National Centre for Excellence in the Teaching of Mathematics

Professional development resource

Unit 3
Lessons 3c and 3d:
Using stories and diagrams to model division and multiplication

Lesson summary

In these lessons we look at different ways in which division (Lesson 3c) and multiplication (Lesson 3d) can be interpreted, by examining division and multiplication stories, by using diagrams to represent the stories, by changing the numbers in the stories and by creating new stories.

In Lesson 3c we use a story involving quotitive division (grouping or measurement) and a story involving partitive division (sharing). Both concern the underlying expression, 3.4 ÷ 16.7, where the divisor is larger than the dividend. This runs counter to our usual expectations and thus will tend to obscure the underlying structure. To make the structure more visible, we substitute 'simpler' numbers into the stories and use a double number line and ratio table to represent the stories.

In Lesson 3d, we start with a 'simple' expression, 3 × 8, and write stories to fit the expression. We then relate the stories to diagrams that express multiplication in a variety of ways, eg as repeated addition, as an array or area, as a cartesian product and as scaling. We then substitute more difficult' numbers into the expression and attempt to modify the stories and diagrams accordingly.

Focus of student learning

In Lesson 3c we use stories and diagrams to help students see that division can involve
- grouping or sharing
- non-whole numbers
- a smaller number being divided by a larger number.

In Lesson 3d we use stories and diagrams to help students see that multiplication can involve
- repeated addition, an array or area, a Cartesian product, scaling
- non-whole numbers
- numbers less than 1.

Lesson preparation

- It is important to try the **Pre-task** with the class a few days before teaching Lesson 3c.
- The file **UNIT-3cd-slides.pdf** contains
  - slides of the **Pre-task**
  - slides of the Lesson 3c Stage 1 and 2 tasks and diagrams
  - slides of the Lesson 3d Stage 1 and 2 tasks and diagrams.

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Lesson 3c Pre-task

Dividing a rod

This pre-task is probably best done as a whole-class activity. The purpose is to see how readily students are able to think of division in terms of quotition (grouping or measuring) or partition (sharing). Treat the task as an assessment- rather than a teaching-activity: stop at the point where students struggle to come up with appropriate models.

We use the idea of a 100 cm rod, marked-off in equal parts, to model the two forms of division. We start with a pair of numbers where both forms are relatively easy to model, and then modifiy the numbers in such a way that the models become more difficult to implement or conceptually more 'strained'.

The drawing shows
a 100 cm long rod.
Mark-off the rod to show

$100 \div 25$

The drawing shows
a 100 cm long rod.
Mark-off the rod in a
different way to show

$100 \div 25$

The drawing shows
a 100 cm long rod.
Mark-off the rod to show

$100 \div 2.5$

The drawing shows
a 100 cm long rod.
Mark-off the rod in a
different way to show

$100 \div 2.5$

The drawing shows
a 100 cm long rod.
Mark-off the rod to show

$100 \div 0.25$

The drawing shows
a 100 cm long rod.
Mark-off the rod in a
different way to show

$100 \div 0.25$

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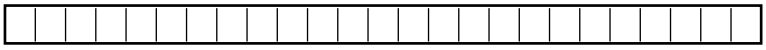
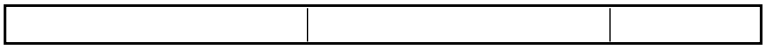
Lesson 3c Pre-task commentary

Dividing a rod



We can think of 100 ÷ 25 in terms of 'sharing' the 100 cm rod among 25 people. This is a fairly straightforward idea, albeit not that easy to put into practice, unless one already knows that the result will be 'a 4 cm piece each'. such a sharing activity might be easier to implement in a discrete context, eg sharing a pack of 100 playing cards, where the result would emerge from the act of dealing the cards.

It is also fairly straightforward to think of this in terms of 'grouping' or 'measurement':
How many 25 cm lengths can be cut from the 100 cm rod?



Students might baulk at 'sharing' the rod amoung 2.5 people, though some might go along with the idea of cutting the rod into 2.5 'equal' pieces, with the result that each (whole) piece is 40 cm long.

The idea of 'grouping' still makes perfect sense here, though it has become more ardous to put into practice:
How many 2.5 cm lengths can be cut from a 100 cm rod?

The idea of 'sharing' the rod amoung 0.25 people is even more tenuous. However, one could remould the task in this kind of way:
I need a wooden rod to fix a banner to my wall. A 100 cm rod is 0.25 of what I need. How long does the rod need to be?

The idea of 'grouping' still makes sense here, though it has become even more ardous to put into practice:
How many 0.25 cm lengths can be cut from a 100 cm rod?

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Lesson 3c

Stage 1.
In this lesson we look at two single-step tasks that turn out to involve division. We vary the numbers in the tasks, but start with 'difficult' numbers, which means that students might not initially recognise that the tasks involve division, even though the tasks are set in accessible, 'real' contexts.

The Stage 1 task involves *quotitive* division ('grouping' or 'measurement'), while the Stage 2 task (page 4) involves *partitive* division ('sharing'). It is likely that students will be more familiar with the latter and so will more readily recognise that the second task involves division (even though they often resort to quotitive division to *perform* division calculations, eg when using chunking or the short or long division algorithm).

For each task, we start by asking students to *estimate* the solution and to consider which keys to press on a *calculator*.

We explore how we can support students' thinking about the structure of the task through
- making sense of the context
- using 'easier' numbers
- using representations (DNL and ratio table)
- inventing similar tasks.

1 Jake has some broken Farley's rusks.
He gives his baby sister a 3.4 g portion.

How much of a rusk is this,
if a single rusk weighs 16.7 g?

a) i. Ask students to estimate the answer (encourage the use of fractions and decimals).

ii. Discuss how the task could be solved using a calculator.
"Can we get the answer by using the numbers 16.7 and 3.4 and a single operation?"

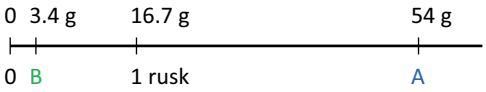
b) i. Ask students to find ways of changing the numbers in the story to make it 'easier'
[ie easier to see *what* to do, and easier to see *how* to do it].
Ask students to
- estimate the solution to the revised task
- decide which keys to press on a calculator.
"Can the original task be solved in the same way?"

ii. Present students with this revised version of the task:
Again, ask students to
- estimate the solution to the revised task
- decide which keys to press on a calculator.
"Can the original task be solved in the same way?"

Jake has some broken Farley's rusks.
His brother grabs a 54 g portion.

How many rusks is this,
if a single rusk weighs 16.7 g?

c) Present students with this double number line.
A. "Could we use the DNL to help us solve the revised task?"
"Could the DNL help us to *explain how* to solve the task?"
B. "Could we use the DNL to help us solve the original task?"
"Could the DNL help us to *explain how* to solve the task?"



d) Present students with tables A and B ('condensed' ratio tables).
A. "Could we use table A to help us solve the revised task?"
"Could the table help us to *explain how* to solve the task?"
B. "Could we use table B to help us solve the original task?"
"Could the table help us to *explain how* to solve the task?"

A

16.7	54
1	?

B

16.7	3.4
1	?

dek

Lesson 3c commentary

Stage 1. Note: **This stage can easily take up a whole lesson**.

1 a) i. Some students might want to operate on the numbers without giving due consideration to the context. This might lead them to consider operations like 16.7+3.4, 16.7–3.4 or 16.7×3.4. If they hit on division, they might well choose 16.7÷3.4 (ie larger number divided by smaller number) rather than 3.4÷16.7.

If students think about the context they are likely to see that the portion is less than a whole rusk, so that the amount of rusk is a fraction (or decimal) less than 1. Some might go on to see that the amount is about $1/5$ or 0.2 of a rusk (eg because 16÷3 ≈ 5 and 1÷5 = $1/5$ or 0.2, or because 3÷16 ≈ $1/5$ or 0.2). However, with these 'difficult' numbers (in particular, that for 3.4÷16.7, the divisor is larger than the dividend), students may well not recognise, at this stage, that the task involves (quotitive) division: How many 16.7s in 3.4?

ii. The calculator can be used successfully in a variety of ways.

For example, students might use trial and improvement to find a number, *A*, such that 16.7 × *A* = 3.4.

Or they might calculate like this: 17.6÷3.4≈4.91, 1÷4.91≈0.20.

However, the calculator can also be used to nudge students towards a more formal approach, ie towards 3.4÷16.7, by asking

Can we get the answer by using the numbers 16.7 and 3.4 and a single operation?
What's the operation, and which number do you key in first?

It can be interesting to consider alternative formulations of *How much of a rusk is this?*

- *How much rusk is this?* allows for the desired fractional answer measured in rusks, but would also admit answers measured in other units, eg grams (ie the given amount, 3.4 g) or pence (eg 2p, as a pack of 18 rusks costs about £2);
- *What fraction of a rusk is this?* spells out what we are looking for and might scaffold the task for students;
- *How many rusks is this?* is a more general formulation and might make the task more challenging.

b) i. The aim here is to encourage students to discern structure by starting from a more familiar/concrete situation and reasoning by analogy. Once students have simplified the task to a point where they can see clearly what operation(s) to perform, can they see that the original task can be solved in the same way?

To start with, students might come up with a revised version of the task such as this:

• A whole rusk weights 15 g. Jake gives his sister a 3 g portion. How much of a rusk is this?

This is likely to lead to the two-step solution 15 ÷ 3 = 5, 1 ÷ 5 = $1/5$ = 0.2,
though for some students it might lead to the direct solution 3 ÷ 15 = $1/5$ = 0.2.

More importantly, try also to elicit revised versions such as this:

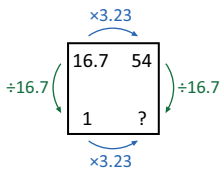
• A whole rusk weights 15 g. Jake gives his sister a 60 g portion. How many rusks is this?

This time, the familiar form of division (ie larger number divided by smaller number) leads straight to the direct solution 60 ÷ 15, and might help students see that not only does this task involve division, it involves quotitive division: How many 15 g rusks make a 60 g portion? or How many 15s in 60?

ii. This is somewhat more abstract than 'How many 15 g rusks make a 60 g portion?'. Do students still recognise that this is of the form 'How many Qs in P?' and that this can be expressed as P÷Q?

c) A. The DNL might support students' thinking by providing a *model of* the revised task in b) ii, and so help students find an informal or formal solution (eg, students might see that 16.7 goes into 54 roughly 3 times). The DNL might also provide some students with a *model for* the underlying, formal mathematical structure.

B. The DNL might enable some students to see that the revised and original tasks have the same structure ('How many Qs in P?') and can be solved in the same way (P÷Q).



d) A. Some students might work formally with the table, eg by finding the multiplier that maps 16.7 g onto 54 g (ie 54÷16.7) and applying this to 1; some might try to find the more abstract divisor that maps 16.7 g onto 1 rusk (ie 16.7) and then apply this to 54.

It might help students to label the rows of the table. Students might then extend the rows, eg by working from 16.7, 1 towards 54, ? by rated addition:

Weight (g)	16.7	54	33.4	50.1
Number of rusks	1	?	2	3

[It is also possible to extend the *columns* of the given table, and work from 16.7, 54 towards 1, ?, but this is much more abstract - what would intermediate values like 8.35, 27 mean?!]

B. As with table A, students might find the multiplier that maps 16.7 onto 3.4 and apply it to 1; or find the multiplier that maps 16.7 onto 1 and apply it to 3.4; or they might extend the table's rows [or columns].

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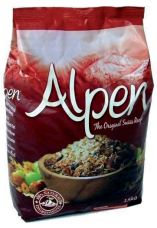
Lesson 3c, continued

Stage 2
The second task (below) involves *partitive* division, ie 'sharing'.
We could bring out the sharing structure by expressing the task like this:
3.4 kg of muesli are shared equally between 16.7 tubs. What does each tub get?

2 Heidi buys a large catering pack of muesli which she pours into storage tubs to keep fresh.

The muesli weighs 3.4 kg and fills 16.7 tubs.

How much muesli does a full tub hold?

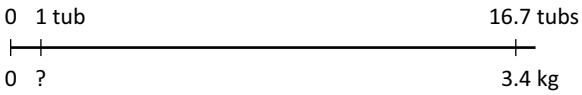


a) i. Ask students to estimate the answer (encourage the use of fractions and decimals).
ii. Discuss how the task could be solved using a calculator.
"Can we get the answer by using the numbers 3.4 and 16.7 and a single operation?"

b) Ask students to find ways of changing the numbers in the story to make it 'easier'
[ie easier to see *what* to do, and easier to see *how* to do it].
Ask students to
- estimate the solution to the revised task
- decide which keys to press on a calculator.

"Can the original task be solved in the same way?"

c) Present students with this double number line.
"Could we use the DNL to help us solve the original task?"
"Could the DNL help us to *explain how* to solve the task?"



d) Present students with this table (a 'condensed' ratio table).
"Could we use the table to help us solve the original task?"
"Could the table help us to *explain how* to solve the task?"

1	16.7
?	3.4

3 Invent some more division stories.

Use nice and not-so-nice numbers.

Classify the stories according to whether they have the same structure as task 1 or task 2.

Lesson 3c commentary, continued

Stage 2. Note: **This stage can also easily take up a whole lesson.**

2 a) i. Student might more readily recognise that this second task is a division task as it involves 'sharing'. However, they might express this as 16.7 ÷ 3.4 (larger number divided by smaller) rather than 3.4 ÷ 16.7.

If students think about the context they are likely to see that a full tub holds less than 1 kg of muesli, ie the number of kg is a fraction (or decimal) less than 1. Some might go on to see that the amount is about 0.2 kg (eg because there are about 5 tubs per kg). However, with these 'difficult' numbers (in particular, that for 3.4÷16.7, the divisor is larger than the dividend), students may well not recognise, at this stage, that the task involves (partitive) division:

3.5 kg are divided into 16.7 (equal?) portions. How big is each (whole?) portion?

ii. The calculator can be used successfully in a variety of ways.

For example, students might use trial and improvement to find a number, *A*, such that 16.7 × *A* = 3.4.

Or they might calculate like this: 17.6÷3.4≈4.91, 1÷4.91≈0.20.

However, the calculator can also be used to nudge students towards a more formal approach, ie towards 3.4÷16.7, by asking

Can we get the answer by using the numbers 16.7 and 3.4 and a single operation?

What's the operation, and which number do you key in first?

b) One way to make the task more accessible is to convert 3.4 kg to 3400 g (thus changing 3.4 ÷ 16.7 to 3400 ÷ 16.7).

You should also try to elicit variants like these:

- The muesli weighs 3 kg and fills 15 tubs.
- The muesli weighs 30 kg and fills 15 tubs (or the muesli weighs 34 kg and fills 17 tubs)
- The muesli weighs 34 kg and is shared between 17 people. What does each person get?

It is also interesting to consider this kind of modification:

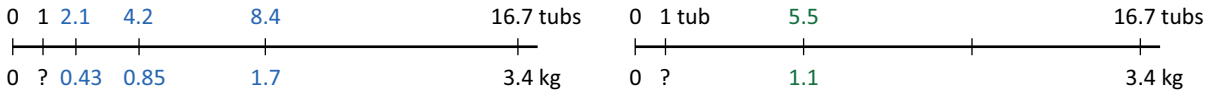
- The muesli weighs 3 kg and fills 0.15 sacks.
- The muesli weighs 3 kg and is shared between 0.15 people. What does each person get?

Can we still think of this as 'sharing'?!

What if we rephrase it like this:

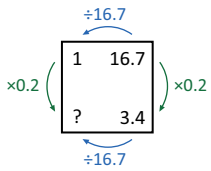
- 3 kg of muesli provides 0.15 of a person's share. What is the weight of a whole person's share?

c) The DNL might support students' thinking by providing a *model of* the original task and so help students find an informal or formal solution (eg, students might see that if we repeatedly halve, starting from 16.7 tubs and 3.4 kg, we will soon get close to 1 tub and about 0.2 kg [below, left]; or, if we divide 16.7 tubs and 3.4 kg by 3, we get about 5 tubs and 1 kg, which is roughly 0.2 kg, or 200 g, per tub [below, right]).



The DNL might also provide some students with a *model for* the underlying, formal mathematical structure (eg, the between-the-lines multiplier that maps 16.7 onto 3.4 [ie 3.4÷16.7 ≈ 0.2] maps 1 onto 0.2 approx).

d) Some students might work formally with the table, eg by finding the divisor that maps 16.7 onto 1 [ie 16.7] and applying this to 3.4; some might try to find the more abstract multiplier that maps 16.7 tubs onto 3.4 kg [ie 3.4÷16.7 ≈ 0.2] and then apply this to 1.



It might help students to label the rows of the table. Students might then extend the rows, eg by working from 16.7, 3.4 down to roughly 1 , ? by repeated halving (below, left), or by initially dividing by 3 (below, right).

[It is also possible to extend the *columns* of the given table, and work from 1, 16.7 towards ?, 3.4, but this is much more abstract - what would intermediate values like 0.5, 8.35 mean?!]

Number of tubs	1	16.7	8.35	4.18	2.09	1.05
Weight (g)	?	3.4	1.7	0.85	0.425	0.2125

Number of tubs	1	16.7	5.5	5	1
Weight (g)	?	3.4	1.1	1	0.2

3 This task provides lots of opportunities, in this and future lessons, to consolidate and develop some of the foregoing ideas. There is plenty of scope for making the task more or less challenging.

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Lesson 3d

Stage 1.
Here we write stories for expressions involving multiplication, starting with the simple whole number expression 3 × 8.

We relate the expressions and stories to various representations and consider whether the stories and representations still work for less simple numbers.

1 Look at this expression:

Here is a story that fits the expression:

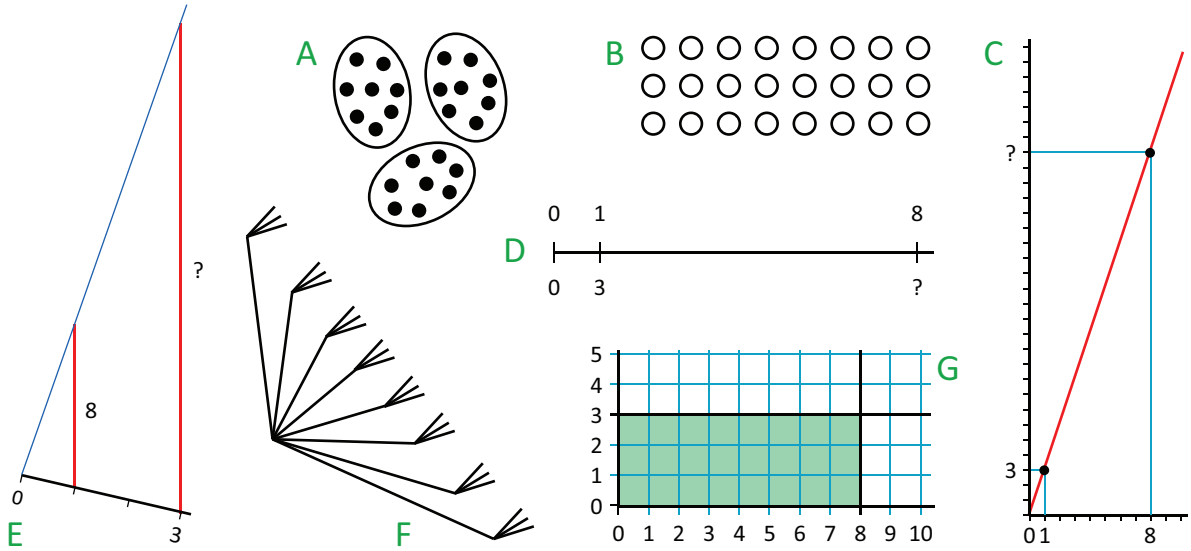
Mike has found 8 conkers.
Julia has found 3 times as many.
How many conkers has Julia found?

$$3 \times 8$$

a) Ask students to write some more stories that fit 3 × 8.

b) Ask students to examine each of these representations for 3 × 8.

Discuss how well each representation fits the various stories.



c) Change the numbers in the expression in some way, eg to

3.2 × 8 or 3.2 × 8.6 or 3 × 0.8 or 0.32 × 0.86, etc.

i. Discuss whether we can modify our stories to fit the new expression.

ii. Discuss whether we can modify the representations to fit the new expression.

Stage 2.

2 Extension: Draw some diagrams to represent this expression:

$$5/8 \times 1/4.$$

3 Extension: Draw some diagrams to represent this expression:

$$5/8 \div 1/4.$$

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Lesson 3d commentary

Stage 1. Note: **This stage can easily take up a whole lesson**.

Multiplication can be thought of in various ways, for example as repeated addition, or in terms of an array or area, or as scaling.

When students are asked to invent stories for a binary expression like 3 × 8, it is likely that they will think first of situations involving repeated addition. Such stories can be challenging to construct because the two numbers are not of the same 'kind', in contrast to stories for addition or subtraction [eg, for 3 + 8, we could write a story involving 3 eggs and 8 eggs, but this wouldn't work for 3 × 8]. The task can be more challenging still (particularly for repeated addition) when the numbers are not both whole.

1 a) The given story involves scaling*.

Students are likely to come up with stories like these:

Julia's coat has 3 pockets. She puts 8 conkers in each pocket. How many conkers does she have altogether?

CDs cost £3 each. What do 8 CDs cost?

CDs cost £8 each. What do 3 CDs cost?

*Scaling can be made more explicit my using *continuous* quantities and replacing 'times as *many*' by 'times as *much*', as in

Mike has grown an 8 ounce marrow.
Julia has grown a marrow that weighs 3 times as much.
What does Julia's marrow weigh?

Stories like the two below (involving an *array* and the *Cartesian product*) are less likely to arise at this stage:

Mike has planted 3 rows of cabbages, with 8 plants in each row. How many plants is this altogether?

A shop sells plastic cups in 3 sizes and 8 different colours. How many different plastic cups are there to choose from?

b) There is no unique, prescribed way of interpreting these representations. However (with a bit of explanation and embellishment) they can all be interpreted as models for 3 × 8. Indeed, they can all be seen as representations of our original story (about Mike's and Julia's conkers), though some do so in a more direct and some in a more abstract way.

A: This provides a very direct model of repeated addition: 3 lots of 8 (but not 8 lots of 3).

B: This provides a model for 3 lots of 8 *and* 8 lots of 3. We can view it as 3 rows of 8 or as 8 columns of 3 (and we can imagine rotating the array through 90˚ without changing the total). If we label the rows (eg small, medium, large) and the columns (eg red, orange, yellow, green, blue, indigo, violet, purple) we can use it to model the Cartesian product.

C: Algebraically, this provides a model of scaling by 3, in particular of scaling 8 by 3 (the red line shows $x \to 3x$ or $y = 3x$.) We could model our original story, though it is fairly abstract - in contrast to A, it does not model the individual conkers.

If we view the diagram geometrically (see E), this provides a model of scaling by 8: we can think of this as an enlargment about the origin, with scale factor 8, of a vertical blue line of length 3 units

D: This again is quite abstract. By operating between-the-lines, we can view this as scaling any number (especially 8) by 3; by operating along-the-lines, we can think of this as operating on 3, eg by scaling or by repeated addition (8 lots of ...).

E: This provides a model of scaling by 3 - the red line of length 8 units is enlarged by a scale factor 3.

F: We can think of this as a scaling by 3 (though not a standard geometric scaling): we start with 8 branches, each of which sprouts 3 branches. More prosaically, we can think of it as 8 lots of 3.

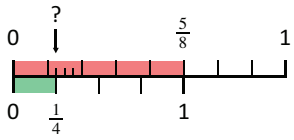
Tree diagrams provide a powerful model for exponential growth. Consider, for example, 3^4: though it might be difficult to draw, it is easy to imagine 3 branches, each of which sprouts 3 branches, and again, and again.

G: This provides an area model of multiplication. It can be thought of as an extension of the array which readily accommodates non-whole numbers (see part c). It can also provide a powerful model of the fact that multiplication is distributive over addition: eg, that 23 x 46 = (20+3)(40+6) = 20x40 + 20x6 + 3x40 + 3x6. However, the model is not unproblematic: many students who know the rule for calculating the area of a rectangle do not understand what they are doing - that in effect they are counting unit squares (and parts of squares).

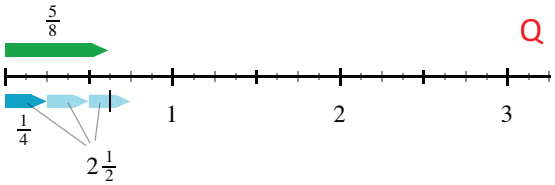
c) As we shift from whole numbers to rational numbers, so the stories will need to shift from involving discrete to involving continuous quantities. At the same time, the representations A, B and F will begin to break down.

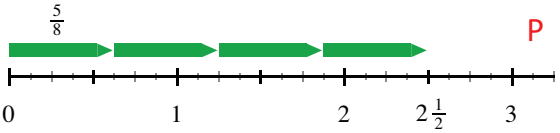
Stage 2. Extension tasks 2 and 3 involve numerical expressions wthout a context and might best be tackled in a separate lesson. You might also want to use slightly less challenging expressions to start with.

2 It is possible, though challenging, to use diagrams like C, D, E and G to represent the expression $\frac{5}{8} \times \frac{1}{4}$. Students might devise a more 'common sense' representation like this one (right).



3 It is again possible, though perhaps even more challenging, to use diagrams like C, D, E and G for the given expression $\frac{5}{8} \div \frac{1}{4}$. More 'common sense' representations are shown below. The first (Q) shows quotitive division (How many $\frac{1}{4}$s in $\frac{5}{8}$?), the second (P) shows partitive division, or sharing (If a $\frac{1}{4}$ portion is $\frac{5}{8}$, what is a whole portion?).





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Adapting the lesson

- These tasks have a strong diagnostic element and should be accessible to most students.
- Suggestions for simplifying the tasks, or for making them more challenging, are built into the lessons.
- Allow plenty of time for students to think about the tasks and to discuss them and to probe them further. Don't worry if the lessons overrun.
- It is worth revisiting ideas from these lessons on a regular basis (!)

Suggestions for lesson study

You might want to consider these aspects of students' thinking:

- How easily do students construct division and multiplication stories for expressions where the numbers are not whole numbers?
- How readily do students see the 'structure' of a story?
- How readily do students grasp that changing the numbers in a story/expression to 'easier' numbers need not alter the structure?
- How readily do students grasp that changing the numbers in a story/expression to 'more difficult' numbers might put a strain on a given model or structure?

Research background to the lesson

School students have a strong tendency to think *Multiplication makes bigger, division makes smaller*. This derives from their early experiences, where multiplication tends to be based on repeated addition, where the numbers are usually whole numbers and where, in the case of division, the divisor is smaller than the dividend and the result is a whole number.

The CSMS Decimals test contains an item which asks students to ring the larger expression in each of these pairs:
8×4 and 8÷4 8×0.4 and 8÷0.4 0.8×0.4 and 0.8÷0.4
In a survey* of a representative sample of Y9 students in 2008/9, only 19 % answered all three parts correctly.

Another CSMS Decimals item involves this 'story':
The cost of 6.22 litres of petrol was £4.86. What would the price of one litre be?
Students are asked to choose one of these six expressions for calculating the answer:
6.22+4.68 6.22÷4.68 6.22×4.68 4.68÷6.22 4.68–6.22 4.68×6.22
Only 17 % of the Y9 students chose the correct expression, 4.68÷6.22, with many going for 6.22÷4.68, where the divisor is smaller than the dividend.

Thus we focus in these lessons on multiplication and division involving non-whole numbers, including numbers less than 1, and we look at a range of models for multiplication, not just repeated addition.

*Hodgen, J., Brown, M., Küchemann, D. & Coe, R. (2010) Mathematical attainment of English secondary school students: a 30-year comparison. Paper presented at the British Educational Research Association (BERA) Annual Conference, University of Warwick.

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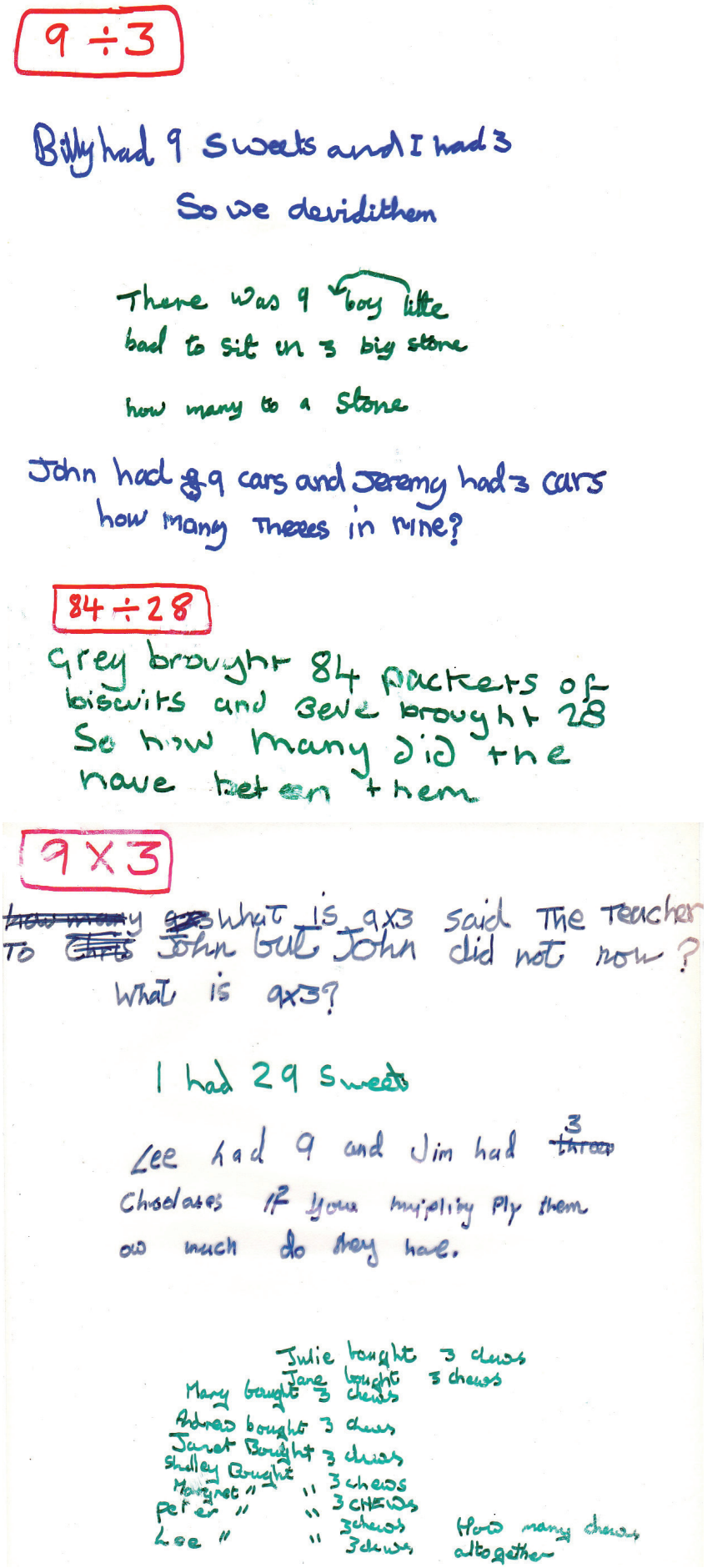
Research background, continued

Writing 'stories' for a binary operation like 8 × 3 is not always easy, especially when the operation is multipication or division (rather than addition or subtraction). Here are some examples given by lower secondary school students in the mid 1970s, for 9÷3, 84÷28 and 9×3. Some of the stories model the given operation effectively but others do not.

The task becomes even more demanding for non-whole numbers and especially for numbers less than one and when the divisor is less than the dividend. To get a feel for this, **it is worth trying to write such stories for oneself**. It is also worth compiling a bank of such stories as a resource for teaching.

In a study* with a representative sample of over 1000 Y7 students, the proportion of students writing appropriate stories for various whole-number expressions was as follows:

84–28	78 %
9÷3	58 %
84÷28	41 %
9×3	42 %
84×28	31 %

Interestingly, division seems to be less demanding here than multiplication.

*Brown, M. & Küchemann, D. (1977) Is it an 'add', Miss? (Part 2). *Mathematics in School*, 6, 1, January 1977.