MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Multiplication and Division

2.18 Using equivalence to calculate

Teacher guide | Year 5

Teaching point 1:

For multiplication, if there is a multiplicative *increase* to one factor and a corresponding *decrease* to the other factor, the product stays the same.

Teaching point 2:

For division, if there is a multiplicative change to the dividend and a corresponding change to the divisor, the quotient stays the same.

Overview of learning

In this segment children will:

- explore how, for multiplication, the product will stay the same if one factor is multiplied by a number and the other factor is divided by the same number
- explore how, for division, the quotient will stay the same if the dividend and the divisor are both multiplied (or divided) by the same number
- learn to recognise where these concepts can be applied to make calculation easier.

This segment explores how the factors in a multiplication equation can be manipulated while preserving the product (*Teaching point 1*) and how the dividend and divisor in a division equation can be manipulated while preserving the quotient (*Teaching point 2*). In each case, the exploration begins by examining pairs, or sequences, of equations that have the same product/quotient, using simple numbers in familiar contexts so that children can more easily focus on the structures.

In the case of multiplication, arrays and the grid method are used initially to help children understand and reach the generalisation: ***'If I multiply one factor by ___, I must divide the other factor by ___ for the product to stay the same.'***

In the case of division, the bar model and pictorial representations are used to help children understand and reach the generalisations:

- ***'If I multiply the dividend by ___, I must multiply the divisor by ___ for the quotient to stay the same.'***
- ***'If I divide the dividend by ___, I must divide the divisor by ___ for the quotient to stay the same.'***

Note that, before beginning this segment, children must already be secure in accurately using the language of dividend, divisor and quotient.

The generalisations are applied to balancing equations, writing sequences of related equations, and simplifying calculations involving larger numbers to make them easier to solve.

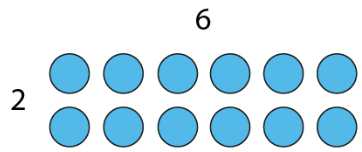
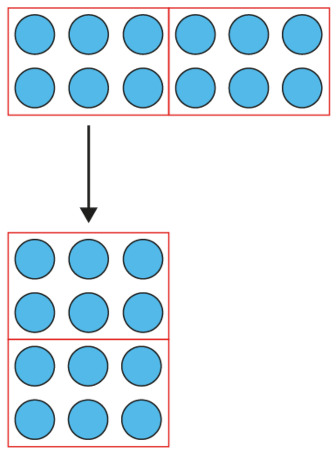
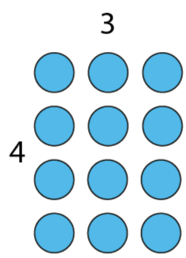
By the end of this segment, children should be able to confidently identify and explain relationships between pairs/sequences of expressions and equations, and work flexibly to make sensible choices about calculation strategies. Learning from *Teaching point 2* is critical to children's understanding of equivalent fractions (*Spine 3: Fractions*, segment *3.7*).

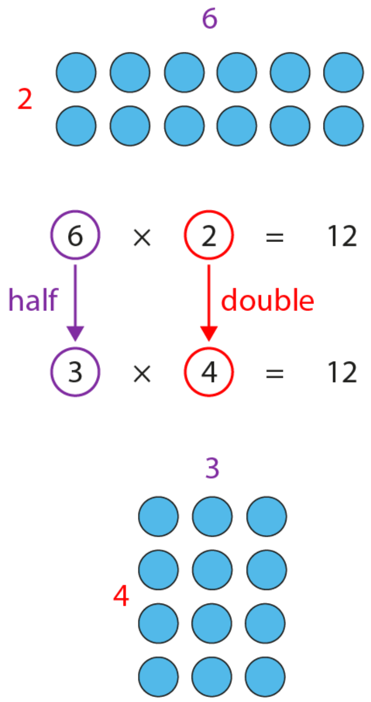
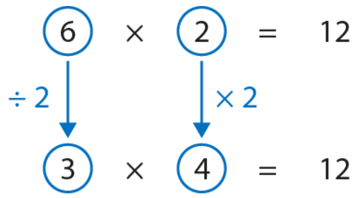
An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Teaching point 1:

For multiplication, if there is a multiplicative *increase* to one factor and a corresponding *decrease* to the other factor, the product stays the same.

Steps in learning

	Guidance	Representations
1:1	Children have already considered the effect on the product of halving one factor and doubling the other, in the context of the relationship between: • the five and ten times tables (segment *2.5 Commutativity (part 2), doubling and halving, Teaching point 4*) • the two and four times tables, and the four and eight times tables (segment *2.7 Times tables: 2, 4 and 8, and the relationship between them, Teaching points 2* and *4*) • the three and six times tables (segment *2.8 Times tables: 3, 6 and 9, and the relationship between them, Teaching point 3*). Steps *1:1–1:5* of this teaching point build on this understanding, until children can generalise and use the knowledge to support calculation strategy. Begin by exploring a known fact, such as $6 \times 2 = 12$. Use an array to represent the fact (opposite this is represented as two rows of six counters), then show how we can split the array in half and rearrange the two portions to represent a different multiplication fact. Ask children to identify the new fact ($3 \times 4 = 12$), writing the corresponding equation (note that both equations are written in the form *row length × column height = product* to facilitate comparison). Compare the two equations, supported by the arrays, and make the following observations:	Step 1 – use an array to represent a known fact:  $6 \times 2 = 12$ • *'Six times two is equal to twelve.'* Step 2 – move half of the array:  Step 3 – identify the new fact:  $3 \times 4 = 12$ • *'Three times four is equal to twelve.'*

	• One of the factors (6) has been halved; the number of columns in the array has been halved. • The other factor (2) has been doubled; the number of rows in the array has been doubled. • The product has remained the same; the number of counters in the array has remained the same. Note that, opposite, *6 × 2 = 12* is represented as two rows of six, and then the *rows* are split and rearranged to make four rows of three. When exploring other facts, you could work by splitting the *columns* in half and rearranging (in this case, *6 × 2 = 12* could be represented as six rows of two, and then the columns split and rearranged to make three rows of four). Repeat for a range of similar calculation pairs, keeping within times table facts (and keep to whole numbers by avoiding halving odd factors). Work towards the following generalisation: ***'If I double one factor, I must halve the other factor for the product to stay the same.'***	Step 4 – compare: 
1:2	Return to the previous example, and discuss how we can also use the following generalisation: ***'If I multiply one factor by two, I must divide the other factor by two for the product to stay the same.'*** This generalisation will be developed further in step *1:6*, when 'two' can be replaced by any number.	

1:3

Now explore doubling one factor and halving the other with *two-digit* × *single-digit* calculations; you can use the grid representation introduced in segment *2.10 Connecting multiplication and division, and the distributive law*, *Teaching point 3*. Demonstrate how what we have learnt about the effect of doubling one factor and halving the other can be used to simplify a calculation such as 5×18:

- Represent the first calculation (here, 5×18) with the grid model.
- Split the rectangle in half, and rearrange, to represent doubling one factor (here, 5) and halving the other (here, 18).

Ask children:

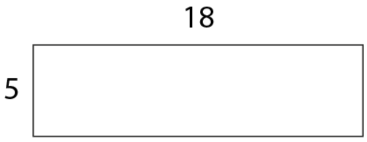
- *'What has changed?'*
  (one factor has halved, as has the width of the rectangle; the other has doubled, as has the height of the rectangle)
- *'What has stayed the same?'*
  (the area of the rectangle, which represents the product, has stayed the same)

Use the simplified calculation to find the product of the original calculation. You can also represent the equivalence using a number line, as shown below.

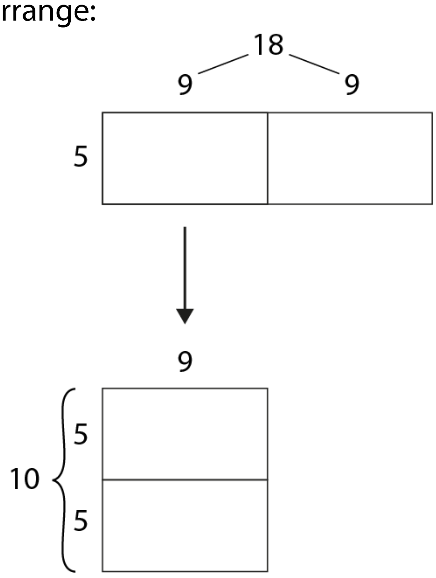
Grid model:

Step 1 – represent the calculation as a rectangle:

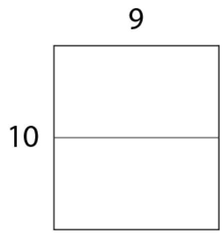
5×18



Step 2 – split the rectangle in half and rearrange:

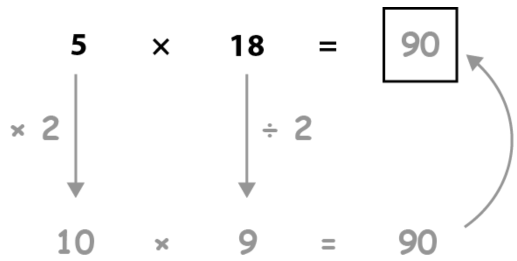


Step 3 – identify the new fact:

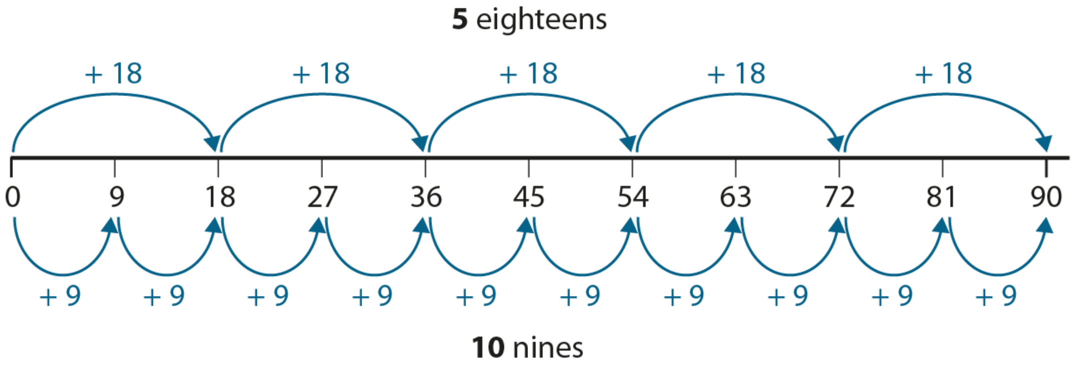


$10 \times 9 = 90$

Step 4 – complete the original calculation:

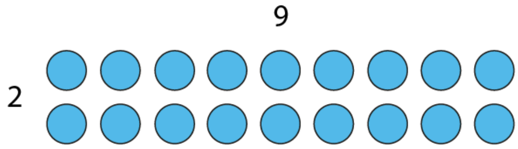
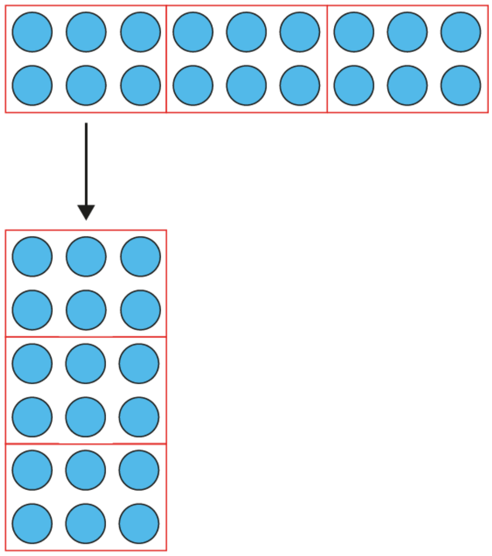
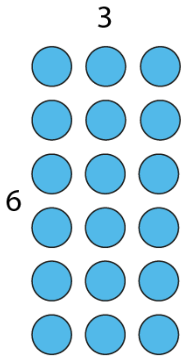


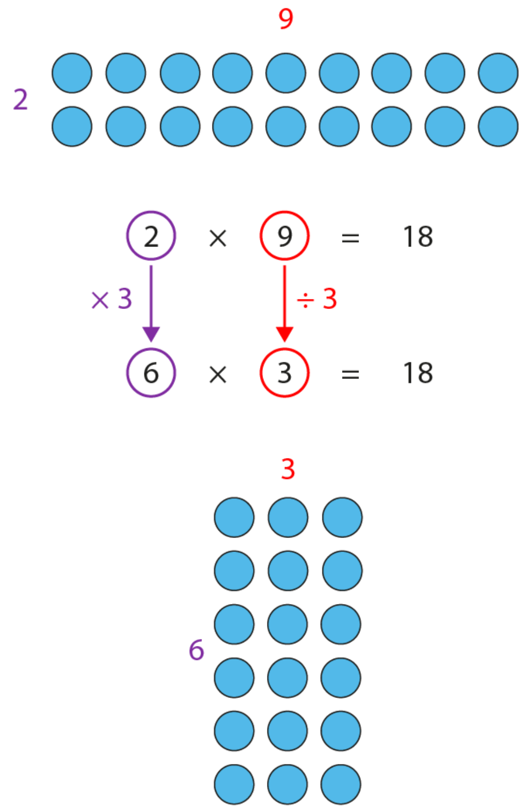
$5 \times 18 = 90$

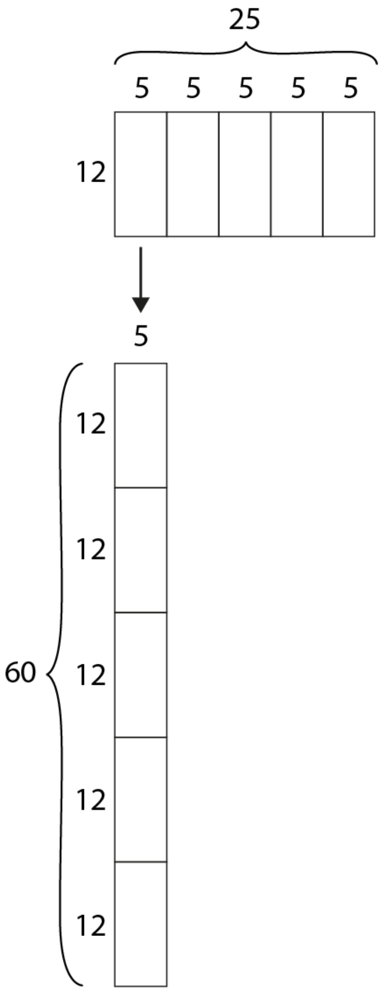
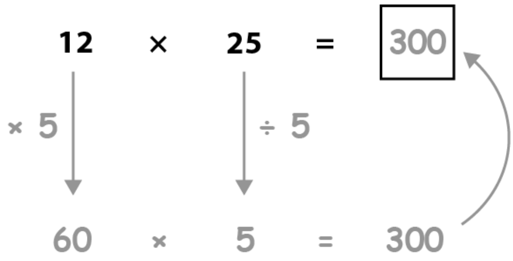
	Number line: 	

1:4	Work through a variety of problems, as a class, without the array or grid scaffolding, including:
	• balancing equations (e.g. $12 \times 2 = 6 \times ?$); you could work through the following sequence:
	  ◦ $2 \times 2 = 1 \times 4$
	  ◦ $6 \times 2 = 3 \times 4$
	  ◦ $8 \times 2 = 4 \times 4$
	  ◦ $10 \times 2 = 5 \times 4$
	  ◦ $12 \times 2 = 6 \times 4$
	(note that $4 \times 2 = 2 \times 4$ has been omitted, but you could include it and discuss how else we know that 4×2 is equal to 2×4)
	• writing a new equation, given an existing one, by doubling one factor and halving the other
	• using this approach to simplify and complete a given calculation (e.g. using 3×70 to calculate 6×35).
	Throughout, continue to use the generalisations from steps *1:1* and *1:2*.	

1:5	At this point, provide children with independent practice involving doubling one factor and halving the other to preserve the product, including: • balancing equations • writing sequences of multiplication equations • applying the generalisation as a strategy for calculation. The dòng nǎo jīn problems opposite apply the calculation strategy when the multiplicand or multiplier is 'something-and-a-half'. Children should draw on what they know about two halves making a whole to double the mixed numbers, thus simplifying the calculations. You could then ask children to generate their own examples using halves. Note that, when working with such examples, the term 'factor' can no longer be used (factors are whole numbers). In such cases, avoid the generalisations from steps *1:1* and *1:2*, and use the following stem sentence: ***'If I multiply ___ by two, I must divide ___ by two for the product to stay the same.'***	Balancing equations: *'Fill in the missing numbers.'* $4 \times 8 = 2 \times \square$ $3 \times 8 = \square \times 4$ $32 \times \square = 16 \times 10$ $24 \times 4 = \square \times 8$ $\square \times 2 = 24 \times 4$ $96 \times 1 = 48 \times \square$ $2000 \times 10 = \square \times 5$ Writing sequences of multiplication equations: *'Use what you have learnt about doubling one factor and halving the other, to write equations that have the same product as:'* $248 \times 1 = 248$ Calculation strategy: *'Use what you have learnt to carry out these calculations.'* $6 \times 22 = \square$ $4 \times 15 = \square$ $125 \times 6 = \square$ Dòng nǎo jīn: $5\frac{1}{2} \times 6 = 11 \times \square$ $\square \times 1\frac{1}{2} = 8 \times 3$

1:6	Now begin to explore multiplying one factor by a number other than two and dividing the other factor by the same value. Begin with an example where the factors are multiplied/divided by three (e.g. *2 × 9* vs *6 × 3*). Use an array, in a similar way to that described in step *1:1* and as shown opposite. When comparing the pair of equations, adapt the generalisation from step *1:2*: ***'If I multiply one factor by three, I must divide the other factor by three for the product to stay the same.'***	Step 1 – use an array to represent a known fact:  2 × 9 = 18 • *'Two times nine is equal to eighteen.'* Step 2 – split the array into three and rearrange:  Step 3 – identify the new fact:  6 × 3 = 18

		Step 4 – compare: 
1:7	Work through some other examples, multiplying/dividing factors by other single-digit numbers. For larger factors, you can use the grid representation, as in step *1:3* and on the next page. Note that the example provided on the next page doesn't necessarily reflect the 'best' strategy for the calculation (transforming to *6* × *50* is probably a more efficient choice for calculating *12* × *25*), however, it is explored to reveal the structure. Use the following stem sentence: ***'If I multiply one factor by ___, I must divide the other factor by ___ for the product to stay the same.'***	

		
1:8	Similar to step *1:4*, work through a variety of problems, as a class, without the array or grid scaffolding, including: • balancing equations • writing a new equation, given an existing one • using this approach to simplify a given calculation. Include examples where the factors are multiplied/divided by 10 or 100. Throughout, continue to use the stem sentence from step *1:7*.	
1:9	To complete this teaching point, provide children with independent practice as shown on the next page. Note that, in many cases, there is more than one calculation strategy; sometimes there is not necessarily a 'best' choice, while in other cases one choice may clearly be preferable. It is worth discussing such cases as a class (for example, in the case of 160×4 on the next page, you may wish to discuss whether children prefer transforming it to 16×40 or to 80×8).	

In the dòng nǎo jīn problem below, both factors are two-digit integers; children may adjust from *10* $\times$ *110* to get *20* $\times$ *55* or adjust from *11* $\times$ *100* to get *22* $\times$ *50*.

Balancing equations:

'Fill in the missing numbers.'

$24 \times 8 = 12 \times \square$

$24 \times 8 = \square \times 32$

$24 \times 8 = 4 \times \square$

$60 \times 4 = 6 \times \square$

$70 \times 8 = \square \times 80$

$4 \times 600 = 400 \times \square$

Writing multiplication equations:

'Write an expression equivalent to the one below by dividing one factor by six and multiplying the other factor by six.'

18×4

True/false problems:

'Decide whether each equation is correct or incorrect. Explain your answers.'

	Correct (✓) or incorrect (✗)?
$6 \times 20 = 3 \times 40$	
$7 \times 12 = 21 \times 4$	
$8 \times 60 = 4 \times 30$	
$12 \times 6 = 4 \times 18$	
$16 \times 9 = 4 \times 18$	
$20 \times 16 = 80 \times 4$	

Calculation strategies:

- *'Fill in the missing numbers and expressions.'*

18×6 may be easier to calculate as $9 \times \square$

24×6 may be easier to calculate as $\square \times 12$

6×33 may be easier to calculate as

$\square \times \square$

160×4 may be easier to calculate as ________

- *'Try writing your own sentence like the ones above.'*

Dòng nǎo jīn:

'Put one digit into each box so that this equation is correct.'

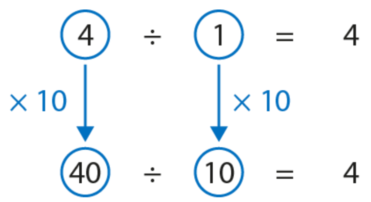
$\square\square \times \square\square = 1100$

'Can you find another solution?'

Teaching point 2:

For division, if there is a multiplicative change to the dividend and a corresponding change to the divisor, the quotient stays the same.

Steps in learning

	Guidance	Representations
2:1	This teaching point explores how the quotient is preserved when the dividend and the divisor are scaled by the same amount. Division represents a proportional relationship between the dividend and the divisor and, just as when working with fractions, where multiplication/division of the numerator and denominator by the same value results in an equivalent fraction, multiplication/division of the dividend and divisor results in an equivalent calculation. Unlike similar comparison for multiplication in *Teaching point 1*, children have not already explored the effect of doubling or halving both the dividend and the quotient; so, instead of beginning by applying a scale factor of two or one-half to the dividend and divisor, begin with scale factors of ten and one-tenth. This builds on what children learnt about the effect of scaling the dividend by ten in segment *2.13 Calculation: multiplying and dividing by 10 or 100*, and also reveals the patterns more clearly; the process essentially represents unitising in tens. Begin by exploring a pair of division equations such as *4 ÷ 1 = 4* and *40 ÷ 10 = 4* using a quotitive division context about length to introduce the problem; for example, *'A four/forty-metre length of ribbon is cut into one/ten-metre lengths; how many lengths are made?'* Represent the calculations using a bar model, demonstrating how the same	Step 1 – represent the two contexts: • *'A four-metre length of ribbon is cut into one-metre lengths; how many lengths are made?'* [4 m | 1 m | 1 m | 1 m | 1 m] 4 ÷ 1 = 4 • *'Four lengths are made.'* • *'A forty-metre length of ribbon is cut into ten-metre lengths; how many lengths are made?'* [40 m | 10 m | 10 m | 10 m | 10 m] 40 ÷ 10 = 4 • *'Four lengths are made.'* Step 2 – compare: 

Bar models (Representations, Step 1):

4 m			
1 m	1 m	1 m	1 m

40 m			
10 m	10 m	10 m	10 m

bar model, just with different numbers, can represent both problems. Emphasise that in both cases the ribbon has been divided into four equal pieces.

Then compare the two equations, supported by the bar models, and make the following observations:

- The dividend (4) has been multiplied by ten (the value represented by the whole is ten times the size).
- The divisor (1) has been multiplied by ten (the value represented by each of the parts is ten times the size).
- The quotient has remained the same (the number of parts has remained the same).

Encourage children to explain why the quotient is the same by recognising that although we have ten times the length of ribbon, the pieces we are cutting it into are also ten times the length. You can use a number line or Gattegno chart to further support this understanding, counting in ones to four vs counting in tens to forty.

Then extend, writing a new equation with the dividend and divisor multiplied by ten again, continuing until children are comfortable with the pattern (i.e. *400 ÷ 100 = 4, 4000 ÷ 1000 = 4).* Use the following generalisation: ***'If I multiply the dividend by ten, I must multiply the divisor by ten for the quotient to stay the same.'***

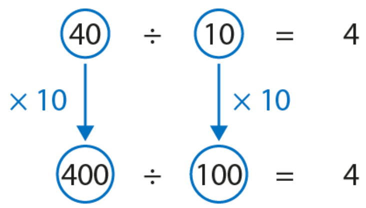
Step 3 – extend:

- 'A four hundred-metre length of ribbon is cut into one hundred-metre lengths; how many lengths are made?'

400 m			
100 m	100 m	100 m	100 m

400 ÷ 100 = 4

- 'Four lengths are made.'



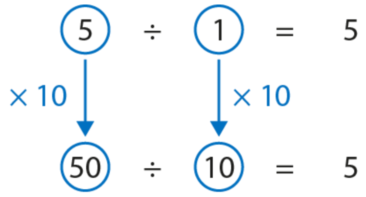
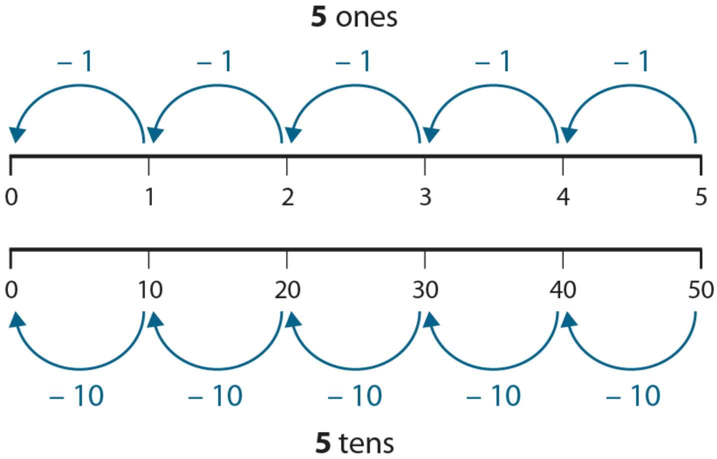
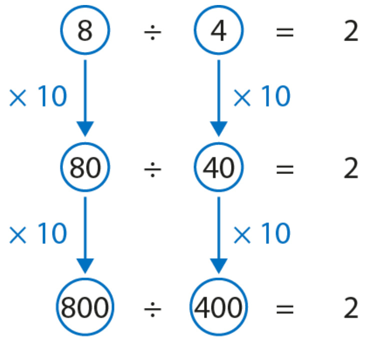
Gattegno chart:

1,000	2,000	3,000	4,000	5,000	6,000	7,000	8,000	9,000
100	200	300	400	500	600	700	800	900
10	20	30	40	50	60	70	80	90
1	2	3	4	5	6	7	8	9

2:2

Work through some different contexts, comparing pairs of equations, including partitive division problems such as:

- *'If £5 is "shared equally" between one person, the person will get £5.'*
- *'If £50 is shared equally between ten people, they will get £5 each.'*
- etc…

	Encouraging children to explain why the quotient is the same by recognising that although there is ten times the amount of money, there are also ten times as many people to share it between. You can use stacked number lines or the Gattegno chart for support. Continue to use the generalisation introduced in step *2:1*. Also explore abstract sets of equations (in the absence of a context), such as: • 2 ÷ 1 = 2 • 20 ÷ 10 = 2 • 200 ÷ 100 = 2 Some children may also directly compare, for example, *2 ÷ 1 = 2* with *200 ÷ 100 = 2*; amend the generalisation accordingly.	Comparing equations: • *'If £5 is "shared equally" between one person, the person will get £5.'*  • *'If £50 is shared equally between ten people, they will get £5 each.'* Number line: 
2:3	Now explore sets of equations that do not involve a divisor of 1, 10 or 100, but for which the dividend and divisor are still *scaled* by a factor of 10; for example: • 8 ÷ 4 = 2 • 80 ÷ 40 = 2 • 800 ÷ 400 = 2 You could ask children to write their own sequences of equations in which the dividend and divisor are both scaled by a factor of ten, with the quotient remaining the same.	

2:4

Although the effect of making the dividend and divisor one-tenth the size (dividing each by ten) is implicit in the previous steps, take a moment to consider this explicitly. Using a similar approach to the previous steps, first consider the same set of equations as in step *2:1*, but now carry out the comparison 'in reverse' (see *Example 1* opposite). Use the generalisation:

'If I divide the dividend by ten, I must divide the divisor by ten for the quotient to stay the same.'

It is important to avoid descriptions such as 'ten times smaller'. Instead, describe relationships using either:

- division by ten (as in the generalisation); for example, *'The dividend was seven thousand. It has been divided by ten. The dividend is now seven hundred.'*

or

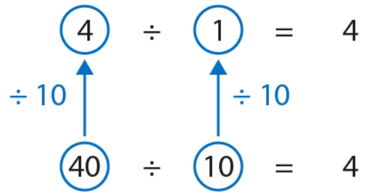
- fractional language (connecting to what children learnt in segment *2.17 Structures: using measures and comparison to understand scaling* and *Spine 3: Fractions* segment *3.6* about multiplication by a unit fraction being equivalent to division by the denominator); for example, *'Seven hundred is one-tenth the size of seven thousand.'*

As in step *2:3*, also explore sets of equations that do not involve a divisor of 1, 10 or 100 but for which the dividend and divisor are still *scaled* by a factor of one tenth (see *Example 2* opposite).

You could ask children to write their own sequences of equations in which the dividend and divisor are both scaled by a factor of one-tenth, with the quotient remaining the same.

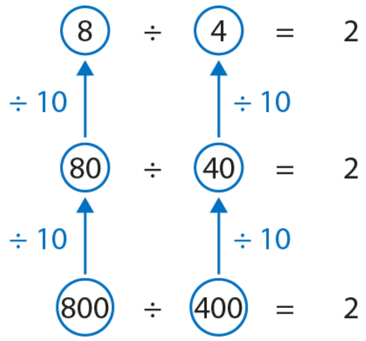
Example 1 – equations and bar models:

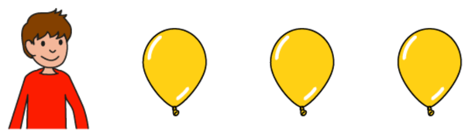
4 m			
1 m	1 m	1 m	1 m

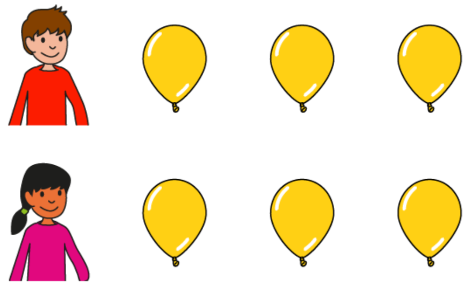
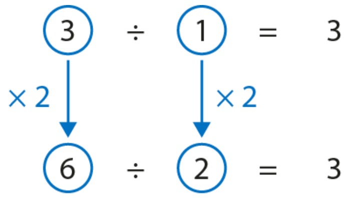
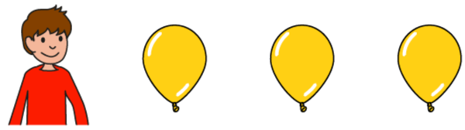
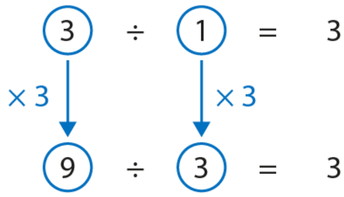
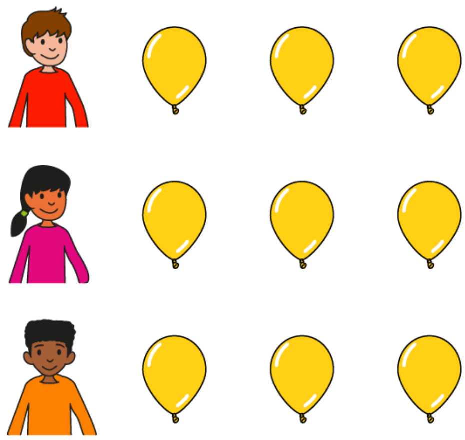


40 m			
10 m	10 m	10 m	10 m

Example 2 – equations only:



2:5	Provide children with some practice, using the generalisations they have learnt so far in this teaching point, including: • balancing equations • writing sequences of division equations • applying the generalisations as a strategy for calculation.	Balancing equations: *'Fill in the missing numbers.'* $6 \div 1 = 60 \div \square$ $60 \div 10 = \square \div 100$ $6 \div 2 = 60 \div \square$ $60 \div \square = 600 \div 200$ $120 \div 2 = 1200 \div \square$ $120 \div 3 = 1200 \div \square$ $120 \div 4 = 1200 \div \square$ $120 \div 6 = 1200 \div \square$ $120 \div 60 = 1200 \div \square$ Calculation strategy: *'Use what you have learnt to carry out these calculations.'* $90 \div 30 = \square$ $150 \div 50 = \square$ $420 \div 70 = \square$
2:6	Now begin to explore scaling the dividend and divisor by a factor other than ten (or one-tenth). Begin with an example in which the dividend and divisor are both doubled, such as *3 ÷ 1 = 3* and *6 ÷ 2 = 3*. A partitive context is exemplified opposite. As in step *2:2*, encourage children to explain why the quotient is the same in each case, by recognising	Step 1 – represent the two contexts: • *'Three balloons are "shared equally" between one child. How many balloons does the child get?'*  $3 \div 1 = 3$

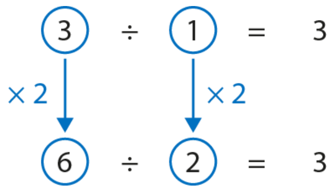
	that although we have twice as many balloons, there are also twice as many people to share them between. Use the generalisation: ***'If I multiply the dividend by two, I must multiply the divisor by two for the quotient to stay the same.'***	• *'<u>Six</u> balloons are shared equally between <u>two</u> children. How many balloons does each child get?'*  $6 \div 2 = 3$ Step 2 – compare: 
2:7	Extend, by considering the first problem in the previous step (*3 ÷ 1 = 3*) but now scaling the dividend and divisor by a factor of three (*9 ÷ 3 = 3*). Continue like this (*3 ÷ 1 = 3* vs *12 ÷ 4 = 3*, *3 ÷ 1 = 3* vs *15 ÷ 5 = 3…*), working towards the following stem sentence: ***'If I multiply the dividend by ___, I must multiply the divisor by ___ for the quotient to stay the same.'***	  

2:8

Now examine the sequence of equations that you built up in the preceding steps:

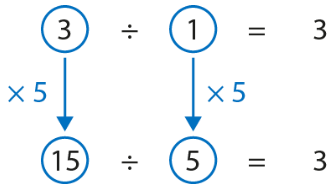
- 3 ÷ 1 = 3
- 6 ÷ 2 = 3
- 9 ÷ 3 = 3
- 12 ÷ 4 = 3
- 15 ÷ 5 = 3

Ask children to explain why these calculations all have the same quotient, prompting them to recognise that each equation in the sequence must be compared to the first equation (*3 ÷ 1 = 3*). Some children may try to explain by comparing each equation to the preceding one, referring to an additive change (i.e. going down the list, three is added to the dividend each time and one is added to the divisor). It is important to reinforce that the quotient is the same because both the dividend and the divisor in the original equation have been *multiplied* by the same value, in each case.



- *'If I multiply the dividend by <u>two</u>, I must multiply the divisor by <u>two</u> for the quotient to stay the same.'*

⋮



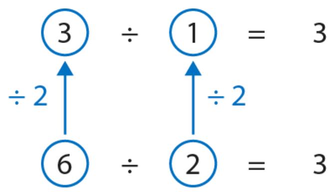
- *'If I multiply the dividend by <u>five</u>, I must multiply the divisor by <u>five</u> for the quotient to stay the same.'*

2:9

Again, although the effect of dividing the dividend and divisor by the same whole number is implicit in the previous steps, take a moment to consider this explicitly. Begin by examining the calculation pair from step *2:6*, this time comparing the calculation with the smaller dividend and divisor (*3 ÷ 1 = 3*) to the one with the larger dividend and divisor (*6 ÷ 2 = 3*).

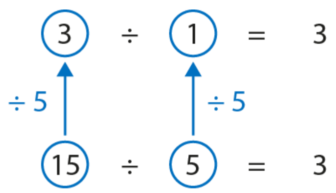
Use the generalisation: ***'If I divide the dividend by two, I must divide the divisor by two, for the quotient to stay the same.'***

As noted in step *2:4*, avoid descriptions such as *'two times smaller'*. Instead, describe relationships using either division by a whole number (divided by two) or fractional language (one-half the size).



- *'If I divide the dividend by <u>two</u>, I must divide the divisor by <u>two</u> for the quotient to stay the same.'*

⋮



- *'If I divide the dividend by <u>five</u>, I must divide the divisor by <u>five</u> for the quotient to stay the same.'*

<table>
<tr><td></td><td>Then work through the sequence of equations from step 2.8, using the following stem sentence: 'If I divide the dividend by ___, I must divide the divisor by ___ for the quotient to stay the same.'</td><td></td></tr>
<tr><td>2:10</td><td>Provide children with some practice, related to scaling the dividend and divisor by the same scale factor, including:
balancing/matching equations
writing sequences of division equations
applying the generalisations as a strategy for calculation.</td><td>Balancing/matching equations:
'Draw a line to match each expression on the left with an equivalent expression on the right.'

450 ÷ 18   480 ÷ 24
110 ÷ 5   300 ÷ 100
420 ÷ 70   225 ÷ 9
240 ÷ 12   440 ÷ 20
1200 ÷ 400   100 ÷ 2
300 ÷ 6   210 ÷ 35

Writing sequences of division equations:
'Write a sequence of division equations that all have a quotient equal to four.'

☐ ÷ ☐ = 4
☐ ÷ ☐ = 4
☐ ÷ ☐ = 4

____

__________________</td></tr>
</table>

		Calculation strategy: *'Use what you have learnt to carry out these calculations.'* 132 ÷ 11 = ☐ so 264 ÷ 22 = ☐ 420 ÷ 10 = 42 so 210 ÷ 5 = ☐ 72 ÷ 8 = 9 so 144 ÷ 16 = ☐ 12 ÷ 3 = 4 so 144 ÷ 36 = ☐
2:11	Complete this segment by providing children with practice applying the learning from both teaching points to contextual problems, for example: • *'Meg has three bags, each containing eighty-four marbles. Bryan has twice as many bags, but the same number of marbles altogether. How many marbles does Bryan have in each bag?'* • *'Thirty-five stickers are shared equally between seven children. If eight times as many stickers are shared equally between eight times as many children, how many stickers does each child get?'*	