11 Geometry

Mastery Professional Development

11.2 Reasoning with the properties of a circle

Guidance document | Key Stage 4

Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The fifth of the Key Stage 4 themes (the eleventh of the themes in the suite of Secondary Mastery Materials) is *Geometry*, which covers the following interconnected core concepts:

11.1 Transformations and relative position

11.2 Reasoning with the properties of a circle

11.3 Trigonometry

11.4 3D shapes

This guidance document breaks down core concept *11.2 Reasoning with the properties of a circle* into three statements of **knowledge, skills and understanding**:

11.2 Reasoning with the properties of a circle

- 11.2.1 Proportionality within circles
- 11.2.2 Reason and prove using properties of circles (including circle theorems)
- 11.2.3 Equation of a circle

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

- 11.2.1 Proportionality within circles
  - 11.2.1.1 Use proportionality to calculate the length of an arc
  - 11.2.1.2 Use proportionality to calculate the area of a sector
- 11.2.2 Reason and prove using properties of circles (including circle theorems)
  - 11.2.2.1 Identify and reason with lines associated with a circle (including segments, chords and tangents)
  - 11.2.2.2 Use chains of reasoning to show that the angle at the centre is twice the angle at the circumference
  - 11.2.2.3 Use chains of reasoning to show that the angle in a semicircle is 90 degrees
  - 11.2.2.4 Use chains of reasoning to show that the angles in the same segments from a common chord are equal
  - 11.2.2.5 Use chains of reasoning to show that opposite angles in a cyclic quadrilateral sum to 180 degrees

11.2.2.6 Use chains of reasoning to show that two tangents drawn from a point to a circle are equal

11.2.2.7 Use chains of reasoning to show the alternate segment theorem

11.2.2.8 Use and apply circle theorems to solve problems

11.2.3 Equation of a circle

11.2.3.1 Appreciate that the equation of a circle emerges from the use of Pythagoras' theorem

11.2.3.2 Solve problems involving circles centred on a coordinate grid

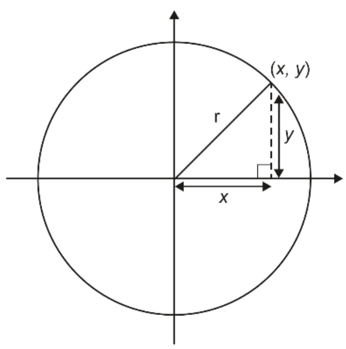
Overview

This core concept builds on students' understanding of circles, bringing together prior knowledge of angle facts and Pythagoras' theorem to explore related theorems and equations. Emphasis is placed on reasoning and proof, with a focus on moving beyond memorisation towards a deeper understanding of the properties of circles.

Students were first introduced to some basic elements of a circle – the radius, diameter and circumference – at Key Stage 2. At Key Stage 3, they learnt to quantify these alongside area. At Key Stage 4, other parts of the circle, including arc and sector are defined, and students' understanding of the formulae for the circumference $C = \pi d$ and area of a circle ($A = \pi r^2$) is developed to facilitate the calculation of lengths of arcs and areas of sectors. Placing an emphasis on the relationship between the arc length formula $\frac{\theta}{360} \times \pi d$ and the circumference formula (and the area of a sector formula $\frac{\theta}{360} \times \pi r^2$ and the formula for the area of a circle) is key to building on existing knowledge and providing an opportunity for students to apply their understanding of proportionality to circles.

Students' deductive reasoning skills are further developed and formalised at Key Stage 4. The introduction of the tangent to a circle, chords and segments and their associated properties, provides an opportunity for students to use and apply their understanding of right-angled, isosceles and similar triangles and related angle facts, to a variety of problems that involve circles. Circle theorems can be introduced as a natural consequence of the basic circle and angle properties: it is important that students understand them as the result of reasoning with these facts, and not merely as theorems to memorise. Proving the circle theorems, particularly how they can all be shown to emerge from the single property that the angle subtended at the centre is twice that subtended from the same chord at the circumference, offers an opportunity for students to consolidate their learning on algebraic manipulation and effective mathematical communication. Recognising which properties to draw from and when to refer to another of the newly-introduced circle theorems, as well as the more familiar angle facts, is key to students developing a deep understanding of the angle relationships in the geometry of a circle. Once these are established, students have a much broader repertoire to draw on when calculating missing angles and solving problems.

The introduction of the equation for a circle at Key Stage 4 provides an opportunity for students to apply their understanding of Pythagoras' theorem in a different geometrical context. Key here is recognising that a right-angled triangle can be formed by connecting the centre of a circle to a point on its circumference (hypotenuse), using vertical and horizontal line segments to form the other two sides. Students can then use Pythagoras' theorem to generalise that, for any variable point with coordinates $(x,\ y)$ on the circumference of a circle (with centre $(0,\ 0)$ and radius r), the equation $x^2 + y^2 = r^2$ can be formed. This can then be used to solve problems involving the radius, and also provides a foundation on which to build an understanding of the general equation of a circle $(x - h)^2 + (y - k)^2 = r^2$ with centre (h, k) and radius r, which is explored in further mathematical study.



Once students have mastered the ideas in this core concept, their understanding of the properties of circles may seem unrecognisable when compared to their earliest understanding of a 'round shape with one side'. However, as more complex ideas are accumulated, it is easy to lose sight of the fundamental definition of a circle as a locus of points that are equidistant from a single point. Teachers should take care to check in with students, and ensure that this understanding is not lost, so that any new learning builds on this idea of the circumference of a circle as an infinite set of points, and the area of a circle as the region bound by these points.

Prior learning

Students will have first encountered circles in their very early explorations of shape. As they began to study mathematics more formally in EYFS and Key Stage 1, they will have classified shapes and identified the features that set circles apart from other two-dimensional (2D) shapes. By Key Stage 2, they should have begun to use key terminology such as radius, diameter and circumference; in Key Stage 3, they should have started to quantify these measures. Students will have explored the relationships between radius, diameter and circumference to come to an understanding of the formula $C = \pi d$ (or $C = 2\pi r$). Also at Key Stage 3, they will have added the formula $A = \pi r^2$ to their repertoire of formulae for the area of 2D shapes.

Using the properties of circles to reason geometrically at Key Stage 4 depends on an understanding of previous work on angles, triangles and Pythagoras' theorem. Students should recognise from Key Stage 2 instances of angles where they meet at a point and form a straight line, and know the associated angle sums. Also at Key Stage 2, they will have learnt to classify triangles, find unknown angles, and know and apply the angle sum of 180°. At Key Stage 3, students used the sum of the angles in a triangle to deduce the angle sum for any polygon, and it is important to check that students are able to recognise situations where this knowledge can be applied. For example, they recognise that this means the angles in a quadrilateral sum to 360°; can explain why this is; and can apply the properties of particular quadrilaterals to find missing angles.

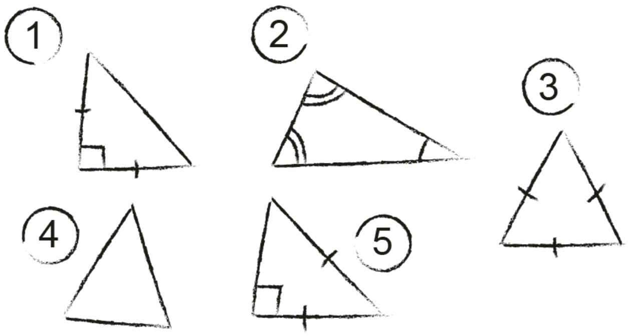
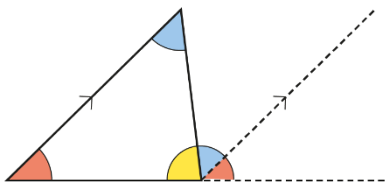
Students' work on area and perimeter of sectors links both these strands of learning – area/perimeter and angle properties – and relies on their confidence with proportional reasoning. Students need to be able to express and use proportional relationships, working fluently between fractional and ratio notation. This understanding will have emerged from students' earliest work on multiplicative relationships in Key Stages 1 and 2 and been deepened over Key Stage 3.

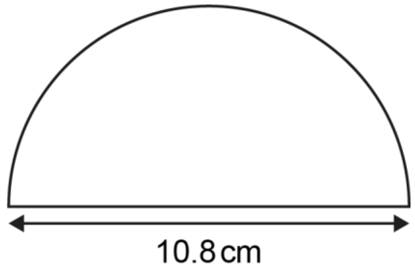
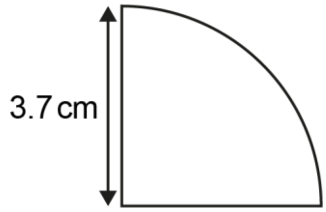
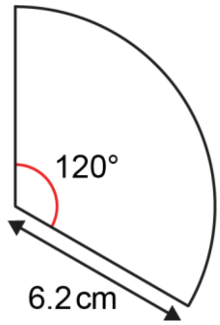
Working with the equation of a circle brings together students' understanding of graphical representations of equations, as well as their application of Pythagoras' theorem. If they have been introduced to the trigonometric functions sine, cosine and tangent using the unit circle at Key Stage 3, this can be used as a basis on which to build when exploring how the equation of a circle emerges from the use of Pythagoras' theorem.

The core concept documents *1.4 Simplifying and manipulating expressions, equations and formulae*, *3.1 Understanding multiplicative relationships*, *3.2 Trigonometry*, *4.2 Graphical relationships, 6.1 Geometrical properties* and *6.2 Perimeter, area and volume* from the Key Stage 3 PD materials all explore the prior knowledge required for this core concept in more depth.

Checking prior learning

The following activities from the NCETM secondary assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Secondary assessment materials page 40	The triangles below have been sketched badly but are labelled correctly.  a) Which of them are definitely isosceles, and which might be isosceles? b) Which one is impossible? c) Explain how you know
Checkpoints 'Geometrical properties - polygons', Checkpoint 16: Seating plan 2	Eight chairs are equally spaced around a circular table. Three people sit on the chairs. Imagine three line segments join the three people.  a) What type of triangle would the line segments form? b) Is it possible for the people to move so that they form an isosceles triangle? c) Is it possible for the people to move so that they form an equilateral triangle? d) Two extra chairs are placed at the table, and all the chairs are adjusted so the spacing is equal. e) How many different isosceles triangles can the line segments form now? How many chairs would need to be around the table to form an equilateral triangle?
Key Stage 3 PD materials document *'6.1 Geometrical properties',* Key idea 6.1.1.2, Example 6	Look at this diagram and think what it might be showing.  a) Share your ideas with your partner and agree what the diagram is telling you b) Work with your partner to construct a clear written argument which explains (using the diagram) why the angle sum of a triangle is 180°.

Key Stage 3 PD materials document *'6.2 Perimeter, area and volume'*, Key idea 6.2.2.3, Example 6	a) What is the area of this semi circle?  b) What is the area of this quadrant?  c) What is the area of this sector? 

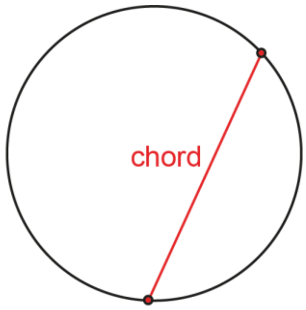
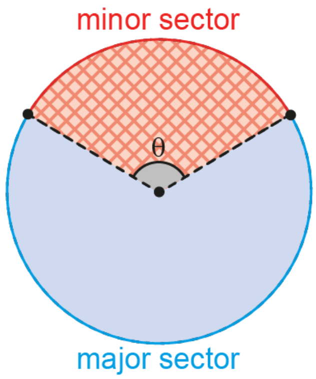
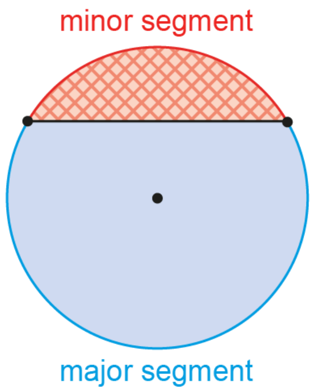
Key vocabulary

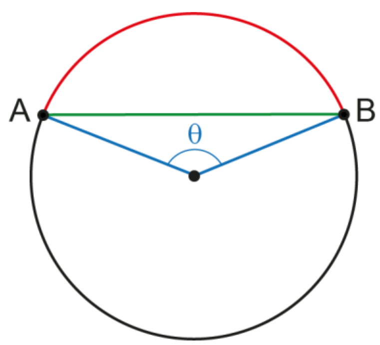
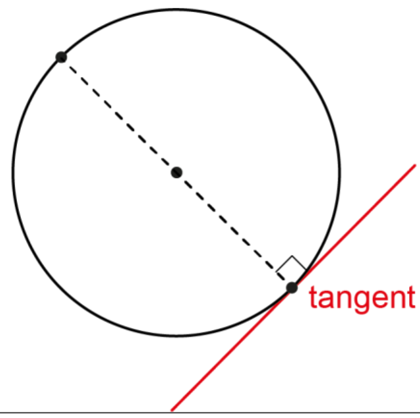
Key terms used in Key Stage 3 materials

- arc
- line segment
- pi (π)

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Definition
chord	A straight line segment joining two points on a circle or other curve. 
cyclic quadrilateral	A four-sided figure whose vertices lie on the circumference of a circle.
sector	The region within a circle bounded by two radii and one of the arcs they bound. Example: the smaller of the two sectors is the minor sector and the larger one the major sector. 
segment	The part of a line between two points. Within a circle, the region bound by an arc and the chord joining its two end points. Example: the smaller of the two regions is the minor segment, and the larger is the major segment. 

subtend	To form an angle when the end points (of a line or arc) are joined to another point. An arc or line segment subtends an angle when rays drawn from each endpoint meet. Example: in the context of a circle and in the diagram on the right the angle subtended by both the chord AB and the centre of the circle is $\boldsymbol{\theta}$. This could also be described as the angle that is subtended by the arc AB at the centre. 
tangent	A line is a tangent to a curve when, at a point of contact, it just touches the curve at that point and does not cross over to the other side of the curve. The tangent of a circle forms a right angle with the radius/diameter. 

Key ideas

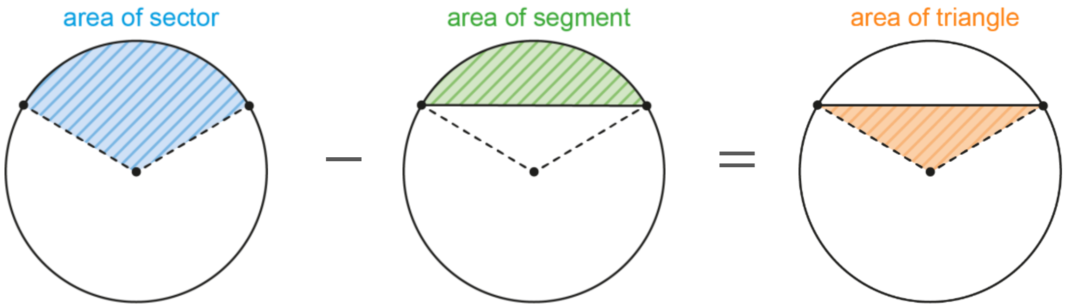
In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

11.2.1 Proportionality within circles

Students first worked with problems involving similarity at Key Stage 2, and over Key Stage 3 should have further developed their understanding of similar figures being enlargements of one another. A key awareness of proportionality is that, like squares, all circles are similar and so every circle is an enlargement of every other circle. This is utilised at Key Stage 4 when calculating the length of an arc and the area of a sector.

Students should be aware of the constant relationship that exists between the circumference and diameter of a circle, namely π, and may appreciate that between any two circles, the ratio of the diameters is equal to the ratio of the corresponding circumferences. As such, if the diameter is enlarged by a particular scale factor, then the circumference will also be enlarged by that same scale factor. Similarly, when a sector is defined, the fraction of the circumference that forms the arc is equivalent to the angle of the sector as a fraction of the whole turn. Students should appreciate that there are proportional relationships between angles of sectors in circles and associated lengths and areas. For example, the ratio of an arc length compared to the length of the circumference is equivalent to the ratio of the angle of the sector compared to the angle of a full turn. Similarly, the ratio of the area of a sector compared to the area of the full circle is equivalent to the ratio of the angle of the sector compared to the angle of a full turn. Students need to be conscious of the difference between perimeter and circumference: when working with a circle, the terms are interchangeable, but to find the perimeter of a sector, there is a need to add the two radii that bound it to the arc length.

The same proportional reasoning can be applied to the area of a sector. Students can confuse sectors and segments, and so exploring the area of a segment can be a helpful way to consolidate learning on area of a sector. Visual representations, such as those shown below, clearly demonstrate that the area of a segment is the difference between the areas of the corresponding sector and triangle.



Focusing on conceptual understanding in this way, rather than identifying and memorising formulae for area and perimeter, deepens students' perception of the proportionality between circles, and strengthens their proficiency at solving problems involving arcs and sectors of a circle.

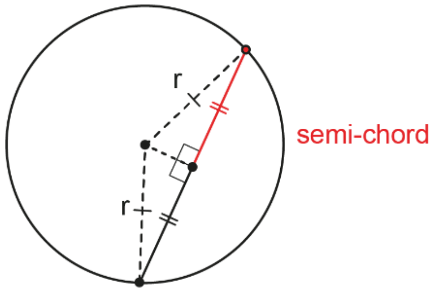
11.2.1.1 Use proportionality to calculate the length of an arc

11.2.1.2 Use proportionality to calculate the area of a sector

11.2.2 Reason and prove using properties of circles (including circle theorems)

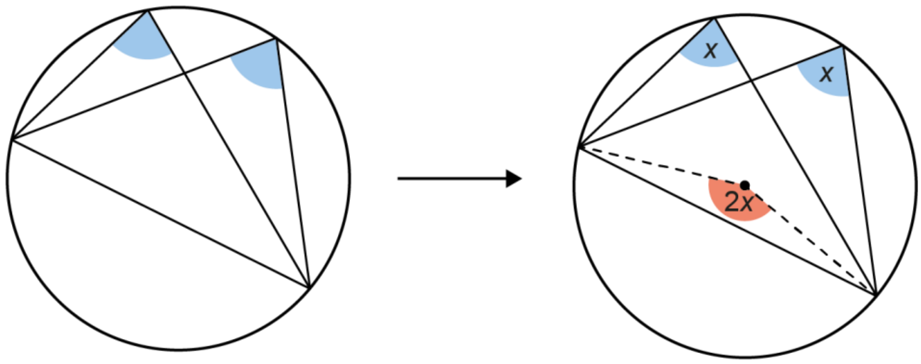
Students' understanding of the properties of circles is significantly developed at Key Stage 4 with the introduction of chords, allowing consideration to be given to angles within triangles constructed in the area enclosed by the arc of a circle and a chord, i.e., within a segment.

Students need to be able to relate different lines and appreciate what properties they have, and in what circumstances. This includes an awareness that the diameter is a particular example of a chord, i.e., the chord which passes through the centre of the circle, and that when the two ends of a chord are both joined to the centre of the circle, an isosceles triangle is formed. A semi-chord is half a chord, and so forms one side of a right-angled triangle with hypotenuse of length r (the radius of the circle), as shown on the right.

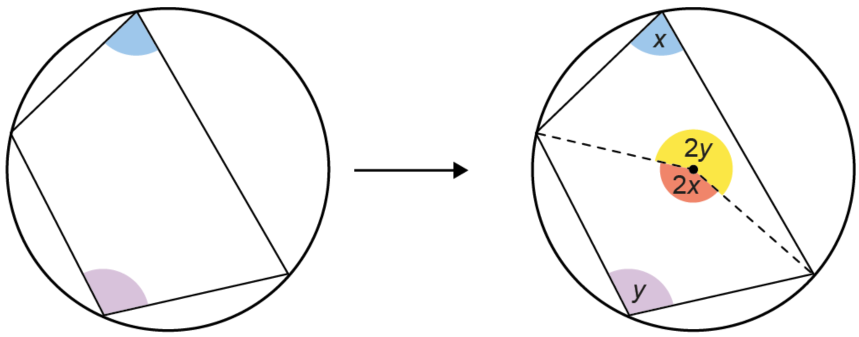


Students need to strengthen their confidence in reasoning from certain known facts to those not yet known, both in solving problems and in using chains of reasoning to deduce facts about angles constructed within circles. They will have met a 'theorem' before in their work on Pythagoras' theorem, but may not have fully explored the definition of a theorem as a statement that has been proven to be true this time. In the case of circle theorems, students can easily become overwhelmed by the number of them and confuse the different properties. It is therefore important to emphasise the common root of all circle theorems: that, for two angles subtended by the same arc, the angle at the centre is twice the angle at the circumference. This theorem can be used to explain all other circle theorems, and students' awareness of this is key to developing an appreciation of how one circle theorem relates to another. For example, the right angle in a semicircle can be seen as a special case of the angle at the centre being twice the angle at the circumference – when the angle at the centre is 180° (as a semicircle is bound by a diameter) the angle at the circumference must be 90°. The relationship between angles subtended by the same arc/chord at the centre and circumference could also be used to show the following:

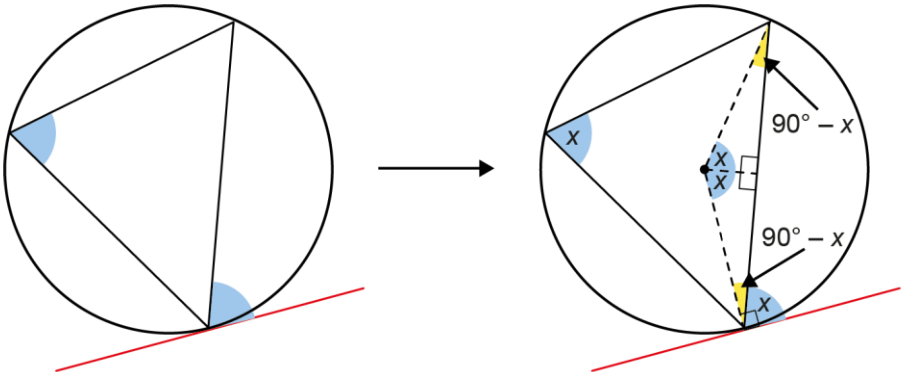
- The angles in the same segment from a common chord are equal.



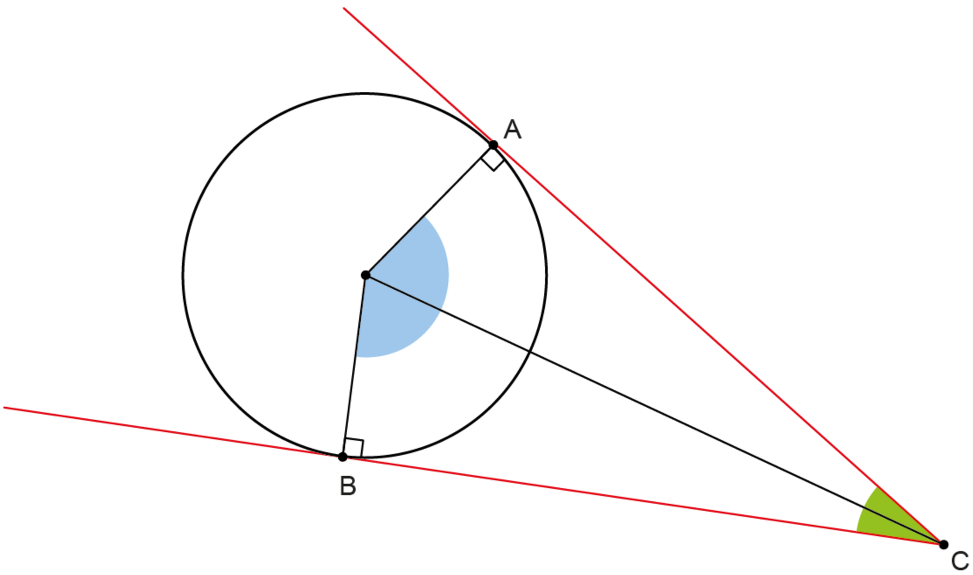
- Opposite angles in a cyclic quadrilateral sum to 180°.



- The alternate segment theorem.



The introduction of tangents at Key Stage 4 extends beyond the context of circle theorems, and it is important that students recognise their usefulness within the areas of graphs and rates of change, as well as their use when exploring geometric properties within circles. Understanding that the tangent to a circle is perpendicular to the radius/diameter at the point of contact with the circumference of the circle, is significant when demonstrating the existence of the two congruent right-angled triangles formed between two radii and two tangents that originate from a single point.



When exploring circle theorems, it is crucial for students to be aware that drawing extra lines – such as exploiting well-chosen radii – can help to support geometric reasoning, and may be essential for exposing some relationships. With this experience, students' ability to build on prior knowledge about angles and triangles and to make connections in a variety of contexts is developed. This provides ample opportunity for students to generate chains of reasoning and to arrive at solutions to a variety of problems.

- 11.2.2.1 Identify and reason with lines associated with a circle (including segments, chords, radii and tangents)
- 11.2.2.2 Use chains of reasoning to show that the angle at the centre is twice the angle at the circumference

11.2.2.3 Use chains of reasoning to show that the angle in a semicircle is 90 degrees

11.2.2.4 Use chains of reasoning to show that the angles in the same segments from a common chord are equal

11.2.2.5 Use chains of reasoning to show that opposite angles in a cyclic quadrilateral sum to 180 degrees

11.2.2.6 Use chains of reasoning to show that two tangents drawn from a point to a circle are equal

11.2.2.7 Use chains of reasoning to show the alternate segment theorem

11.2.2.8 Use and apply circle theorems to solve problems

11.2.3 Equation of a circle

One of the key advantages of the Cartesian coordinate system is that any point in space is defined using two orthogonal coordinates. Students (and perhaps teachers) may use these coordinates without fully appreciating that these orthogonal coordinates describe the horizontal (x-ordinate) and vertical (y-ordinate) distance of the point from the origin. Consequently, this defines a right-angled triangle with the hypotenuse bounded by the origin and the point in question. This means that Pythagoras' theorem can be used to find the distance of a point from the origin using the coordinates of that point.

The introduction of the equation of a circle at Key Stage 4 relies on the understanding that any point with coordinates (x, y) on the circumference of a circle with centre $(0, 0)$ and radius r can be joined to the centre by a straight line that forms the hypotenuse of a right-angled triangle with side lengths x and y. This hypotenuse coincides with the radius of the circle at that point and therefore defines the equation of a circle with centre $(0, 0)$ as $x^2 + y^2 = r^2$. Once the equation of a circle has been established, it is important that students recognise the way in which the centre and radius are represented within the equation, and how this relates to the points at which the circle crosses the x- and y-axes. This enables students to solve problems such as determining the equation of a circle with centre at the origin given its radius, and finding the radius of a circle given its equation.

11.2.3.1 Appreciate that the equation of a circle emerges from the use of Pythagoras' theorem

11.2.3.2 Solve problems involving circles centred on a coordinate grid

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

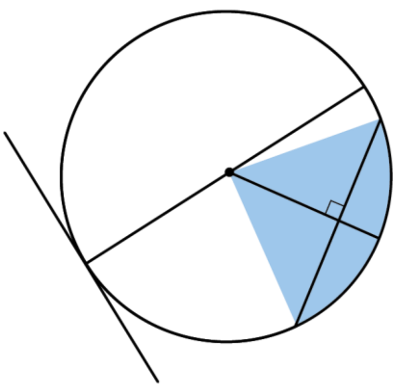
11.2.2.1 Identify and reason with lines associated with a circle (including segments, chords and tangents)

Common difficulties and misconceptions
In the same way that students can sometimes confuse the radius and diameter when working with the circumference and area of a circle, they can struggle to distinguish between other lines and regions associated with circles. It is important that students are given time and practice to develop this subject-specific language before there is an expectation that they use it to make sense of circle theorems. A chord is a line segment joining any two points on the circumference of a circle. Students need to be able to distinguish between a chord in general and a diameter specifically: to recognise that, while every diameter is a chord, not every chord is a diameter. From this, students can deduce that diameter is not only a special case of a chord, it is also the longest possible chord in a circle. Tangents may also be confused with chords, so noticing the similarities and differences between them is essential for understanding the precise relationships described by circle theorems. Students might need guiding to the way a chord joins the two endpoints of an arc, so lies within a circle; whereas a tangent touches the circle at only one point. Teachers may wish to explore the links between the tangent function and its representation as a tangent from the unit circle. Connections to other related words/usages can help students' understanding; for example, the common root of 'tangent' and 'tangible' is the Latin *tangere*, to touch. Students can often confuse segments and sectors due to the similarity of their names, and the fact that both are formed from the interaction of straight lines with the circumference of a circle. It is important that they are able to make the distinction between a segment (which is formed by cutting a circle with a chord) and a sector (which is the area enclosed by two radii and an arc).

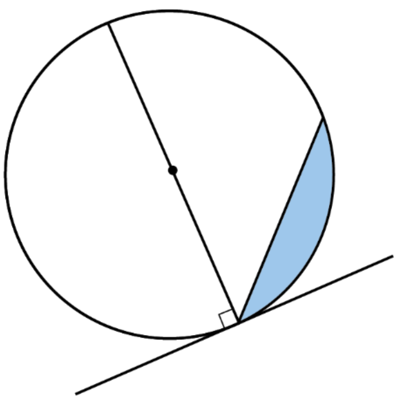
The language of ‘subtend’ can also be challenging if it is introduced alongside later learning on circle theorems. Precise use of complex language in familiar situations can help students to understand the meaning, rather than using unfamiliar language in unfamiliar situations. Support students to gain confidence with this new vocabulary by using it in contexts that are simple and familiar, such as explaining that the hypotenuse always subtends the right angle when exploring right-angled triangles.

Students need to	Guidance, discussion points and prompts
Identify lines and regions associated with a circle *Example 1:* *On a circle, Hector marks a diameter, a chord and a tangent at one point that the chord meets the circumference. The area of the minor segment created by the chord is shaded.* *Below this question are three diagrams, labelled A, B and C.* a) *Which of the diagrams represents the description* ***exactly****? Explain your answer* b) *For the remaining two diagrams, accurately describe what is shown.* c) *What other lines, angles and relationships can you identify on each diagram?*	The subtle **variation** between A, B and C helps students to distinguish between chords and tangents and sectors and segments, and to recognise that a radius is half of a diameter. The use of non-examples encourages students to think deeply about circles, and can be used to expose and address any confusion. When discussing the diagrams, asking students to describe what is the same and what is different may be beneficial. Check that students are using the correct **language** when referring to different parts of the circle, as a way of assessing understanding. For example, in diagram A, a confusion between a chord and the diameter may exist: while the diameter is a chord (the longest possible), the question asks for a circle with a diameter and chord marked, suggesting that a chord is marked in addition to the diameter. In diagram B, the chord and diameter meet at the same point on the circumference, and so the tangent is marked at the point the chord (and the tangent) meet the circumference as required. The right angle marked between the diameter and tangent provides an opportunity to reinforce the relationship between the radius and diameter. The difference between a sector and segment can be emphasised, and the distinction between a minor and major segment discussed if appropriate. Discuss with teachers other possible right and wrong diagrams to represent the description given in *Example 1*. Are there any that highlight a possible confusion in a different way the diagrams already given? Are there any other misunderstandings that teachers think their students might have? If so, how might they be addressed?

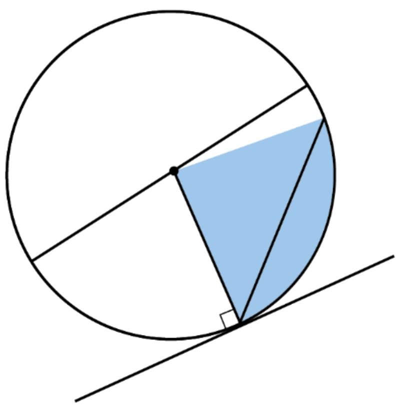
A:



B:

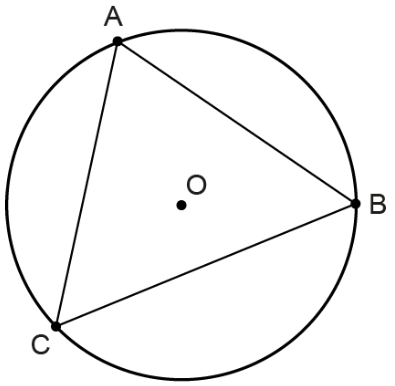


C:



Example 2:

The points A, B and C are on the circumference of a circle. O is the centre.



a) *Use two-letter line segment notation and three-letter angle notation to complete the sentences below. How many different ways can you do this?*
   (i) *Angle ___ is subtended by the chord ___.*
   (ii) *The chord ___ subtends the angle ___.*

b) *How would your answers to part a change if the point O was included? How might you differentiate between the two different angles subtended by each chord?*

In *Example* 2, students are presented with a **representation** that will later become familiar as the geometrical structure for the alternate segment theorem. Teachers do not need to reference this at this stage – instead, the focus is on building confidence with the language and notation.

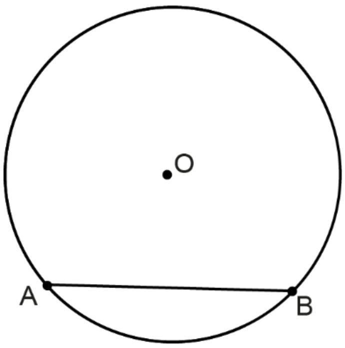
The **variation** between parts (i) and (ii) may seem onerous, but it is intended to help students notice the geometrical structures involved. By stating the relationship between the chord and the subtended angle in two different ways, students can start to build a deeper understanding of situations where the word *subtend* is appropriate, as well as how to use it with grammatical accuracy.

The repeated **language** structures are designed to help students build confidence with the new vocabulary of subtend before it is used to introduce any more challenging or complex concepts. In part b, students become aware that the angle described earlier is not the only possible angle subtended by the chord, dispelling any misconception about the word subtend being specifically associated with the circumference. Students may notice that the chord BC subtends an infinite number of angles, and the point at which the rays meet needs to be defined. Encourage students to create precise and unambiguous sentences: they should describe the angle subtended at the centre as AOC, and at the circumference as ABC. When students revisit this structure to explore the relationship between these two angles, they will be supported by their prior experience of describing each angle in this way.

Identify triangles formed within circles and use their properties

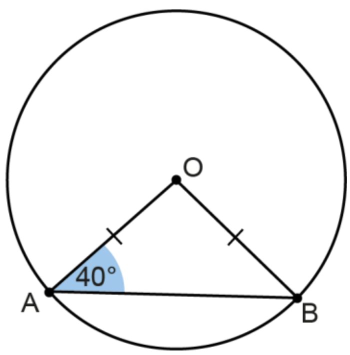
Example 3:

A and B are points on the circumference of a circle with centre O. Points A and B are joined with a chord.

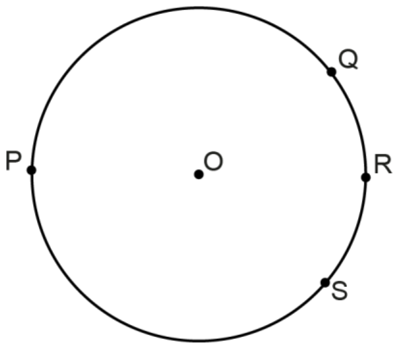
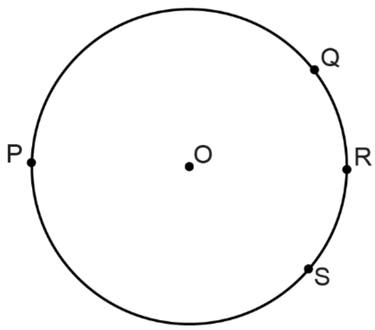
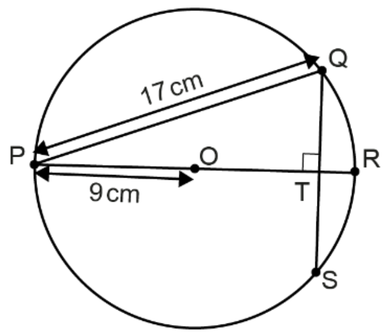


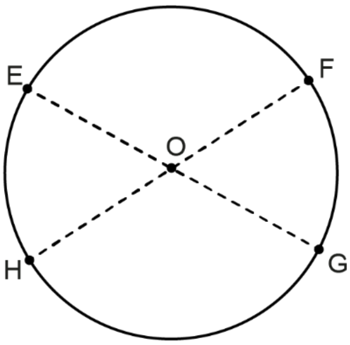
a) *If angle OAB = 40°, determine the size of angle AOB.*

In *Example 3,* students are presented with some information about a circle and asked to calculate a missing angle. The geometric **representation** is only partially complete, to check whether students recognise the importance of adding extra lines to help establish geometric facts. It is important to check whether students identify that **any** line from the centre of a circle is a radius. They need to use this to establish existence of an isosceles triangle; explain how they know it exists; and what markings, if any, they use to denote it (as demonstrated in the image to the right).



To determine angle AOB = 100° in part a, students need to apply properties of isosceles triangles and the sum of the angles in a triangle, **deepening** their understanding of angle facts in relation to triangles. Later work on circle

b) *How would your answer to part a change if angle OBA was 40°?* c) *How would your answers to parts a and b change if the point O was not the centre of the circle?*	theorems relies heavily on students' ability to connect rules about angles in triangles to the lines in circles. Highlight to teachers the importance of the partial representation in exploring students' understanding of the properties involved, particularly the significance of the radius. Discuss the effect, if any, that providing no geometrical representation has on cognitive demand.
Example 4: *P, Q, R and S are points on the circumference of a circle with centre O.*  a) *How many different triangles could you construct by connecting point O to two of the points on the circumference?* b) *How many of your triangles in part a are isosceles? How do you know?* c) *How many different triangles could you construct by connecting point Q to two of the other points O, P, R or S?* d) *How many of your triangles in part c are isosceles? How do you know?*	*Example 4* further explores the different triangles that can be created using points on and in a circle. The **variation** between parts a/b and parts c/d draws attention to the information that we can deduce from a diagram, based on the properties of a circle. Students may be surprised, in part b, that all of their triangles are isosceles. This is an opportunity to discuss that all radii are equal in length, since every point on the circumference is equidistant from the centre. Links should be made with learning on loci, to support a deep and connected understanding of different strands of geometrical reasoning. Teachers could use part d as an opportunity for **deepening** students' understanding about what is known and what can be assumed from diagrams. Students may be tempted to state, for example, that triangle PQS is isosceles because 'it looks like it is'. In *Example 5*, the same diagram is explored in more detail to establish that PQS is indeed isosceles, but not enough information is provided here to draw this conclusion. Comparing and contrasting the two examples, and identifying what additional information is provided in *Example 5*, can help to reinforce the point about making assumptions.
Example 5: *P, Q, R and S are points on the circumference of a circle with centre O.*  *The following three properties are given:* • *The radius of the circle OP = 9 cm*	*Example 5* relies on an understanding that the perpendicular from any chord, which passes through the centre of the circle, will bisect the chord. This provides an opportunity for students to identify a right-angled triangle and recognise how application of Pythagoras' theorem enables the length of the chord to be calculated. The information required to establish this is not provided on the diagram, ensuring that students work on the particular skill of moving between a **representation** and a worded description (as shown to the right). 

• *The chord PQ = 17 cm.* • *The chord QS is perpendicular to the diameter PR at T.* *a) Annotate the diagram with the given information.* *b) What other lengths can you determine?* *c) If OT:TR = 2:1, find the length of chord QS.*	When determining length QS, students need to think carefully about the properties of right-angled triangles, **deepening** their undestanding of the ways in which their existing knowledge can be applied. They need to identify the right-angled triangle, determine the length OT using the ratio provided, and then apply Pythagoras' theorem to find length QT. Students may think this is sufficient to answer the question, and need prompting to use the fact that QS is bisected by the diameter PR, so the length of QT needs to be doubled to give QS = 16 cm. Discuss potential ways to advise teachers whose students struggle to recognise that QT = TS. Students may, for example, assume this to be the case, but fail to provide an explanation. Discuss the understanding required to be able to explain why PR bisects QS (congruent triangles, for example), and highlight the importance of emphasising the way in which chains of reasoning can be used to prove relationships.
Recognise that drawing extra lines can help expose underlying geometry *Example 6:* *Points E, F, G and H lie on the circumference of the circle formed from O.*  *a) Are lines EG and FH the same length? Explain how you know.* *b) What shape is EFGH?* *c) Explain why angle EFG is a right angle.*	*Example 6* explores the geometric properties of shapes through a **representation** of their diagonals. Students need to recognise the additonal information that the presence of the circle provides: lines EG and FH are both diameters and so are equal in length. They may say that shape EFGH is a quadrilateral; while this is true, it is important to encourage them to think more deeply about the quadrilateral and its properties. Recognising that the diagonals of the quadrilateral are equal and bisect each other is key to identifying that shape EFGH is a rectangle; it is important that students can explain how they know this to be the case. Explaining that the lengths of the opposite sides are equal provides an opportunity for students to apply their knowledge of the properties of triangles, and it is important that accurate **language** is used when students refer to such things as 'radius', 'isosceles' and 'congruent'. Students may draw on the realisation that the diagonals are equal and bisect each other when explaining why angle EFG is a right angle. This gives an opportunity to discuss how this property can be used to distinguish a rectangle from other quadrilaterals. Once it has been established that shape EFGH is a rectangle, teachers may highlight how the properties explored in this example can be used to support the construction of rectangles, further **deepening** students' understanding. If we draw two intersecting lines and a circle centred at the point of intersection, a rectangle can be formed by joining the four points where the circle cuts the lines. Teachers can then ask students what the necessary conditions would need to be in order to draw a square. Discuss other possible methods for explaining why angle EFG is a right angle. For example, how might this diagram be revisited to revise circle theorems once they have been taught?

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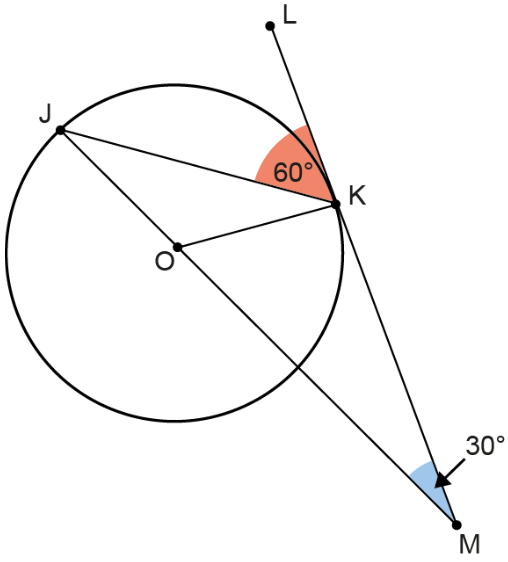
Know that a tangent to a circle is perpendicular to its radius

Example 7:

J and K are points on the circumference of a circle with centre O.

The tangent LM touches the circle at K.

Line JM is a straight line that passes through the centre of the circle O.



Angle KMO = 30° and angle JKL = 60°.

Calculate the size of angle JOK.

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It is important for students to realise that properties of circles can be applied without the need to prove them every time – once a fact is known, it can be used and applied to solve problems. However, it is equally important for students to be aware that they cannot assume that such a property exists based on appearances. Highlight the **language** in this question that allows us to apply the fact that a tangent makes an angle of 90° with the radius of the circle at the point they meet.

Students may approach determining the size of angle JOK in different ways, depending on whether they choose to establish the size of angle KOM = 60° first, or determine that angle JKO = 30°. To establish that angle JOK is 120° requires students to demonstrate **deepening** understanding of geometric properties, including being able to use chains of reasoning that involve several different angle relationships. Here, this includes the angles on a straight line, sum of angles in a triangle, properties of isosceles triangles and complementary angles.

Discuss with teachers how this example could be used with students who do not recognise that a tangent to a circle is perpendicular to its radius. What existing knowledge of circle properties could be applied? How could this concept be revisited once circle theorems, such as angles in a semi circle or the alternate segment theorem, are more established? How could assuming that the angle is not 90° be used to construct a proof by contradiction? What existing knowledge would students need for this? For example, the longest side in a triangle is always opposite the largest angle.

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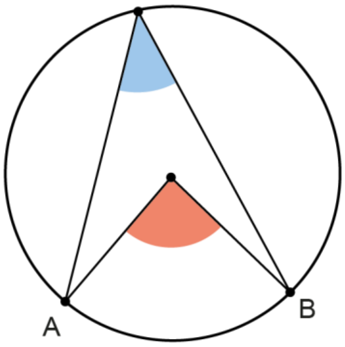
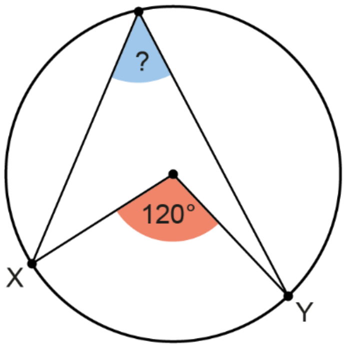
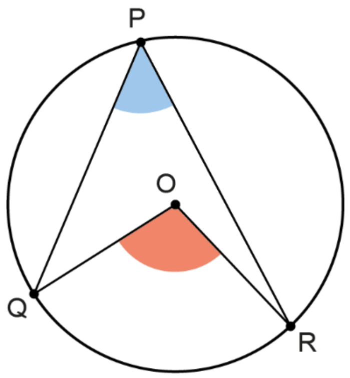
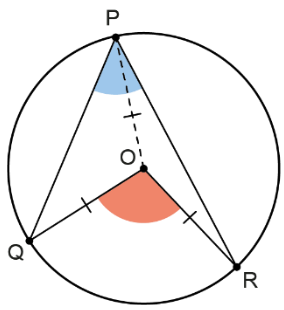
11.2.2.2 Use chains of reasoning to show that the angle at the centre is twice the angle at the circumference

Common difficulties and misconceptions
Students need to understand that the angle subtended at the centre by a chord is double the angle subtended at the circumference by the same chord. This relationship also applies to angles subtended by an arc with the same endpoints. Understanding that the angle at the centre is twice that at the circumference is a crucial step in the development of students' grasp of circle theorems. All other circle theorems rely on a recognition of this relationship. It is important that students develop an intuitive sense of the general result, before being presented with a formal argument. The mistakes that students commonly make when working with this relationship are often as a result of trying to memorise a theorem, with little understanding of why the relationship exists. Common incorrect assumptions include, for example, that the angle at the circumference is twice the angle at the centre, that the angles are equal, or that they sum to a particular value (90° if the angle at the centre is acute, 180° if it is obtuse or 360° is there is a reflex angle at the centre, for example). This highlights the need for students to develop a deep understanding of the reasoning that supports the circle theorems. To show that the angle at the centre is twice the angle at the circumference, students need to fluently find angles in a triangle and on a straight line, and revision of these principles may be helpful for less confident students.

Students need to	Guidance, discussion points and prompts
Recognise that a relationship exists between the angle at the centre and the angle at the circumference when both are subtended from the same two points *Example 1:* *Draw a circle and mark two points A and B on its circumference.* *Join points A and B to both the centre and an additional point on the circumference to form angles as in the diagram below.*  a) *Measure the two angles.* b) *Repeat part a with the same circle, but different positions for A and B. Repeat several times with different positions for A and B.* c) *Now repeat part b with a different size of circle.* d) *What do you notice?*	Before exploring the elements of a formal proof to show that the angle at the centre is twice the angle at the circumference, it is helpful to first establish curiosity about what the result might be. In *Example 1*, students are asked to draw a number of different-sized circles, mark their own points A and B and think about the different angles subtended at the centre and the circumference. **Variation** is often used by teachers constructing examples to draw students' attention to certain concepts and mathematical structures. By giving students an opportunity to vary things for themselves, their awareness of the relationships that remain unchanged under these variations can be developed. As students make variations to the circle, they may observe that the greatest angle that can be subtended at the centre is 180°, when AB is the diameter of the circle and the angle subtended at the circumference is a right angle. Such observations should be encouraged and discussed to promote the use of correct technical **language**. Students should, for example, be supported in using *subtend*, *subtended* and *subtending* when describing angles. Some students may struggle to use their observations to develop a sense of generality of the situation. To support students' developing awareness, teachers may like to ask: • *'Does it matter what size the circle is?'* • *'Does it matter where the points A and B are?'* • *'Have you tried examples where the angle at the centre is very small and where it is very large?'* • *'Have you tried points on the circumference which are close to A and close to B?'* While the key realisation for students is that, irrespective of these variations, the angle subtended at the centre seems to be twice the size of the angle subtended at the circumference, the journey to establishing this is important, as it supports students in **deepening** their understanding of mathematical structure.
Example 2: a) *Draw a circle and mark two points X and Y on its circumference such that the angle at the centre is 120°.* b) *Draw a point, Z, on the circumference. Connect points X and Y to this point.* c) *What is the measurement of the angle at the circumference?* d) *Draw a new point, Z_2, and connect points X and Y to this point. What is*	Before students are introduced to a formal proof that the angle at the centre is twice the angle at the circumference, it is important that they explore the situation and develop an awareness that there might be something which is always true. *Example 2* develops the idea of student-led **variation** introduced in *Example 1*, with a tighter structure to support students to see that, wherever the point is on the major arc, it is always half of the angle at the centre. It is important that students are given sufficient time to explore a number of different scenarios for part d, in order to promote interest in the possible presence of a constant relationship between the size of the two angles. Part e

the measurement of the new angle at the circumference? e) *Experiment with other points on the circumference, making sure you keep the position of X and Y the same. What do you notice?* f) *Draw a new circle and repeat parts a and b with a different angle at the centre of the circle. What do you notice?*	demonstrates that the relationship holds for all angles at the centre; students could compare their diagram with others to reinforce this. In this example, the **representation** is left for students themselves to construct. Teachers should reassure students that their individual images may be orientated differently, but should include the features shown in the image to the right. If students struggle with parts a and b, it may help to show them a completed diagram.  *Examples 1* and *2* both acknowledge that students' understanding of a variable may still be developing; as a result, they are likely to struggle to construct a sequence of deductions using an unknown angle. *Example 2* is distinguishable from *Example 1* in that there is a value provided for the angle at the centre of the circle. Working analytically with particular angles aims to ease students' difficulties, **deepening** their understanding before the general case is explored as part of a formal proof. Discuss the possibilities that open up when using dynamic geometry software in the classroom. What would need to be in place to ensure students can use the software effectively? Highlight the need for thoughtful preparation and discuss different ways in which the use of the software could be scaffolded.
Know that the angles formed at the centre and circumference can be used to identify isosceles triangles within circle diagrams *Example 3:* a) *Using compasses and a ruler construct the following diagram.*  b) *Using the points labelled O, P, Q and R (and without adding any other*	*Example 3* provides an opportunity for students to use their existing knowledge of angles and triangles to reason about the angles at the centre and circumference of a circle. While teachers may prefer to have the diagram prepared ready for students to use, asking students to construct the **representation** for themselves allows multiple circles containing different-sized angles to be explored simultaneously. Teachers should consider what classroom management strategies could support effective comparison and discussion of the students' diagrams, and how to collate the collective knowledge of the class. An important learning point is identifying the existence of two isosceles triangles, each with two sides equal to the radius of the circle (as shown to the right). This is key to **deepening** students' understanding of why the relationship between the angle at the centre and the angle at the circumference exists. 

points) identify as many triangles as you can.

c) *Compare your diagram with a partner's. Together, write down all the things that you know about these triangles and their angles.*

d) *Measure the two angles QPO and RPO and* ***without measuring any other angles****, work out as many other angles in the diagram as you can.*

e) *What are the sizes of the two shaded angles?*

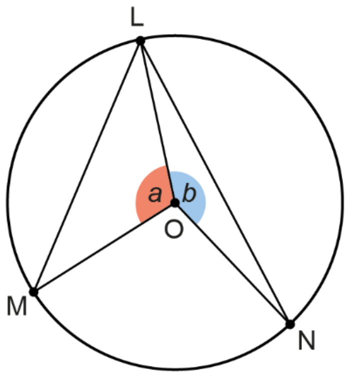
When discussing students' findings as a whole class, draw attention to the generality relationships that exist, irrespective of the actual particular values being used. If, for example, a student has measured angle QPO as 30° and deduced that angle PQO is also 30°, and then used their knowledge of the angles in a triangle to calculate QOP = 150°, encourage them to generalise by asking:

- *'If the angle you measured was 40° (or 50° or 80°, etc.) what would the angles be?'*
- *'What calculation would you do to find the size of angle QOP if the original angle QPO was 40° (or 50° or 80°, etc.)?'*

Discuss the benefits of working with particular values, but with a focus on the structure of the calculations involved, rather than the answers. What other areas of the curriculum might benefit from a similar approach?

Example 4:

In the diagram below, O is the centre of the circle and points L, M and N are on the circumference.



a) *What else do you know about this diagram?*

b) *Using only the points M, N, O and L, write expressions for as many other angles in this diagram as you can.*

Example 4 draws attention to the fact that two isosceles triangles exist within the chords used to construct an angle at the centre and circumference of a circle. In part a, recognising the three radii is key to identifying equality of the two pairs of angles: LMO and MLO, and ONL and OLN. This provides an opportunity for students to explore the relationships between different angles, **deepening** their understanding of the properties required to prove the circle theorem. Students may begin by identifying that angle MON = 360° $- a - b$ (or 360° $- (a + b)$). While this is a good starting point, it is important to encourage them to think more deeply about the properties of a circle to be able to express angles LMO, MLO, ONL and OLN in terms of angles a and b. Once the relationships between the angles in the two triangles have been established, comparison of angles MLN ($180° - \frac{1}{2}a - \frac{1}{2}b$) and MON ($360° - a - b$) can be used to examine the relationship between the angle at the centre and circumference of the circle.

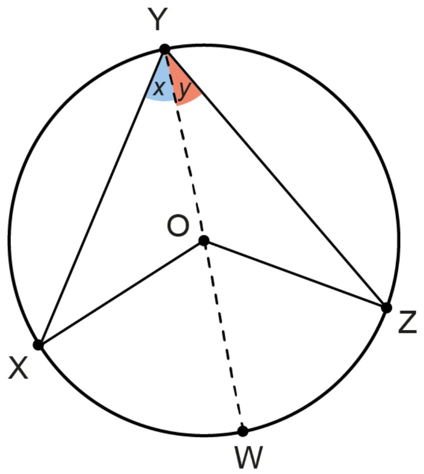
Discuss possible **variations** that could be used with the same starting point, such as changing the angles given. How might defining angle MLO as a and angle OLN as b make the relationship between the angle at the centre and circumference of the circle more accessible to students?

Apply geometrical properties to show that the angle at the centre of a circle is twice the angle at the circumference

Example 5:

The table below this example shows the steps to a proof that the angle at the centre of a circle is twice the angle at the circumference.

It refers to this diagram.



Complete the table stating reasoning for each of the steps 2 to 8. Step 1 has been completed for you.

Students can often rely on memorising steps of a proof, without **deepening** their thinking about the reasoning behind each of the steps. In *Example 5*, students are given the steps for proving that the angle at the centre of a circle is twice the angle at the circumference, and are asked to complete the reasoning for each step. Completion of the table relies solely on students' previous knowledge of angle facts from Key Stage 3: properties of isosceles triangle; sum of the angles in a triangle; and angles on a straight line.

Emphasise the importance of asking teachers to compare the proof outlined in the table with the one they commonly use with students. If it is different, ask teachers to compare the conceptual understanding that their usual proof requires compared to the Key Stage 3 content. Emphasise the importance of building on and developing students' existing knowledge to promote a connected, deep understanding.

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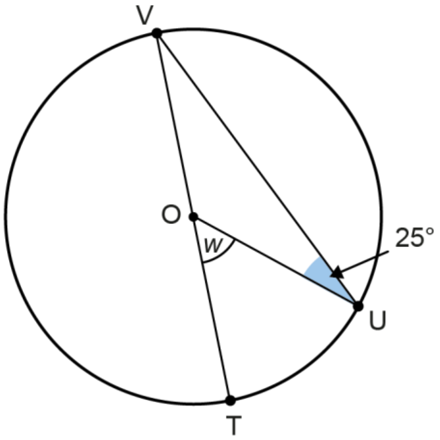
	Geometric property	Reasoning
1	$OX = OY$	Radius of the circle
2	$OXY = OYX = x$	
3	$XOY = 180 - 2x$	
4	$XOW = 2x$	
5	$OZ = OY$	
6	$OZY = OYZ = y$	
7	$ZOY = 180 - 2y$	
8	$WOZ = 2y$	
9	$XOZ = 2x + 2y = 2(x + y)$	Angle at the centre = 2 × angle at the circumference.

Example 6:

A circle with centre O has points T, U and V on the circumference.



a) *If angle OUV = 25°, calculate angle w.*

b) *Comment on anything you notice.*

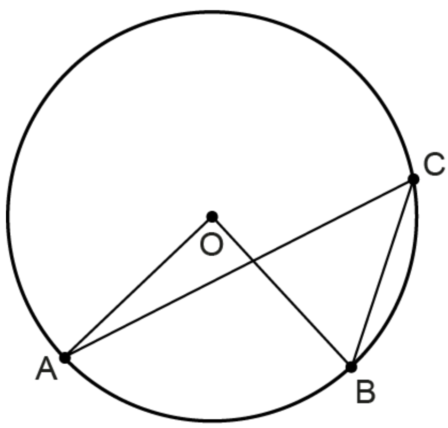
Example 6 highlights the difference between recalling and applying a relationship, and using other known angle facts to show this to be the case. Recognising that triangle UOV is isosceles allows angle OVU to be identified as being equal to 25°, and using the sum of the angles in a triangle and on a straight line enables the required angle to be determined. Students may focus on the relationship between angle w and angle OUV, as well as between angle w and angle OVU. It is important for them to recognise that the relationship between the angle at the centre and the angle at the circumference still exists, even when the **representation** is non-typical. This diagram is the limit of the cases where the standard proof can be applied.

Opportunities for students to explore the relationship between the angle at the centre and circumference presented in alternative ways, are key to consolidating their understanding of the mathematical structure of the theorem. Discuss with teachers how the specific case explored in this example could be used as a basis for the general case. Replacing OUV = 25° with OUV = x, for example, provides a succinct way of showing that $w = 2x$.

Recognise that there are three different geometric configurations required for a full proof, and understand these additional proofs

Example 7:

Mhari is looking at a circle with a chord AB subtending an angle at the centre of the circle and another angle at the circumference.



She doesn't believe that angle ACB is half the angle at the centre, as it looks very different to other examples she has seen.

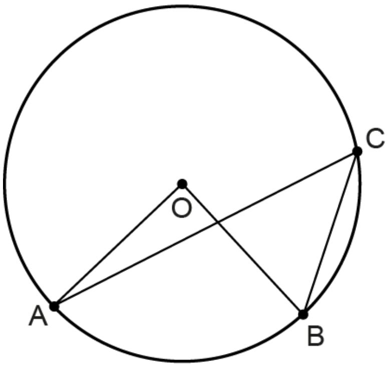
What do you think?

Examples 7 to *9* demonstrate instances of the theorem which are visually and geometrically different from previous examples: one where the lines forming the angle at the circumference overlap the ones forming the angle at the centre, and one where the angle at the centre is a reflex angle. It is important to plan to address this **variation** directly as part of a learning sequence. Many textbooks will only show the first proof and expect students to extrapolate and accept that the theorem will hold for all cases. Whilst it is true that the theorem holds true, the proofs of these cases are not the same. Students are often confused about when a proof applies and when it does not, and this is an example of when students are right to be confused! Glossing over the different cases is unhelpful and can leave students mystified about the nature of proof. It is sufficient for teachers to show these two proofs to students so that they are convinced that the theorem holds true; students do not need to learn them.

Dynamic geometry is a useful **representation** that can be used to help convince students that the theorem always holds true.

Example 8:

Becky and Carol each look at the diagram from Example 7, which is also shown below.



They try to work out what properties and angles in the diagram are connected. They draw extra lines on the diagram to help them.

Becky draws a straight line from C, through O but stops between O and the circumference. She labels the end of this line as D. She labels angle OAC as x.

Becky says that triangle is isosceles.

a) How does she know this?

Becky goes on to say that she now knows what angles ACO, AOC and DOA are.

b) How might Becky have done this?

Carol also draws Becky's extra line on her diagram, but she decides to focus on triangle BOC instead. She labels angle OBC as y.

Carol says that triangle BOC is also isosceles.

c) Is she correct? How do you know?

Carol can now work out angles BCO, COB and DOB.

d) How might Carol have done this?

Becky looks at both of their diagrams and combines facts from the two. She is correct when she states that:

angle AOB = angle DOB − angle DOA

e) What is angle AOB?

Carol thinks she can now work out angle ACB in a similar way.

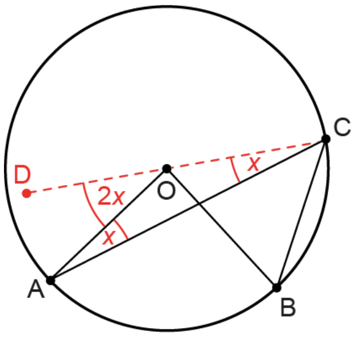
What is angle ACB?

The proof of this second configuration is more challenging for students as it necessitates the subtraction of angles. The **representation** can become very 'busy' with lots of annotations, so teachers should encourage students to draw two separate diagrams if they find it confusing when Becky's and Carol's markings are overlaid.

The **variation** in this case is between this example and previous examples, rather than within the example itself. In part a, point D is deliberately not on the circumference, as this has the potential to confuse with other circle theorems.

Before using this example with students, it is worth ascertaining the levels of confidence with this proof among your team. To support this, the intended steps for this example are outlined in italics below. Discuss with colleagues which aspects of the proof are most likely to cause issues for students. For example, in part b students may need to be encouraged to recognise that DOA and AOC are two angles forming a straight line.

- *In parts a and b, we consider triangle AOC (which is isosceles because OA and OC are radii):*

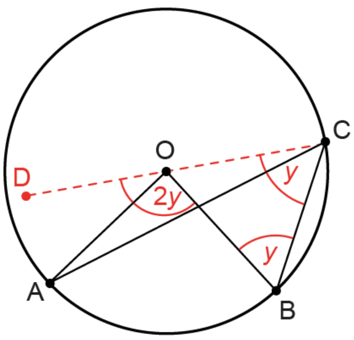


Angle OAC = OCA $= x$

Angle AOC $= 180° - 2x$

So angle DOA $= 2x$

- *In parts c and d, the steps are the same as in a and b but we consider instead triangle BOC (which is isosceles because OB and OC are radii):*

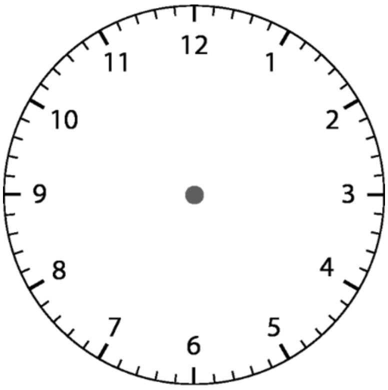
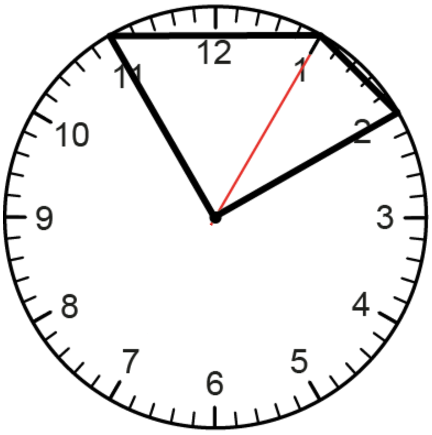


Angle OBC = OCB $= y$

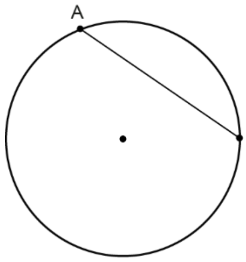
Angle BOC $= 180° - 2y$

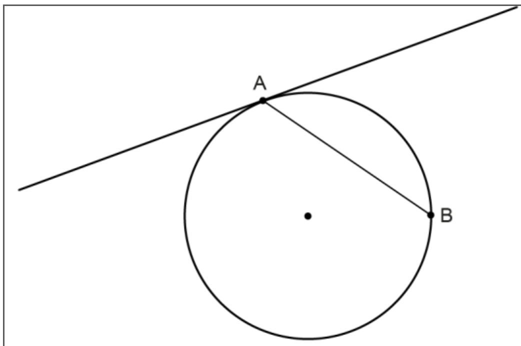
So angle DOB $= 2y$

- *In parts e and f, information from the two diagrams can be used to establish that angle AOB* $= 2y - 2x$

Example 9: *Anwar creates a quadrilateral on the clock face below.*  *He draws lines from the centre of the clock to 11, from 11 to 1, from 1 to 2 and from 2 back to the centre. He then indicates the reflex angle at the centre.* *Anwar says that he has drawn an example of an angle at the centre and an angle at the circumference. He expects the angle at the centre to be twice the angle at the circumference.* *a) Do you think he is right? Why or why not?* *b) Draw a line from 1 on the clock through the centre. Does this change your answer to part a?*	*Example 9* is similar to the standard proof but explores the case where the angle at the centre is greater than 180°. Students may struggle to visualise this case, as they can experience more difficulties with reflex angles. Using a familiar **representation** such as a clock, where the hands constantly turn through a variety of angles, including reflex angles, may help students with this. The image described in the question rubric, including the line drawn during part b, is offered below:  Part a offers students the opportunity to reson without any structure, whilst part b anticipates that they might need some support in structuring their thinking. Key to accessing this is the additional line drawn between 1 and the centre; students need to feel confident to annotate diagrams to expose geometical properties that are not immediately evident. The proof (as outlined in italics below) also relies on fluency with algebraic manipulation, **deepening** the connections that students make between work on geometrical and numerical/algebraic structures. • *Drawing a line from 1 through the centre creates two isosceles triangle, formed by radii meeting each of the the chords.* • *For the isosceles triangle formed by the centre, 11 and 1, label the angles at 11 and 1 as x. This means that angle at the centre is $180° - 2x$* • *For the iscosceles triangle formed by the centre, 1 and 2, label the angles at 1 and 2 as y. This means that angle at the centre is $180° - 2y$.* • *It can then be discerned that the reflex angle at the centre is $360° - (180° - 2x) - 180° - 2y) = 2x + 2y$* • *The angle formed by the chords at 1 on the clock can be identified as $x + y$.* • *Therefore, the relationship holds: even in the case of a reflex angle at the centre of the circle, that angle willl be twice the angle at the circumference.*

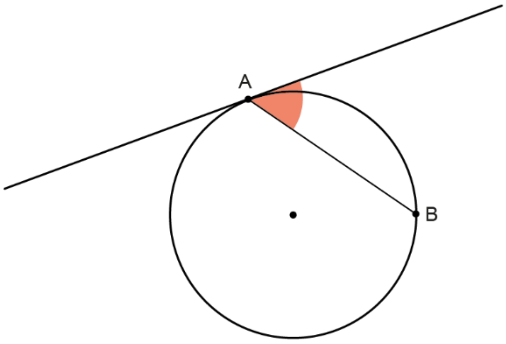
11.2.2.7 Use chains of reasoning to show the alternate segment theorem

Common difficulties and misconceptions	
The alternate segment theorem states that the angle between a tangent and a chord is equal to the angle subtended by the same chord in the alternate segment. The word 'subtend' has been gradually introduced throughout these materials but, as a term that is introduced only at Key Stage 4, it may still create a barrier to students' understanding of the concept. Time should be taken to establish the position relations before exploring the quantity relations: that is, to establish when two angles can be described as being in the alternate segments formed by the chord, and when that they cannot. This should include the common misconception that the angle is formed between the chord and the circumference, rather than the chord and another straight line (i.e., the tangent in one segment, and another chord in the other segment). Students should have opportunities to explore the geometric structure described, including through using dynamic representations so that they can see the effect of any changes to the key angles. Demonstrating this relationship necessitates an understanding of other properties of circles and triangles: that the triangle formed by two radii and a chord is isosceles; the angle between a tangent and a radius is 90°; and the angle at the centre is twice the angle at the circumference. Students often struggle to identify the relevant lines needed for the alternate segment theorem, and can sometimes rely on appearances, rather than geometrical facts, when determining unknown angles. Incorrect assumptions may include that the angle at the circumference is 90° (even though the associated chord does not go through the centre of the circle); and that because the tangent is perpendicular to the radius, the angle in the alternate segment is the remainder of the angle subtracted from 90°. Students may also be confused with the sum of opposite angles in a cyclic quadrilateral and subtract the angle from 180°. Once students have fully grasped both the position and quantity relations, it is important that they have time to practise identifying the alternate segment theorem in context. This should include examples where there are distracting or extraneous lines, as well as examples where there are insufficient lines and they need to construct their own.	
Students need to	**Guidance, discussion points and prompts**
Identify the segments and angles formed by a chord *Example 1:* *Points A and B are on the circumference of a circle. A chord is drawn to connect them.*  *a) Have any angles been formed by the chord? Explain your answer.* *A tangent to the circle is drawn at A (shown below).*	Students are commonly presented with complete **representations** of the circle theorems they are learning, without being given time to build them up from scratch. This can make it difficult for students to identify which particular features are relevant, and for teachers to identify which aspects of the diagram students are struggling with. In the particular case of the alternate segment theorem, the 'classic' diagram includes three chords, and it can be a challenge for students to appreciate which chord the wording of the theorem refers to. This is complicated by the inclusion of additional lines in the proof, so there is also more than one line that meets the tangent. By focusing here on just the relevant line and one of the angles that it forms, students can secure their understanding of this geometrical structure before they begin to work on the properties associated with it. The **variation** in these examples helps students to build an appreciation of what lines are necessary to construct the alternate segment theorem, to support them with later identifiying these features in more complex diagrams. Each image has only one additional feature each time. In part a, the aim is to dispel a common misconception that



b) Have any angles been formed by the tangent? Explain your answer.

One of the angles between the chord and the tangent is shaded.



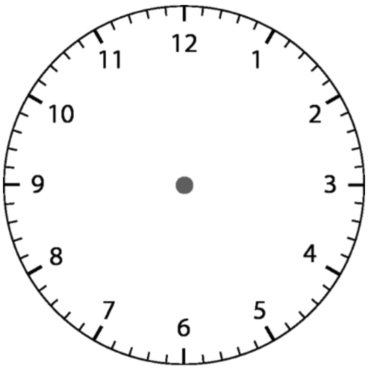
c) Shade the alternate segment to the angle.

the chord forms an angle with the circumference: it is only when the tangent is introduced in part b that there are two straight lines/line segments and therefore an angle is formed.

Part c focuses on the **language** of 'alternate segment'. Here, rather than identifying the angle formed in this segment, students identify and shade the entire segment. This approach can be referenced when students are working on more complex diagrams with multiple chords and then multiple segments formed. In those cases, students may find it helpful to identify the correct chord and then highlight the entirety of the alternate segment (so that the lines can still be seen through the colour), to support them in working out which lines or angles may be relevant for them, and which they can ignore.

Example 2:

Euclid is exploring angles on a clock face.



He picks the numbers 7 and 4 and draws a line between their points on the circumference.

a) What is the term for the type of line Euclid has drawn in this circle?

Euclid then subtends four angles from the ends of his line – at 11, 12, 2 and 3. He says, 'I've made four different triangles with the same shared base.'

In *Example 2*, the **representation** of a clock face is used as a familiar image to support understanding of the infinite angles that can be subtended from a single chord. In order to appreciate the nuances of the alternate segment theorem, students need to accept that all angles subtended at the circumference by the same chord will be of the same magnitude. This example establishes that fact in isolation, so that it does not form a barrier to the alternate segment theorem being proved.

Part c offers an opportunity for further **deepening** students' understanding. Teachers could ask what other points would create the same angles as Euclid's first four, and what other points would create the same angle as described in part c, drawing attention to the properties of the alternate sectors created by the chord.

The **language** of the clock face, covered extensively in the primary maths curriculum, can be a helpful way to create and describe situations involving points, line segments and angles. However, there may be some students for whom telling the time on an analogue clock is still a barrier – perhaps one that has gone unnoticed. Consider what interventions might be needed to support students who struggle with this task for that reason.

Alexandra says, I think they all have the same angles.'

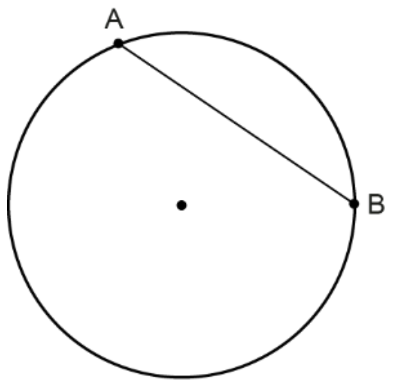
b) What could Alexandra mean? Do you agree?

c) Would her statement still be true if another angle was subtended at 5? Why or why not?

This example, alongside the previous and subsequent examples, focuses on some of the properties that the alternate segment theorem relies on. The intention is that teachers have made explicit the facts and features that are important, and that there are no assumptions from either students or teachers about what facts and features are known. This set of examples could also be used as the basis of a departmental workshop exploring other circle theorems: what facts and features are needed for each theorem to be fully understood? Could the clock face be usefully explored to ensure that these are established before the theorem is learnt?

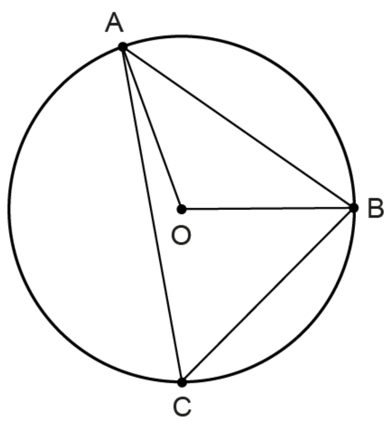
Example 3:

Points A and B are on the circumference of a circle. A chord is drawn to connect them.



Joe marks the centre of the circle as O, and then uses the chord AB to subtend an angle at O. He uses the same chord to subtend an angle at the circumference.

He notices they form two different triangles AOB and ACB:



He says that triangle AOB is isosceles.

a) How does he know this?

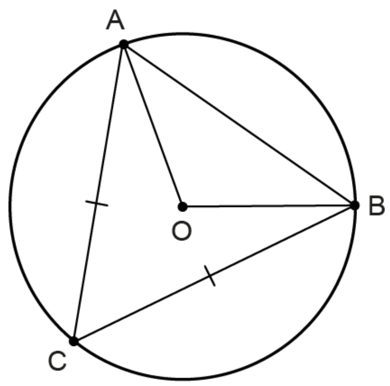
In *Example 3* students consider part of the **representation** that is needed to explore the alternate segment theorem. By focusing just on the relationships between the various angles subtended by the chord AB, students can build confidence with the knowledge that they will later need to apply to prove the circle theorem. Being fluent in identifying the relationships between various triangles formed within circles is essential to be able to work confidently with chains of algebraic reasoning.

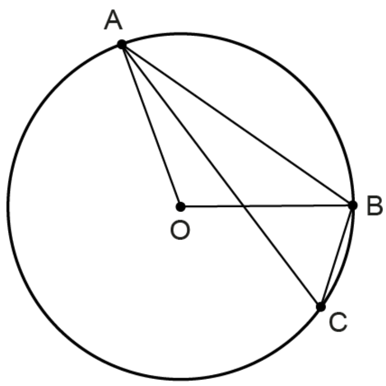
Part a reminds students of the fact that a triangle formed by a chord and the centre will always be isosceles since the other two sides are radii. Parts b to d are to ensure that students are familiar with the angle subtended at the circumference being half the angle subtended at the centre, alongside other relationships. The **variation** in parts b to d offers two different diagrams that represent the same structure; this supports students to identify that the same relationships exist, even when the triangles might be very different visually . Teachers should help students to understand which features of the diagrams have remained constant (i.e., the position of the chord AB that forms the base of the triangle, and the centre).

Part e helps with **deepening** students' understanding by encouraging them to consider the generalisation (i.e., if the two base angles of the isosceles triangle are denoted as θ, then the angle subtended by AB at the centre is $180-2\theta$, and the angle at the circumference is $\frac{1}{2}(180-2\theta)$ or $90-\theta$). If students are already confident with this, then the proof of the alternate segment theorem will only require a few additional steps.

Discuss with your team the ways in which other circle theorems rely on a grasp of this relationship between the angle at the centre and the angle at the circumference. Reflect upon the models and tasks that colleagues currently use in their lessons: are there opportunities for students to appreciate how these relationships interconnect and build upon one another? Highlight the importance of supporting students in seeing the circle theorems as part of a connected whole, and

Joe measures angle OAB and finds it to be 40°. *b) Without measuring anything else, what other angles can he work out?* *c) Joe repeats the process with the same chord AB and two new positions for point C, shown on the diagrams below. What will angle ACB be in each of these? How do you know?* *d) What other angles can be found in each of these without measuring?* *e) Joe decides to assign a general value to one of the angles: ACB = $\boldsymbol{\theta}$. What values are the other angles in the diagram now?*	begin to explore a potential learning trajectory through the various theorems, highlighting the links between them.





Recognise instances of angles formed in alternate segments by a chord and tangent

Example 4:

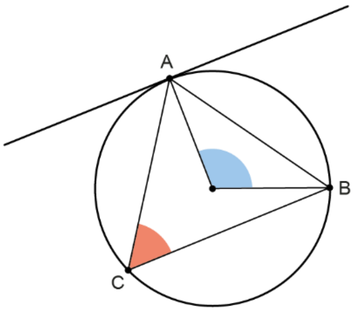
Caroline shades pairs of angles that she thinks she can describe as formed in alternate segments by a chord and a tangent.

Her thirteen diagrams are shown below this question.

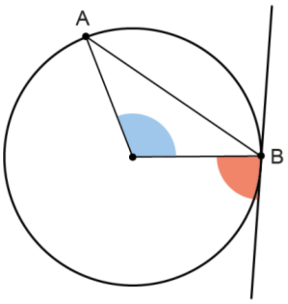
a) Which of Caroline's diagrams A-M correctly show this relationship?

b) Identify the correct alternate segment relationship in the diagrams she has marked incorrectly.

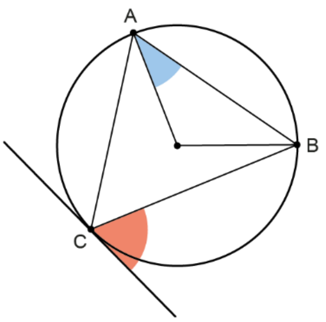
A:



B:



C:



In *Example 4*, the focus is on the position relations between angles in alternate segments. The intention is to unpick the **language** of the alternate segment theorem, so that students are able to identify the relevant geometrical features. Teachers may find it helpful to revisit this task once students have learnt the theorem in more depth, so that they are then also applying the quantity relations to the same diagrams, consolidating understanding of the position relations.

The **variation** in this example raises many common misconceptions, providing an opportunity to address them before the circle theorem is taught. To support teachers in identifying the key learning points for each diagram, details are provided for each below.

A The angle ACB is subtended by the chord AB, but the relationship shown is between that and the angle subtended at the centre by the same chord.

B Neither angle is relevant here: one is subtended by AB, but not at the circumference, and the other is not subtended by a chord. (Teachers might like to show students that, were it a diameter, a relevant relationship could be found).

C The relevant angle is formed in the altenate segment at A, but needs to be subtended from CB, not the radius.

D Students may easily confuse this one with the similarly-worded alternate angles on parallel lines. The angles marked may look the same, but we have no information to say that the chord BC is parallel to the tangent.

E A correct representation using chord AB.

F The angle marked at B is not an angle, as it is not formed from two straight lines. It is also not in an alternate segment to the angle marked at C – an additional point in the minor segment is required.

G The angle marked at B is not an angle – this time the relevant angle ACB in the alternate segment is visible

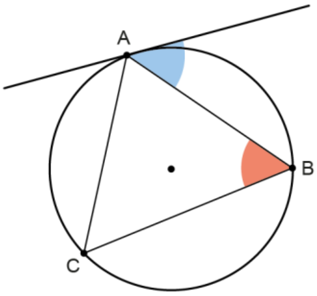
H A correct representation using chord AB.

I Another correct representation from the same diagram, this time using chord AC.

J The same diagram is presented again, with a different correct representation using chord AB – this time the angles are in the opposite segments to diagram H and so another point (D) has been added.

K Additional, irrelevant lines are included in the same diagram as a distraction. The correct angle in relation to the angle shown at the tangent is ABC rather than ACB as marked.

D:

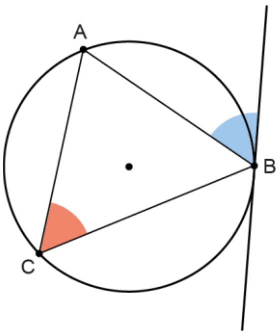


L As with K, additional lines are marked as a distraction. Here, the angle BAC marked, but it should be BAD.

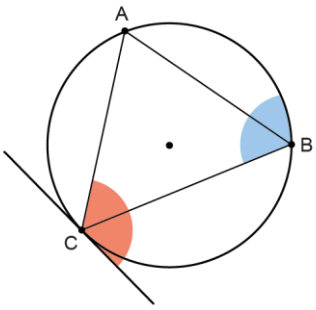
M As with F and G, the angle marked at B is not an angle. The relevant acute angle is at B, but is formed between CB and the tangent.

When teaching any angle relationship, how often do you separate understanding of the position relationships from the quantity relationships? Discuss with your team the potential benefits of spending time identifying when a diagram does and does not demonstrate an instance of a particular relationship, before teaching students to quantify that relationship.

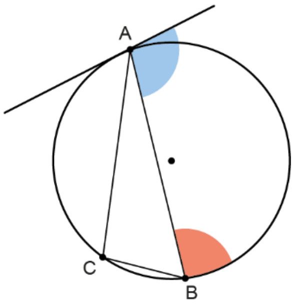
E:



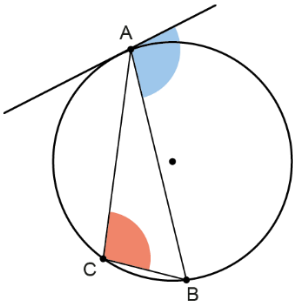
F:



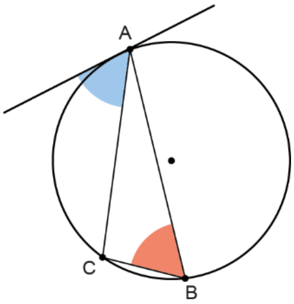
G:



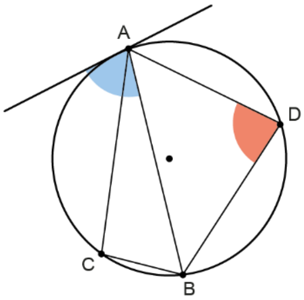
H:



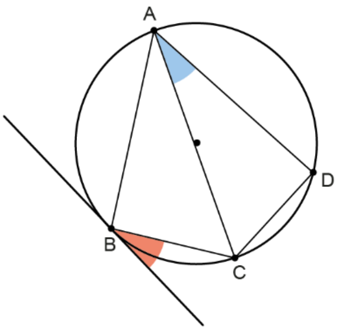
I:



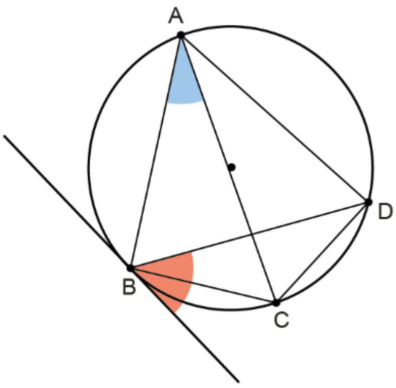
J:



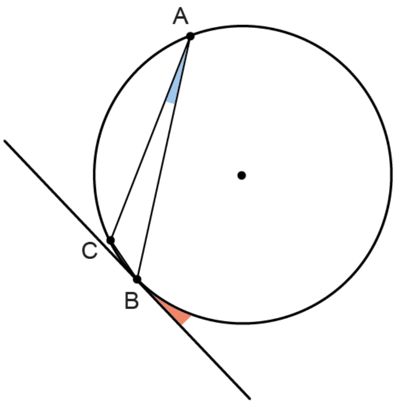
K:



L:



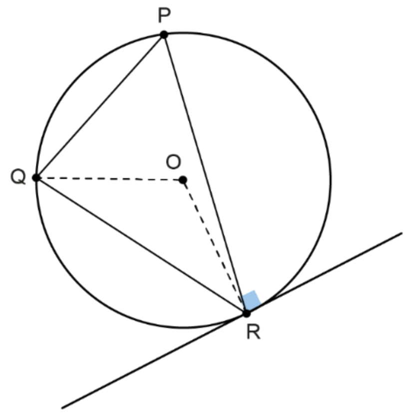
M:



Recognise and use the triangles and lines that can be identified within circle diagrams that show alternate segments

Example 5:

A circle with centre O has points P, Q and R marked on its circumference, which are joined with chords. A tangent meets the circle at R.



What properties and relationships do you know about the triangles and angles in this diagram? Discuss with a partner and write as many as possible.

Example 5 requires students to recall previous angle facts and circle theorems, helping to reveal some of the relationships embedded in the diagram. The key addition in this example is the tangent. This provides opportunities to remind students that the angle between a tangent and the radius at the point of contact is 90°. The **language** of 'tangent' and 'normal' might be introduced - or revisited - at this point.

Encouraging students to come to the board, annotate the diagram and explain their reasoning when discussing this example as a class, will help with **deepening** their understanding of the structures that are useful when finding relationships between the angles represented. It is important to draw out the following relationships:

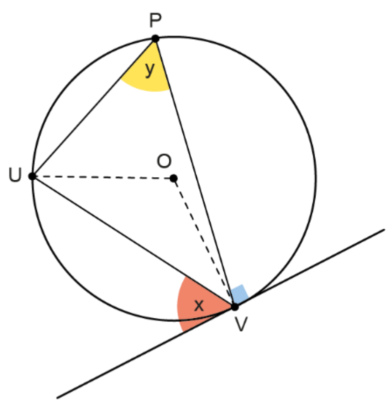
- Triangle ORQ is isosceles since OR and OQ are radii
- Angles ORQ and OQR are equal.
- The angle between the tangent and radius OR is 90°
- The circle theorem that states the angle at the centre is twice the angle at the circumference, can then be applied, with angle QOR being the angle at the centre.

Know how to prove the alternate segment theorem

Example 6:

Ash is working on a diagram involving a circle with chord UV which meets a tangent at V and also subtends an angle on the circumference at U. Two angles, x and y, are marked.

Ash adds in dotted lines OU and OV where O is the centre of the circle.



a) *What do you know about triangle OVU?*

Example 6 explores the steps needed to determine the angle subtended at the circumference by the same chord in the alternate segment, when the angle formed between the tangent and the chord from the point of contact with the tangent is known. Each part utilises procedural **variation** and supports students to see each step as a small and logical next step.

Parts a and b draw attention not to the steps of the proof, but to understanding the geometrical structures that it relies on. It is useful for teachers to take time to ensure that students really understand the **representation**.

In part d, teachers can assess whether students have noticed all the properties they will need for the proof. For example, students need to recognise the facts that:

- The sum of angle x and angle OVU must be 90°, as a right angle is marked between the tangent and OV, and angles on a straight line sum to 180°.
- Angle OVU = angle VUO (by symmetry of isosceles triangles).
- Angle UOV is therefore equal to $2x$

Part e requires students to equate the two expressions obtained for angle UOV, i.e., $2y = 2x$ and hence deduce that $y = x$

b) *What do you know about OV and the tangent at V?*

c) *Write an equation in terms of y for angle OUV.*

d) *Write an equation in terms of x for:*

   (i) Angle OVU

   (ii) Angle VUO

   (iii) Angle UOV

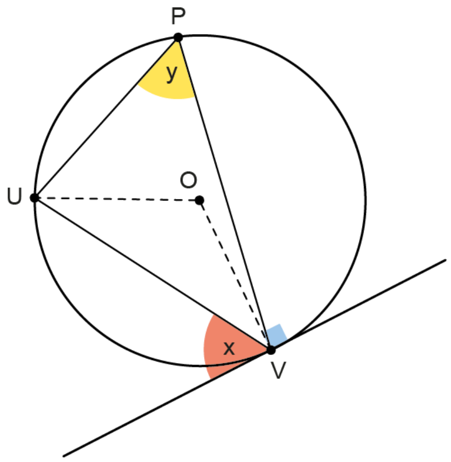
e) *Use your answers to parts c and d to form an equation for y in terms of x. What do you notice?*

f) *Explain in words the relationship that you have proved in parts c to e.*

Compare this example with the next: both explore the chains of reasoning needed to deduce the alternate segment theorem. Which version do teachers feel more comfortable using? Why? What misconceptions might arise from either approach? How do teachers mitigate against these within their preferred method?

Example 7:

The diagram below has been annotated with all the information needed to prove that angle x is equal to angle y.

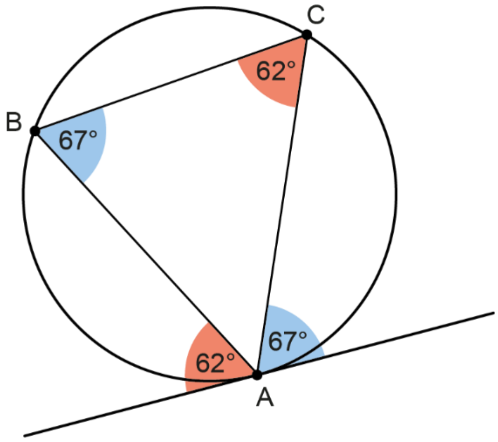
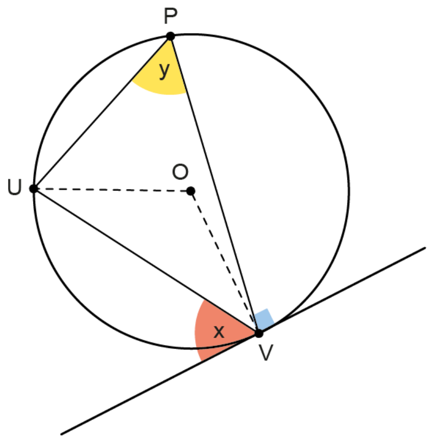


Complete the table below this example, stating the angle fact or circle theorem used each time.

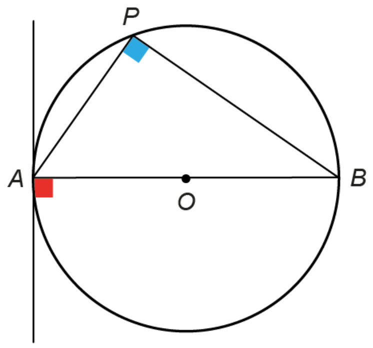
The mathematics required for *Example 7* is identifcal to *Example 6*, but this time the focus is on both the sequence of steps and the associated reasoning. Teachers can use this version for **deepening** students' understanding of how familiar properties of triangles and circles can be used to show the alternate segment theorem, and for developing their understanding of geometric proof.

The **language** in the proof may feel different to what students are used to; teachers may wish to adapt the table, so it aligns more closely with the approaches for proof that students are familiar with. Discuss with students the need to express their ideas clearly, particularly when bringing together multiple calculations and threads of thinking.

If I know…	***I can find out…***	***By using…***
Angle x	*Angle OVU*	
Angle OVU		*Base angles in an isosceles triangle are equal.*
	Angle UOV	

Recognise when there are multiple or unusual instances of the alternate segment theorem, and apply the theorem accurately *Example 8:* *A circle has points A, B and C marked on its circumference. The points are joined by chords AB, BC and AC. A tangent touches the circle at point A.*  *Which of the chords has formed the angles for the alternate segment theorem here? How do you know?*	*Example 8* addresses a very particular point of confusion for students: which is the chord that the alternate segment theorem refers to? As the theorem refers to an angle subtended in the alternate segment, two further chords must be drawn to form this angle. By the very nature of the geometrical structure, one of these chords will also meet the tangent and then also form an equal angle in the alternate segment. This can lead to students becoming muddled about which chord is relevant to which angle. Here, chord AC forms an angle of 67° with the tangent at A, and it is angle ABC that is subtended by this chord in the alternate segment, and so is also 67°. The pair of 62° angles are linked to the chord AB, which also meets the tangent at A and subtends angle ACB. Precision with **language** when describing each line segment and angle is key to clarifying this important point.
Example 9: *Joe is thinking about the alternate segment proof he has been shown.*  *He notices that in this diagram, with U and V fixed and P in the shown segment, angle UPV will always be acute. He wonders if it is possible to have a slightly different diagram so that an angle on the circumference is not acute.* *a) What other types of angle could be subtended at the circumference?*	In *Example 9* students are asked to consider the full range of possible cases, ensuring that they have experience the every possible conceptual **variation** of the alternate segment therorem. If students have already explored the three different geometric configurations for the angle at the centre being twice the angle at the circumference, they will be familiar with the idea that it is not valid to simply extrapolate a proof of one instance to apply to a geometrically different configuration. In this case, the proof can be completed either by verifying the three different cases - as we will do here - or by combining other circle theorem proofs. In part a, students need to identify that a right angle is a special case, and that an obtuse angle could be obtained through moving point P onto the circumference in the minor segment. Using the **language** of major and minor here will be helpful to distinguish the different cases. Teachers might also refer to the major or minor arcs. In part b, students should identify that any chord passing through the centre of the circle is a diameter. For part c, they may have already encountered the theorem stating that any angle subtended by a diameter is a right angle, but it is always worth reminding them that this is a particular instance of the angle at the centre being twice the angle subtended at the circumference.

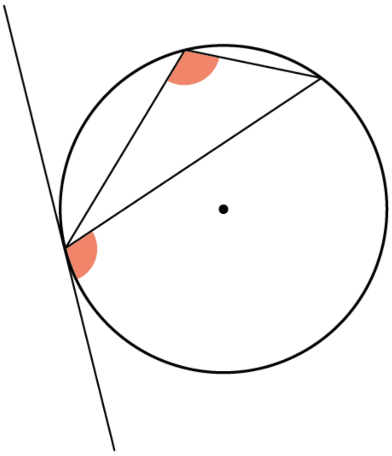
Joe draws a different diagram with the chord AB passing through the centre of the circle O.



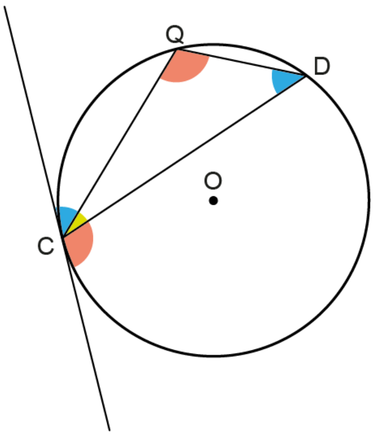
b) *What is special about chord AB here?*

c) *Explain why the angle at P is the same as the angle at A.*

Finally, he draws a diagram with point on the minor segment and realises that this creates an obtuse angle.



He spots some other angles and shades them in different colours.



d) *How might this diagram help him to prove that the red (obtuse) angles are equal?*

For the third case, where the angle is subtended in the minor segment, the angle will always be obtuse. It is worth exploring this and verifying that there is always an obtuse angle in one segment and an acute angle in the other - except when the chord is a diameter, in which case they are equal. A short demonstration with dynamic geometry **representation** accompanied by a discussion is helpful in securing this.

In part d, the proof becomes more complex as it draws upon other proofs to establish equivalence, **deepening** students' understanding of both the alternate segment theorem itself and the conditions required for geometric proof. Students may need prompting to recognise that the blue (acute) angles are a familiar case of the angle subtended in the alternate segment. Once they have verified this, they can use existing angle knowledge to establish that the three different coloured angles shown represent the angles in a triangle and the angles on a straight line, both of which sum to 180°. The yellow (smallest) angle is common to both, so if the blue (acute) angles are already proven to be equal to each other, then the red (obtuse) angles must be equal to each other.

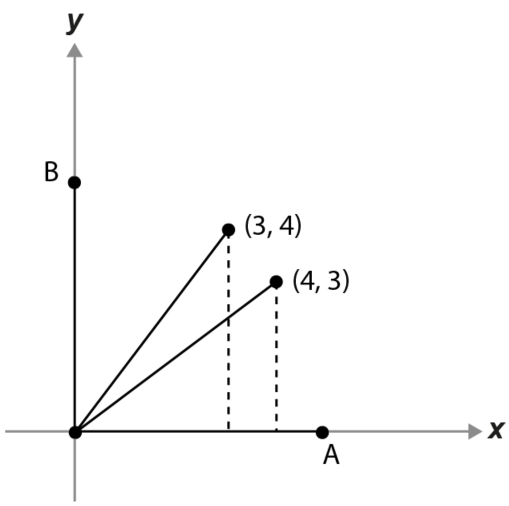
Students frequently struggle with what elements have already been proven and so can be drawn upon in future proofs. How do teachers manage this? When working through an area of mathematics such as this, would it be useful to have a 'proof board' with things that have already been established?

11.2.3.1 Appreciate that the equation of a circle emerges from the use of Pythagoras' theorem

Common difficulties and misconceptions

The equation of a circle sits across several different mathematical themes but can often feel like a 'standalone' topic to students. Ensuring that students realise they are connecting their learning on properties of circles, graphs and Pythagoras' theorem is a key priority.

Although students should be very familiar with graphical representations by Key Stage 4, they are often less accustomed to working with equations that are not expressed in the form $y =$, which can result in some difficulty in identifying which variable is the radius. There are numerous incidental mistakes that can occur – for example, mistaking the diameter for the radius, or overlooking the fact that the radius is square and therefore neglecting to square root. When working with square roots, recognising that the radius cannot be negative, and so the positive value must always be taken is key to understanding the relationship between the numerical values and the geometrical context.

Students need to	Guidance, discussion points and prompts
Use Pythagoras' theorem to determine the distance from the origin to points on the circumference of a circle *Example 1:* *A line segment is drawn from the origin to point A. This line segment is rotated 90° anticlockwise about the origin to point B.* *a) What shape will be formed in the first quadrant as the line segment rotates?* *Two points that it passes through are marked on the diagram below.*  *b) Determine the length of the line segment.* *The line continues to rotate a full 360° about the origin.*	Understanding the relationship between the equation of a circle and Pythagoras' theorem relies on the recognition that a circle is defined as all the points on a plane that are a fixed distance away from the centre, and that this distance is the radius of the circle. *Example 1* unpicks this in an accessible and logical way. The **representation** of a rotating line segment supports students to visualise the formation of an arc between A and B and link their thinking to prior work on transformations and loci. Focusing on the first quadrant means that students can build up their understanding gradually, thinking deeply about how the given points are related, before generalising to the whole circle. Students will find this example more accessible if they already have a degree of fluency with Pythagorean triples. Considering the pairs of integer coordinates that will be found in the other quadrants offers an opportunity to introduce the **language** of 'magnitude'. This provides students with the terminology they need to conceptualise what is the same about, for example, (3, 4) and (−3, −4). Links can also be made to work on vectors. Teachers may be tempted to use part d as an opportunity to formally introduce the formula for a circle, but this is not the intention of this task. Rather, the focus should be on **deepening** students' understanding of the geometrical and algebraic structures, so that they do not merely memorise the formula, but rather have an appreciation of how it arises from an infinite series of right-angled triangles.

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c) *Which other integer coordinates will the end of the line pass through? How do you know?*

d) *If every possible point that the end of the line passed through was marked, what shape would be formed?*

e) *How might you work out the coordinates of some of these other points?*

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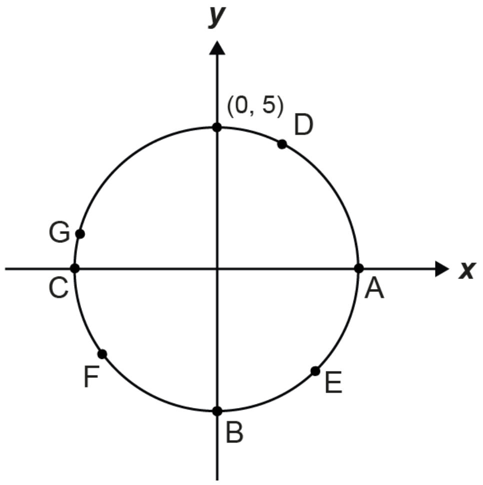
Example 2:

A circle with centre at the origin passes through point (0, 5).



a) *What are the coordinates of points A, B and C?*

b) *How might you determine the possible coordinates of points D, E, F and G?*

c) *How would your answers to parts a and b change if the original coordinate was (0, 12)?*

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Like the previous example, *Example 2* considers the geometric structure of a circle presented on a Cartesian coordinate grid. Students should be encouraged to think about how the magnitude of the points on the circumference relates to the radius, **deepening** their understanding of the algebraic structure of the equation.

In part b, students are not asked to determine the coordinates but to explain how they would do so. The **language** that students use here is likely to be revealing, and it is worth collating a few different explanations and then using whole-class questioning to adapt and refine them. Teachers will need to decide at what stage to formally introduce the equation of the circle. It may be appropriate to bring students' reasoning together and summarise it algebraically or, alternatively, students' responses may reveal that more time and exploration is needed first.

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Know how to identify the radius of a circle with centre at the origin, given its equation

Example 3:

A circle with centre (0, 0) has equation $x^2 + y^2 = 9$.

a) *Which of the following accurately describe the circle's radius?*

A: 81   *B: 9*

C: 3   *D: −3*

b) *Explain how you know.*

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Examples 3 to *5* expose some common mistakes that students often make when using the equation of a circle to identify the size of its radius. Students need to recognise that the equation of a circle with centre (0, 0) has equation $x^2 + y^2 = r^2$, where r is the radius, and so in *Example 3*, $r^2 = 9$. Explaining why the options of 81 and 9 are incorrect helps with **deepening** understanding of the mathematical structure of the equation of a circle. Option D, meanwhile, aims to check that students understand that whilst 9 has two square roots, +3 and -3, only the positive root can be used for the circle's radius.

Teachers may have come across multiple-choice questions where the incorrect options seem to have been selected randomly, rather than designed strategically. Carefully-chosen alternative values can help to expose potential mistakes and incorrect

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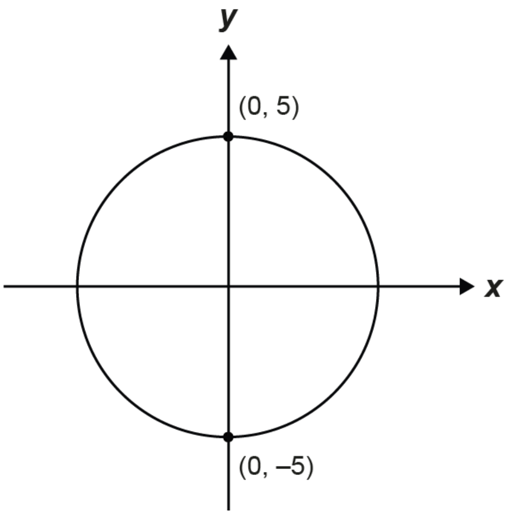
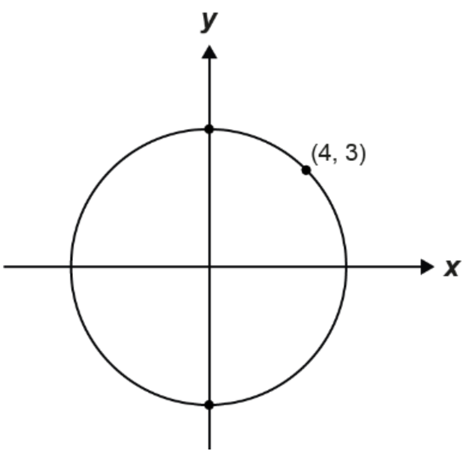
	thinking, as well as being useful discriminators for assessment purposes. Designing multiple-choice questions is a valid and valuable use of departmental time.
Example 4: *A circle with centre (0, 0) has equation* $x^2 + y^2 = 2$. *a) Which of the following accurately describe the circle's radius?* *A: 2* *B:* $\sqrt{2}$ *C: 4* *b) Explain how you know.*	The **variation** between this and the previous example is designed so that r^2 is a square number in one, but not in the other. When students are only exposed to examples that give integer results, as when r^2 is square, they can believe that they have made a mistake if the result is irrational. As a result, they may begin to doubt their understanding and change their approach. It is important that students are exposed to a variety of examples so that they do not rely on assumptions about the appearance of their answer. Discuss the importance of encouraging students to check that their selected radius makes sense in the context of the given equation. What questions could students remind themselves to ask about how their radius should relate to the given values? For example, should the radius be more or less than r^2? If r^2 is even, what does this mean for the value of r?
Example 5: *A circle with centre (0, 0) has equation* $x^2 + y^2 = 2.56$. *a) Which of the following accurately describe the circle's radius?* *A: 1.28* *B: 1.6* *C:* $\sqrt{2.56}$ *D: 6.5536* *b) Explain how you know.*	*Example 5* provides an opportunity for **deepening** understanding of squaring and square rooting. It aims to provoke a discussion about surd form, while also highlighting the distinction between multiplying by 2 and squaring. Students can sometimes struggle to know how to use surd form appropriately. While option C is not incorrect, simplifying to 1.6 should be encouraged. Contrasting $\sqrt{2.56}$ with $\sqrt{2}$ can be helpful in developing students' understanding and prompting them to think about whether it is as appropriate to simplify a surd when the result is not rational. Ask your colleagues whether there are any other types of solution that might be helpful to explore. Emphasise the importance of checking that any additional examples suggested are addressing something different, not just repeating the same structural idea.
Example 6: *What is the radius of a circle with centre (0, 0) and equation* $2x^2 + 2y^2 = 98$*?*	Here, students need not only to be familiar with the equation of a circle with centre (0, 0) (i.e. $x^2 + y^2 = r^2$) but also recognise that $2x^2 + 2y^2 = 98$ is not in this form. Students should connect this to earlier work on algebraic manipulation and know how to write it in the recognisable form, appreciating where **representations** are equivalent in value if not in appearance. Discuss the choice of radius value in this example and the benefits of helping students to focus on developing a particular part of their understanding. What other values might be used to elicit the same reasoning? What values might be helpful for emphasising a different learning point?

Determine the equation of a circle with centre at the origin, given its radius *Example 7:* *Write the equation of this circle:* 	Students should be able to complete *Example 7* successfully if they can recall the equation of a circle with centre (0, 0) and substitute $r = 5$. It is important to use questioning and prompts to check that students' understanding extends beyond memory recall, to a **deepening** grasp of how the equation of a circle relates to the geometrical representation. You may like to ask: • *'Why is $x^2 + y^2$ equal to 25?'* • *'Can you give me the coordinates of another point that lies on the circle?'* • *'If just one of the points, either (0, 5) or (0, -5) were given, would you still be able to determine the equation of the circle? Why or why not?'* In this example, the points at which the circle intersects with the y-axis have been represented, rather than the x-axis intercepts. Consider the effect of having different points provided. Discuss with colleagues how this task might be varied, and what learning the changes would elicit.
Example 8: *Which of the following describes a circle with centre at the origin and a radius of $\sqrt{2}$?* A: $x^2 + y^2 = \sqrt{2}$ B: $x^2 + y^2 = 4$ C: $x^2 + y^2 = 2$ D: $x^2 + y^2 = 2\sqrt{2}$	*Examples* 8 and *9* can be likened to *Examples 3* to *5*, in that they aim to provide an opportunity for **deepening** students' thinking about the mathematical structure of the equation of a circle. They also enable students' understanding of surds to be assessed and developed. Once students have identified the correct equation, teachers may ask them to identify the coordinates of a point, (that does not lie on either of the axes), that lies on a circle with centre at the origin and a radius of $\sqrt{2}$. Highlight the lack of graphical representation in this example with your colleagues. What effect might including a representation have on students' cognitive load? What are the benefits and challenges of presenting the information as both a description and a diagram?
Example 9: *How many of the following equations describes a circle with centre at the origin and a radius of $2\sqrt{3}$?* A: $x^2 + y^2 = 4\sqrt{9}$ B: $x^2 + y^2 = \sqrt{3}$ C: $x^2 + y^2 = 4\sqrt{3}$ D: $x^2 + y^2 = 12$	*Example 9* focuses on students' grasp of surds and simplification of radicals in the context of the equation of a circle. It is important that students recognise the difference between a surd ($\sqrt{3}$) and the radical $\sqrt{9}$, which is equal to 3. The precise use of **language** should be encouraged. Discuss what advice might help teachers who are considering the use of calculators when working on examples of this type. What insight might be lost if students have ready access to a calculator when working with surds? What, if any, insight might be gained?

Know how to use the equation of a circle to determine the coordinates of a point that lies on the circle *Example 10:* *A circle with centre (0, 0) and radius 13 has a point on the circle with coordinates (5, y)* *Write a value for y.*	In *Example 10,* students need to determine the equation of a circle, given its centre and radius, and then use the equation to determine the coordinates of a point that lies on the circle. It is likely that some students will describe the equation of the circle as $x^2 + y^2 = 169$ and others as $x^2 + y^2 = 13^2$ instead. Identifying which **representation** is most helpful in finding the value of y may be an interesting point of discussion: when 13^2 is used and 5 substituted in for x, then the Pythagorean triple is more obvious. Once a value for y has been obtained, see if students recognise that there are two possible values for y when $x = 5$. Discuss how this example could be developed further, for example, by asking students to identify all other possible integer values for x and y.
Know how to use the equation of a circle to determine whether a point lies inside, outside, or on the circle *Example 11:* *A circle has centre (0, 0) and radius 17.* *Does the point (5, 12) lie on the circle, inside the circle, or outside of the circle? How do you know?*	*Example 11* highlights the relationship between the hypotenuse of a triangle with base length x and height y and the radius of a circle, in order to identify how the point (x, y) relates to the circle. This can help with **deepening** students' understanding of the way in which the equation of a circle emerges from the use of Pythagoras' theorem. Students are likely to determine the equation of the circle as being $x^2 + y^2 = 289$ and substitute in for x and y to show that $x^2 + y^2 < 289$ (i.e., (x, y) lies inside the circle). Alternatively, those who have fluency with Pythagorean triples may recognise that the point (5, 12) lies on a circle with a radius of 13. As this is less than 17, the point must therefore lie within the circle. It is important to discuss students' approaches, prompting them to think about how the mathematical structure can help us to draw conclusions, without the need for repeated processes. For further challenge, ask them to give the coordinates of a point that lies on the circle and outside of the circle.
Determine the equation of a circle with centre at the origin, given a point that lies on the circle *Example 12:* *Write the equation of this circle:* 	*Example 12* relies on students recognising that the radius of the circle is equal to the length of the hypotenuse of a right-angled triangle with base length 4 and height 3. It calls back to *Example 1* of this sequence, and teachers may like to explicitly compare the two examples to narrate the building picture of the equation of a circle. Asking students to identify the coordinates of other points that lie on the circle is key to **deepening** students' understanding – both of the properties of a circle, and of the relationship between its equation and Pythagoras' theorem. Discuss the role of the graphical diagram, and how students might respond if the question were expressed in words instead. For example, *'Write the equation of a circle with centre at the origin, that passes through the point (4, 3).'* Would the relationship between the equation of a circle and Pythagoras' theorem be more, less or similarly apparent in this case?

Determine the equation of a circle with centre at the origin, given the endpoints of the diameter *Example 13:* *The endpoints of the diameter of a circle are (25, 0) and (-25, 0).* *a) What is the equation of the circle?* *A point on this circle has the coordinates (7, y).* *b) Write possible values for y.*	Students can sometimes confuse the radius and the diameter of a circle. In *Examples 13* and *14*, they need to think carefully about the circle being described, in order to accurately determine its equation. Encouraging students to consider the graphical **representation** of the circle may be beneficial. It is important that students can explain why the equation of the circle is $x^2 + y^2 = 625$. When determining values for y, students must be able to understand that they do not use the negative root when determining the radius but nonetheless recognise that that a circle with centre at the origin extends to all four quadrants. This means that both (7, 24) and (7, -24) are valid coordinates, even though -25 is not a valid radius. The endpoints given in this example may seem trivial to some teachers; it is easy to reduce this topic to a mechanical exercise of substituting values in/out of equations. However, it is possible to explore the equation of a circle in rich and meaningful ways, with an emphasis on mathematical structure. What was your team's experience of learning the equation of a circle at school? How confident are they in finding meaningful and enriching teaching moments in tasks related to this topic?
Example 14: *The endpoints of the diameter of a circle are (15, 20) and (-15, 20).* *a) What is the equation of the circle?* *A point on this circle has the coordinates (x, 24).* *b) Write possible values for x.*	*Example 14* builds on the thinking explored in *Example 13.* While the graphical **representation** may be considered as imperative here, it is not provided. This requires the students to draw it for themselves, think carefully about the structure of a circle, and continue to keep this in mind as they work more with circles in a graphical context. Appreciating the links between Pythagoras' theorem and the equation of a circle is important if students are to correctly identify the radius as being 25. If students have previously worked through *Example 13,* the **variation** between the examples becomes more apparent. Once it has been established that both examples refer to the same circle, teachers can ask students to identify the coordinates of another point that lies on the circle.

Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 11 Geometry* can be found here:

https://www.ncetm.org.uk/media/3xcpkpft/ncetm_ks4_cc_11_solutions.pdf