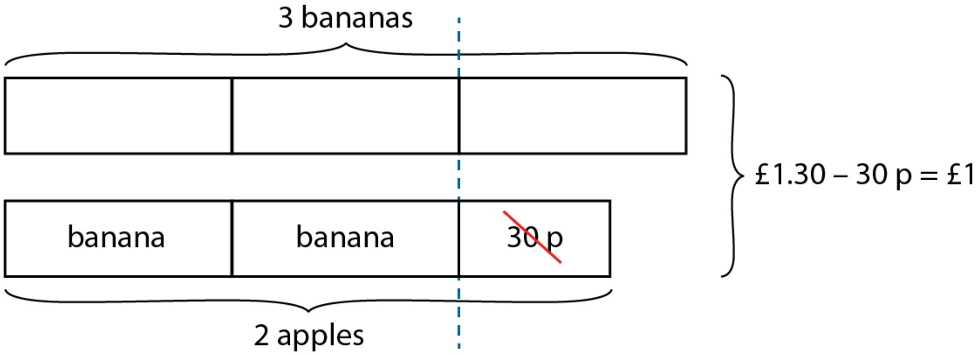
- Step 1 – *'2 apples and 3 bananas cost £1.30.'*



- Step 2 – *'An apple costs 15 p more than a banana. So the cost of two apples is equal to the cost of two bananas plus 30 p.'*



- Step 3 – *'Now we can see that £1.30 minus 30 p is equal to the cost of 5 bananas.'*



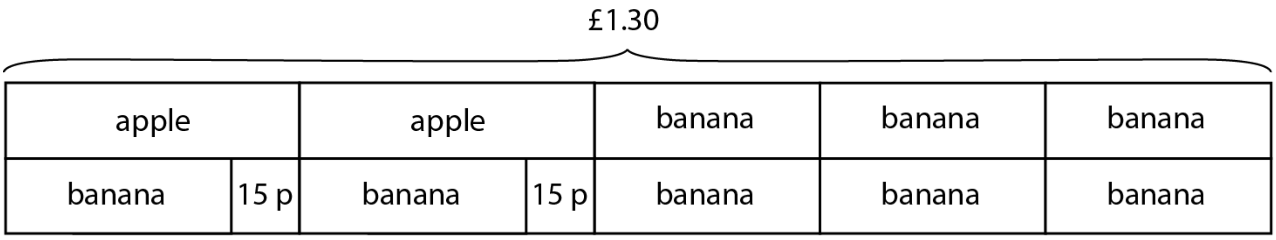
cost of 5 bananas = £1

cost of 1 banana = £1 ÷ 5 = 20 p

- Step 4 – *'An apple costs 15 p more than a banana.'*

  cost of 1 apple = 20 p + 15 p = 35 p

End-to-end arrangement of bars:



- Check – *'2 apples and 3 bananas cost £1.30.'*

  cost of 2 apples = 2 × 35 p = 70 p

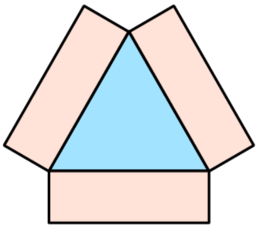
  cost of 3 bananas = 3 × 20 p = 60 p

  cost of 2 apples and 3 bananas = 70 p + 60 p = £1.30

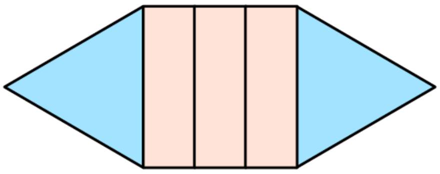
4:5 To complete this teaching point, provide children with practice, as shown below. As in *Teaching point 3*, avoid being too formulaic in the wording and structure of the problems, encouraging children to fully engage with and explore each one they encounter. Continue to encourage children to check their solutions.

- *'1 rubber and 5 pencils cost £3.35.*
  5 rubbers and 5 pencils cost £4.75.
  How much does a rubber cost?
  How much does a pencil cost?'
- *'An adult ticket for the zoo costs £2 more than a child ticket. I spend £33 buying three adult and two child tickets. How much does each type of ticket cost?'*

- *'Here are two different designs made with the same rectangular and triangular tiles.'*



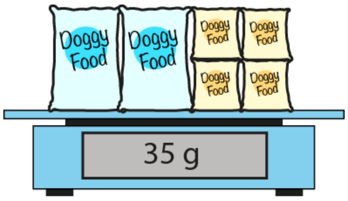
total area = 51 cm^2

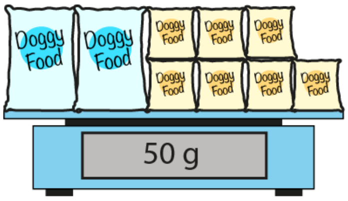


total area = 66 cm^2

  - *'What is the area of one triangular tile?'*
  - *'What is the area of one rectangular tile?'*

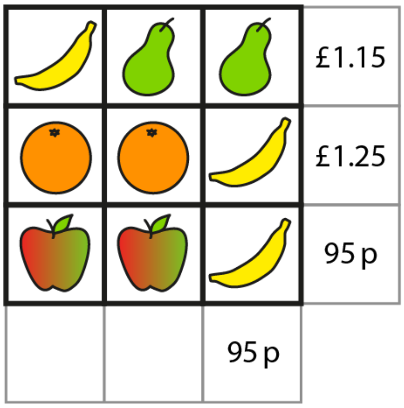
- *'The scales show the masses of some large and small bags of dog food.'*

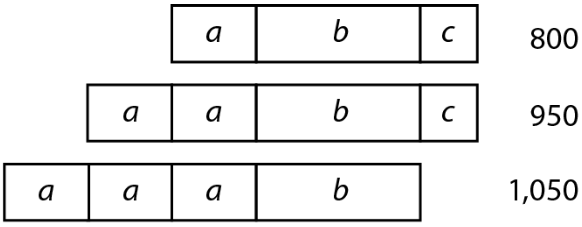


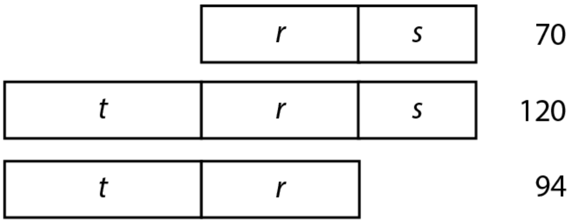


'What is the mass of each of the two different sized bags?'

Dòng nǎo jīn:

Problems	**Teacher's notes – possible strategies**
'Two mugs of hot chocolate and a sandwich costs £6.80. *Two sandwiches and a mug of hot chocolate costs £7.15.* - *How much does a sandwich cost?* - *How much does a mug of hot chocolate cost?'*	1. Draw a bar for each piece of information. 2. Compare bars: a sandwich costs 35p more than a mug of hot chocolate. 3. Add information from step 2 into one of the bars to calculate one of the costs. 4. Substitute known cost into information from step 2 to find the other cost.
'The diagram shows the total cost of the items in each row and column. Fill in the two missing costs.'  	1. Compare top row with right-hand column: a pear costs 20 p more than a banana. 2. Substitute 'cost of pear = cost of banana + 20 p' into top row or right-hand column: cost of a banana = 25 p cost of a pear = 45 p 3. Use the cost of a banana and row two total: cost of orange = 50 p 4. Use the cost of a banana and row three total: cost of apple = 35 p 5. Calculate missing column totals.

'Here is the menu for a very strange café that I visit.' **Menu** **Item** – **Price** Juice – ? Milk – 50 p more than a juice Tea – 75 p more than a juice *'I buy two teas and two milks and the price is £5.70.'* - *'How much does a tea cost?'* - *'How much does a milk cost?'*	1. Draw a bar to represent *'I buy two teas and two milks and the price is £5.70.'* 2. Split the bars so the cost is expressed in terms of the price of juice: (cost of tea = cost of juice + 75 p; cost of milk = cost of juice + 50 p). 3. Observe that: 4 × cost of juice + £2.50 = £5.70 Calculate cost of juice. 4. Calculate cost of milk and cost of tea.
$a + b + c = 800$ $2a + b + c = 950$ $3a + b = 1{,}050$ *'What are the values of a, b and c?'*	1. Draw bars for all three pieces of information, keeping a and b aligned:  2. Compare top two bars: $a = 150$ 3. Either a. subtract a = 150 from the bottom bar and 1,050 (900), then compare to the second bar: $c = 50$ b. substitute $a = 150$ into bottom bar. 4. Substitute $a = 150$ and $c = 50$ into the top bar: $b = 600$

	'The mean of r and s is 35. *The mean of t and r is 47.* $r + s + t = 120$ *Calculate the values of r, s and t.'*	1. From our understanding of the mean: $r + s = 70$ $t + r = 94$ 2. Draw bars for all three pieces of information, keeping unknowns aligned:  3. Compare middle and bottom bars: $s = 26$ 4. Compare middle and top bars: $t = 50$ 5. Substitute $t = 50$ into bottom bar: $r = 44$

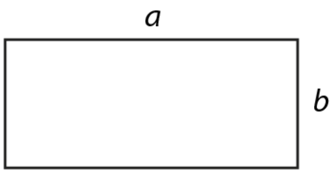
Teaching point 5:

Some problems with two unknowns can't easily be solved using model drawing but can be solved by a 'trial-and-improvement' approach; these problems may have one solution, several solutions or an infinite number of solutions.

Steps in learning

	Guidance	Representations
5:1	We now explore problems with two unknowns that do not lend themselves to model-drawing but instead require a trial-and-improvement approach. The problems considered have one equation and another implicit or explicit condition or property, for example (here all expressed algebraically for teachers only): • $3.2f + 1.5c = 29.9$ the unknowns are both whole numbers (one solution; step *5:1*) • $a + b = 30$ a is a two-digit whole number b is a one-digit whole number (several solutions; step 5:2) • $4s + 6h = 72$ the unknowns are both whole numbers (several solutions; step *5:3*) • $x + 50 = y + 20$ (infinite number of solutions; step *5:4*) The number of solutions depends on the specific criteria and values involved, so always encourage children to think about the problem carefully and, once they have found *a* solution, to consider whether it is the *only* solution. Note, for example that the problems in steps *5:1* and *5:3* have a similar structure, but not the same number of solutions. Begin by presenting a problem with one solution, for example:	*'I spent £29.90 on fish and chips. One fish costs £3.20 and one portion of chips costs £1.50. How many portions of each did I buy?'* (see table below) One solution: *'Seven portions of fish and five portions of chips were bought.'*

Number of portions	Fish	Chips
1	£3.20	£1.50
2	£6.40	£3.00
3	£9.60	£4.50
4	£12.80	£6.00
5	£16.00	**£7.50**
6	£19.20	£9.00
7	**£22.40**	£10.50
8	£25.60	£12.00
9	£28.80	£13.50
10	£32.00	£15.00

	- *'I spent £29.90 on fish and chips. One fish costs £3.20 and one portion of chips costs £1.50. How many portions of each did I buy?'* Note that the fact that both unknowns are whole numbers is implied; you can only buy a whole fish, or a whole portion of chips. Encourage children to work systematically to solve this problem (they first encountered systematic working in segment *1.3 Composition of numbers: 0–5*), making a list of multiples of £3.20 and multiples of £1.50, before looking for a combination of multiples that makes a total of £29.90. Children need to think about how many multiples they might need in their table; they don't need to go over £29.90 in either column. Having found nine or ten multiples of £1.50, children should check to see if they can find a solution, before working out more multiples. Once children have found one solution (seven fish and five portions of chips), ask them if there are any more solutions and how they might check (by completing the multiple table up to £29.90 in the 'chips' column).	
5:2	Now present a problem with several solutions, for example: - *'A rectangle with sides* a *and* b *has a perimeter of 30 cm.* a *is a two-digit whole number and* b *is a one-digit whole number. What are the possible values of* a *and* b*?'* Before starting to find the solution, encourage children to draw a diagram to deduce that $a + b = 15$ *cm*. Then, as in the previous step, work systematically to ensure all possible solutions are found (beginning, for example, with a as the smallest possible two-digit number: ten).	*'A rectangle with sides* a *and* b *has a perimeter of 30 cm.* a *is a two-digit whole number and* b *is a one-digit whole number. What are the possible values of* a *and* b*?'*  perimeter = 30 cm so $a + b =$ half of 30 cm = 15 cm

You may wish to ask children to reason why $a = 15$ and $b = 0$ isn't a solution. These values satisfy the conditions $a + b = 15$, a is a two-digit number and b is a one-digit number, but they don't make sense in the context of the problem (since the rectangle would become a line); some children might incorrectly include this solution if they get 'carried away' filling in the table, forgetting the context of the problem, which imposes its own implicit criterion (namely, that both unknowns must be greater than zero).

a **(two-digit) (cm)**	***b*** **(one-digit) (cm)**	***a* + *b*** **(cm)**
10	5	15
11	4	15
12	3	15
13	2	15
14	1	15

5:3

Now present a problem that combines skills from the preceding problems. In step *5:1*, children learnt to systematically list multiples and spot the single solution to the problem. In step *5:2*, they again worked systematically, but the full set of solutions were the adjacent rows of the table. The slightly more complex problem opposite is similar to that in step *5:1*, but there are several solutions. Note that, again, the fact that the solutions are whole numbers is implied (Zac can't take, for example, half a hexagon).

As in step *5:1*, children must consider how many rows of the table to complete (up to 72 sides for both squares and hexagons). Encourage them to explain how they know they have found all possible solutions (the solutions must be between the two 'end cases' of all squares or all hexagons).

Note that although this problem ($4s + 6h = 72$) has a similar structure to the fish-and-chips problem in step *5:1* ($3.2f + 1.5c = 29.9$), *this* problem has several solutions due to the numbers involved. In general, if there are two unknowns and only one equation (as

'Zac has some squares and hexagons. He chooses some of the shapes and when he counts the number of sides, he finds that the total is 72. How many of each shape could he have?'

Number of each shape	**Squares**	**Hexagons**
0	○ 0	○ 0
1	4	6
2	8	☆ 12
3	⬠ 12	18
4	16	▭ 24
5	20	30
6	□ 24	◇ 36
7	28	42
8	32	□ 48
9	◇ 36	54
10	40	⬠ 60
11	44	66
12	▭ 48	○ 72
13	52	
14	56	
15	☆ 60	
16	64	
17	68	
18	○ 72	

	for both steps *5:1* and *5:3*) there will be an infinite number of solutions. However, we get one solution (in step *5:1*) or seven solutions (here) because the context limits us to positive, whole numbers (we are not allowed, for example, 11.5 hexagons and 0.75 squares, or *14* hexagons and −3 squares). For this problem, there are several solutions even with the restriction to positive, whole numbers, because three squares have the same number of sides as two hexagons; so once we have found one solution (for example, 12 hexagons and zero squares) we can successively replace two hexagons with three squares. The fish-and-chips problem in step *5:1* would have multiple solutions if, for example, one fish cost £3, one portion of chips cost £1.20 and the total was big enough (e.g. £31.20) to allow for you to get a solution (e.g. ten fish and one portion of chips) and then start swapping two fish for five chips (for example, eight fish and six portions of chips). Although children do not need this depth of understanding, it is useful to consider this when you are setting problems and it also highlights the importance of encouraging children to always check that they have found all possible solutions. You can set similar problems using different shapes, leading to different common multiples.	Several solutions: • 0 squares and 12 hexagons • 3 squares and 10 hexagons • 6 squares and 8 hexagons • 9 squares and 6 hexagons • 12 squares and 4 hexagons • 15 squares and 2 hexagons • 18 squares and 0 hexagons
5:4	Now explore the final type of problem – one with an infinite number of solutions. Obviously, we cannot find all possible solutions, so encourage children instead to spot patterns and generalise the relationships between the numbers involved. Present a problem of this type, such as the example opposite. Begin as before,	*‘What could the missing numbers be?’* ☐ + 50 = ☐ + 20

starting with a simple value for one unknown and working systematically, increasing or decreasing that value, to produce a table. After completing around half-a-dozen rows, ask children to comment on the patterns and generalise using what they know about balancing equations, for example:

- *'I notice that the value of the second missing number is always thirty more than the value of the first missing number.'*
- *'This is because the number given on the left-hand side of the equation (fifty) is thirty more than the number given on the right-hand side of the equation (twenty).'*
- *'I know that that the second missing number needs to be bigger than the first so that the equation balances.'*

Once children have generalised, ask them:

- *'Give me a solution where both missing values are three-digit numbers.'*
- *'Give me a solution where the first missing value is a single-digit number.'*
- *'Give me a solution where the second missing number is greater than one thousand.'*
- *'Could the first missing number be three-point-ninety-four? What would the other missing number be in that case?'*

  etc.

Note, that the final question challenges a common assumption that children will make – namely that because the given addends are whole numbers, the unknowns must be whole numbers. There are also solutions where one, or both, of the unknowns are negative numbers, but these are not considered here.

Example table of solutions:

Value of first missing number	**Value of one side of the equation**	**Value of second missing number**
0	50	30
10	60	40
20	70	50
30	80	60
40	90	70

5:5

Now explore some related problems by changing the known addends and seeing how this affects the relationship between the unknowns, for example:

- ☐ + 55 = ☐ + 25

  (same difference)
- ☐ + 60 = ☐ + 20

  (left-hand addend increased by 10, right-hand addend the same)
- ☐ + 70 = ☐ + 20

  (left-hand addend increased again; right-hand addend the same)

Value of first missing number	Value of one side of the equation	Value of second missing number
☐ + 50 = ☐ + 20		
0	50	30
10	60	40
20	70	50
☐ + 55 = ☐ + 25		
0	55	30
10	65	40
20	75	50
☐ + 60 = ☐ + 20		
0	60	40
10	70	50
20	80	60
☐ + 70 = ☐ + 20		
0	70	50
10	80	60
20	90	70

5:6

You could then investigate a similar problem, but now with subtraction expressions. Once again, children should comment on the solutions they have found related to a comparison of the numbers in the equation. Similarly to step *5:5*, you could then present related equations where the difference between the subtrahends is the same (e.g. *? – 70 = ? – 30*) or where one subtrahend is kept the same, while the other is changed (*? – 60 = ? – 30*).

'What could the missing numbers be?'

☐ – 60 = ☐ – 20

Example table of solutions:

Value of first missing number	Value of one side of the equation	Value of second missing number
70	10	30
80	20	40
90	30	50
100	40	60
110	50	70

5:7

Complete this teaching point by providing some independent practice, such as the examples shown opposite and below:

- *'Miriam has some triangles and pentagons. She chooses some of the shapes and when she counts the number of sides, she finds that the total is 45. How many of each shape could she have?'*
- *'A baker is packing cakes into boxes to take to a shop. He has 60 cakes. A small box can hold 8 cakes and a large box can hold 12 cakes.'*
  - *'How many different ways can he pack the cakes?'*
  - *'How can he pack the cakes with the fewest number of boxes?'*
- *'Angie and Ted both have some marbles. If Angie gave away 20 marbles and Ted gave away 40 marbles they would both then have the same number of marbles. How many marbles might Angie and Ted have?'*

Dòng nǎo jīn:

- *'What could the missing numbers be?'*

  $2\,\boxed{} = 4\,\boxed{} - 6$

- *'What are the possible values of whole numbers j and k?'*

  $j + k = 50$

  $3j < 20$

  $2k > 80$