

The ICCAMS Teaching Materials:
TEACHER'S HANDBOOK

Acknowledgements

The ICCAMS teaching programme, mini-assessment tasks and lessons were originally developed as part of the Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) research project led by Professor Jeremy Hodgen in collaboration with Professor Margaret Brown, Professor Robert Coe and Dr Dietmar Küchemann.

The design of the materials was a collaborative process led by Professor Jeremy Hodgen although the individual lesson materials were primarily devised by Dietmar Küchemann. We are also indebted to the many teacher-researchers who contributed to this work.

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Background

This handbook has been produced for teachers and schools participating in the ICCAMS Maths intervention. ICCAMS Maths is a collaborative project between the University of Nottingham and Durham University and is being evaluated by the University of Manchester. The project is funded by the Education Endowment Foundation (EEF).

ICCAMS Maths supports teachers in tackling students' common misconceptions around algebra and multiplicative reasoning and provides teachers with training, lesson plans and resources to help embed formative assessment in Year 7 and Year 8 mathematics classrooms.

Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) was funded by the Economic and Social Research Council (ESRC) to investigate ways of raising students' attainment and engagement by using formative assessment in secondary school mathematics. The project focused on algebra and multiplicative reasoning because these topics are central to the current mathematics curriculum and also to further participation in mathematics, science and STEM more generally.

These materials have been designed to address the algebra and multiplicative reasoning topics in the Years 7 and 8 National Curriculum. The materials are supported by an extensive professional development programme.

There are more than 40 ICCAMS lessons, split between algebra and multiplicative reasoning. The lessons are organised in pairs with a related mini-assessment task. Additional lessons and mini-assessment tasks have been provided to enable teachers to give additional support for low attaining students or as extension activities for high attainers.

During the two-year ICCAMS Maths intervention, the requirement is that teachers will teach 40 ICCAMS lessons together with the related mini-assessment tasks plus 10 or more revisits tasks. This means that, in each of Year 7 and Year 8, teachers will use 10 mini-assessment tasks, teach 10 lessons pairs and teach at least 5 revisit tasks.

How can I use the materials?

The materials are set out to enable you to assess the students before teaching each set of two linked lessons. Over the next few pages we show you how to use these materials. The lessons are designed to help you figure out what students already know, what they struggle with and how best to provide challenge whilst addressing any difficulties. To do this we have several aims:

To provide guidance on using formative assessment in mathematics classrooms. For example, the materials are designed to highlight students' misconceptions, build on what students already know and support the use of appropriate feedback.

To set activities in realistic contexts. By realistic, we do not mean that all the activities are set in real life contexts that students may have encountered, but rather contexts that the learners can imagine and engage with. To do this we have designed activities that are intended to be intriguing and provide opportunities for students to have insights into mathematical structure.

To make connections between mathematical ideas. For example, we suggest ways of teaching algebra that bring together the often fragmented activities of tabulating values, solving equations, drawing graphs, and forming and transforming algebraic expressions and relations.

To encourage students to collaborate and talk. By collaborating and talking, students can gain insights from each other and at the same time enable teachers to find out what the students understand.

To use different representations, such as the double number line to help students better understand mathematical concepts such as ratio.

The mini-assessment tasks

Before teaching a particular pair of lessons, use the mini-assessment task to assess what the students know about the ideas and what they may find difficult. The mini-assessment task should take no more than 10-15 minutes and should be used a day or two before teaching the first of the linked lessons.

Try to observe and listen to the students in order to focus on assessment rather than teaching. There is no need to resolve the mini-assessment tasks. The commentary suggests a few things to look for when you observe the students. You may find it helpful to spend some time after the mini-assessment task thinking about what the students said and what they found easy or hard.

The format of the mini-assessment tasks varies. Sometimes they are in the form of a problem to present to the whole class, but there are also a few short tests to be carried out individually.

The lesson materials

Each lesson description consists of two double-page spreads. You do not need to read all of this before you teach the lessons for the first time.

If you are short of time, the outline on the first page should give you enough information to teach the lesson, although you may find the overview helps you to think about formative assessment and feedback as well as ways of adapting the lessons.

The revisits

A large number of 'revisit' tasks are provided to enable you to consolidate or extend key ideas in some of the lessons. Choose the tasks that are best suited to your class. The requirement for the trial is that each teacher will use at least 10 revisit tasks over the two years of the trial, teaching at least 5 revisits in each of Year 7 and Year 8.

Most of the revisit tasks are intended to take around 10-15 minutes rather than a whole lesson. They are available online as well as in a separate addendum to this handbook.

Frequently asked questions

Does ICCAMS work?

In the original ESRC-funded research study, we tested ICCAMS in 22 classes from 11 schools and found that the rate of learning for students in ICCAMS Maths classes was double that for students compared to a control group. The current trial is intended to evaluate whether ICCAMS Maths works at scale across a range of different schools.

Does ICCAMS Maths encourage mastery?

Our aim is that students will become fluent with maths and will be able to tackle non-routine problems successfully. We emphasise ways of ensuring that students develop conceptual understanding.

Can ICCAMS Maths be used alongside my school's existing scheme of work?

ICCAMS Maths is designed to supplement, not replace, ordinary lessons. The programme meets the requirements of the new National Curriculum and has been used with many different schemes of work (and many different textbooks). The links to the National Curriculum are outlined on pages 12-17.

Is ICCAMS Maths appropriate for low and high attainers?

ICCAMS Maths caters for all students across the attainment range and the lessons have been trialed in a wide range of different classes. The programme is designed to help you improve teaching and learning across all lessons and for all students by assessing what your students understand, what they find difficult and what they need to learn next.

Some lessons and revisits may be particularly appropriate for low or high attainers. For some low attaining classes, you may wish to spend more time on the early multiplicative reasoning lessons. You can do this by using revisit tasks and/or by using the additional lessons in the second pair of lessons (Lessons 2 Extra and 2 Further). Some of the later extension tasks may be particularly appropriate for high attainers (for example, Lesson Pairs 11, 17, 22, 23 and 24).

Do I need to teach the lessons in pairs?

Yes, the lesson pairs are designed to give students extended experiences of linked mathematical ideas.

Can I use the mini-assessment task at the start of an ICCAMS lesson?

No, the mini-assessment task should be taught before teaching either of the linked lessons. They have been designed to allow you time to consider how the students are likely to engage with the lesson and what they might find challenging.

Do I need to read all the lesson materials before teaching?

No, if you are short of time, the outline on page 1 of the lesson notes is intended to give you enough information to teach the lesson. You may find it helpful to read the overview on page 2 as well. The other materials may be more useful when you are teaching a lesson for the second or third time.

Can anyone use the lesson materials?

Please do not share these materials beyond your school, because the materials are being evaluated through the trial.

My students took longer than one lesson to do the activity. Does this matter?

It is important to give students time to engage with and talk about the mathematical ideas in the lessons. But there is no need to reach the 'end' of every task.

Where can I find out more?

Look on the ICCAMS website:

iccams-maths.org

Further Reading

This short pamphlet outlines the ideas that underlie the ICCAMS approach to formative assessment. It can be obtained from GL Assessment.

Hodgen, J., & Wiliam, D. (2006). *Mathematics inside the black box*. London: NFER-Nelson.

The following papers describe the teaching approach in more detail and can be downloaded from the ICCAMS website: <http://iccams-maths.org>

Brown, M., Hodgen, J., & Küchemann, D. (2014). Learning Experiences Designed to Develop Multiplicative Reasoning: Using Models to Foster Learners' Understanding. In P. C. Toh, T. L. Toh & B. Kaur (Eds.), *Learning Experiences that Promote Mathematics Learning*. Singapore: World Scientific.

Brown, M., Küchemann, D. E., & Hodgen, J. (2010). The struggle to achieve multiplicative reasoning 11-14. In M. Joubert & P. Andrews (Eds.), *Proceedings of the Seventh British Congress of Mathematics Education (BCME7)* (Vol. 30, pp. 49-56). University of Manchester: BSRLM.

Hodgen, J., Coe, R., Brown, M., & Küchemann, D. E. (2014). Improving students' understanding of algebra and multiplicative reasoning: Did the ICCAMS intervention work? In S. Pope (Ed.), *Proceedings of the Eighth British Congress of Mathematics Education (BCME8)* (pp. 167-174). University of Nottingham: BSRLM / BCME.

Hodgen, J., Küchemann, D., & Brown, M. (2014). Learning Experiences Designed to Develop Algebraic Thinking: Lessons from the ICCAMS Project in England. In P. C. Toh, T. L. Toh & B. Kaur (Eds.), *Learning Experiences that Promote Mathematics Learning*. Singapore: World Scientific.

Although this book is more than 30 years old, it is an invaluable introduction to how students understand mathematics and the misconceptions that they may have:

Hart, K. (Ed.). (1981). *Children's understanding of mathematics: 11-16*. London: John Murray.

Mini-assessment task notes: commentary

The Mini-assessment task

This is the task to present to students. Use this a day or two before teaching the first of the linked lessons. It should take no more than 10-15 minutes.

Focus on *assessment* rather than teaching. Observe what the students do and say.

There is no need to resolve the mini-assessment tasks.

Algebra 1A8

Which is larger, $3n$ or $n + 3$?

Commentary

The aim of this Mini-assessment is to see what approaches students use to compare algebraic expressions.

- Do students understand the algebraic notation?
- Do they focus on the operations ('multiplication makes bigger')?
- Do they evaluate the expressions for specific values of n ?
- Do they respond to the fact that we don't know the value of n ?
- Do they realise that the difference between the expressions might change as n varies?

Use the Mini-assessment a few days before teaching the two lessons.

page 8

A commentary on what to look for

This suggests things that the students may do or say.

Lesson notes: outline and overview

An outline of the lesson (Page 1)

This brief summary and outline of the lesson is designed to be used as aide-memoire when you are teaching the lesson.

Overview (Page 2)

This gives more background on the rationale and the mathematical ideas underlying the lesson. It also suggests how the lesson can be adapted.

Lesson 3A

Boat hire

Olaf is spending the day at a lake. He wants to hire a rowing boat for some of the time. Freya's boat hire charges £3 per hour. Polly's boat hire charges £10 plus £1 per hour. Whose boat should Olaf choose?

Summary

In this lesson, the boat hire problem is used to explore the two algebraic relationships underlying Freya's and Polly's different hire charges. A variety of representations are used to express the relationships:

- everyday language
- algebraic expressions
- tables of values
- points on a Cartesian graph.

Outline of the lesson

1. Display the *boat hire* problem.
 - Ask students for their immediate responses.
 - Ask students to consider the problem further in small groups.
 - Collect numerical data on the total cost for various numbers of hours (ask learners to discuss arguments and conclusions – but don't pursue these at this stage).
 - Represent the data
 - "spontaneously" on the board
 - in a (randomly ordered) table
 - in an ordered table (try to prompt the need for this, rather than simply produce such a table).
2. Ask students to represent the hire-rates algebraically.
 - For example, as expressions, eg $5a$ and $10 + a$
 - Or as relations, eg $b = 5a$, $b = 10 + a$.
3. Ask students to represent the data as points on a standard (Cartesian) graph.
 - Ask questions like "Are Freya's and Polly's charges ever equal?"
4. Discuss the links between the various representations and between them and the story.

Lesson 3A

Overview

Mathematical ideas

When we work with equations, we often think of a letter as representing a single number or yet unknown. Here, we are working with relations between two variables (the number of hours and the charge) and we think of the letters as representing a set of numbers.

A Cartesian graph is a particularly powerful way of representing variables, since it allows us to represent a range of values simultaneously.

Students' mathematical experiences

Students might discover some of the following:

- for some values of a , Freya's hire charge ($5a$) is larger than Polly's ($10 + a$), but for others it is smaller
- when $a = 2.5$ the expressions are equal
- if a increases by 1, then $5a$ increases by 5, but $10 + a$ only increases by 1
- each set of points on the graph forms a pattern: each lies on a straight line.

Students might discuss

- different slopes and how these relate to the hourly charges
- continuity: whether some or all points on the line fit the relationship.

Some students may want to change the scales of the axes. Discourage them from doing so. The expressions have been deliberately chosen to be represented on an equally-scaled graph.

Key questions

When is Freya (or Polly) cheaper?

How could we record this more systematically?

What is the cost as the number of hours increases?

When would a boat for 1.5 hours?

Adapting the lesson

You might want to adapt a different problem for subsequent lessons. For example, the price of an ice cream cone for different numbers of scoops, or the cost of a swimming club (based on a membership fee and a cost for each visit). Choose the numbers carefully: if you want to graph the relationships, think about where the lines intersect. Do you want this to occur for a simple whole number (if ice cream scoops, say), or something more complex (if ice cream cones)?

Assessment and feedback

Be flexible over the organisation and timing of the lesson. Some teachers have taught this lesson over two periods.

Choose some students to present their work to the class. Ask them to explain their work to the class. Let's have a go at preparing what you'll say.

Allow students time to generate algebraic expressions, but if they really struggle you may want to provide the expressions for them.

Some students may have difficulties constructing a Cartesian graph. Observe the students and decide whether you need to spend more time with either a group or the whole class teaching these skills.

You may need to prompt or challenge some students to consider whether the charges can ever be equal.

Assessment and feedback ideas

Key Questions

Adapting the lesson

This may suggest additional / alternative activities or Revisits to consolidate / extend the ideas.

Lesson notes: annotated outline and background

Extended annotated outline (Page 3)

This is a longer read which will help you plan your teaching interventions. We recommend that you read this before teaching the lesson.

Background (Page 4)

This describes some of the key mathematical or pedagogical issues in the lesson.

Lesson 3A

Outline of the lesson (annotated)

1. Display the *first hire* problem.
 - Ask students for their immediate responses.
2. Ask students to consider the problem further in small groups.
 - Collect numerical data on the total cost for various numbers of hours (and listen to students' arguments and conclusions – but don't pursue these at this stage).
 - Represent the data
 - verbally on the board
 - in a (randomly ordered) table
 - in an ordered table (try to prompt the need for this, rather than simply produce such a table)
3. Ask students to represent the data as points on a standard (Cartesian) graph.
 - For example, as expressions, eg. $5a$ and $10 + a$
 - Or as relations, eg. $h = 5a$, $h = 10 + a$
4. Discuss the links between the various representations and between them and the story.

Algebra 1A

Students will tend to say that one or other hire firm is cheaper. Let them voice their opinions but don't discuss their reasons at this stage. (Their decisions may be based on one or two numerical examples, or they may have focused on the lower hourly rate, in the case of Polly, or the absence of a charge-upfront in the case of Fred's.) The difference of opinion should make the task more engaging for students.

This allows students to think more carefully about their and other students' ideas.

With the students' help, start to organise the data in a more systematic way, contributing to ordered tables. Some students might already have developed a rudimentary understanding of the problem, involving the idea that the choice of hire company depends on how long you want the boat for. That, being so, help to prompt and allow students time to consider how and why the two expressions differ.

Ordered tables allow one to bring out the fact that increasing the hire time by one hour, increases the hire charge by £5 (Fred's) or £1 (Polly's). This can of course be related back to the original story, but the data in the tables to the algebraic and graphical representations of the relationship.

If students find this difficult, then present the expressions in terms, without too much fuss. (Or state the expressions in words, eg. "5 × number of hours")

It is more illuminating to see more with the same words, even though the resulting line for Fred's is then very steep. To avoid getting bogged down in the issue of choosing and drawing suitable axes (which are of course very important), you might want to give students a blank version of the graph with the axes already drawn and numbered.

Lesson 3A

Background

Students develop their understanding of 'variable'

The kind of variable we are learning about (the a called 'a' or 'the number of hours') has these properties:

- It is a number.
- It can take on lots of different values.
- as it changes in a systematic way, the 'dependent variable' (in this case 'the total hire charge') may also change in a systematic way.

Specifically, students might discover that:

- for 'large' values, such as $a=6$, $5a$ is larger than $10+a$
- for some 'small' values, like $a=1$, $5a$ is smaller than $10+a$
- there is a particular value, $a=2$, where the two expressions are equal.
- if a is increased by a specific amount, then $10+a$ increases by the same amount, whereas $5a$ increases by 5 times that amount.

Students develop their ability to 'read' various representations

Each of the properties of variable mentioned above can be seen in the three representations (tables, expressions, graphs). For example,

- if a value of a in a table goes up by 1, then the corresponding value of $5a$ has a difference of 5 (see the second table, right)
- if two points representing $(a, 5a)$ are 1 horizontal unit apart on a graph, they will be 5 vertical units apart (see points P and Q on the graph, below right)
- the two sets of points on the graph form a pattern (in this case, they lie on two straight lines), and so help students get a general sense of the relationships, in a sense of what happens for a range of values rather than just isolated numerical cases
- the pattern (in the straight lines) also suggest that there is a systematic relationship between a and $5a$, and between a and $10+a$ and that any further 'boat-hire' points that we might calculate will also lie on the corresponding line.
- the two lines cross, so there is a strong incentive to find the common point (in this case, $(2, 5, 12.5)$) and to consider what is meant by the point where the lines intersect.

Students create algebraic expressions

The problem can be solved perfectly well without algebraic symbolism, on the other hand, the expressions do provide a convenient shorthand and the context should help

Algebra 1A

make them meaningful to the students. It also allows us to refer explicitly to variables.

Students engage with the notion of continuity

The graph, in particular, prompts the question of 'intermediate' values: given that each set of points lies on a line, what about those other points on that line: do they fit the algebraic relationship, and do they fit the boat-hire story? And do 'all' points on the line fit the relationship? Many hire firms charge in whole numbers of hours (so a boat used for 2 hours 15 mins, say, would be charged for 3 hours). This results in a step function, represented on a Cartesian graph by a set of horizontal lines.

a	$5a$	$10 + a$
1	5	11
2	10	12
3	15	13

ICCAMS: Links to the National Curriculum

ICCAMS has been designed to align with the National Curriculum and this document sets out the main links between the lessons and the Key Stage 3 (KS3) programme of study. Note that ICCAMS is intended to supplement rather than replace a school's scheme of work for Year 7 and Year 8. ICCAMS places more emphasis on conceptual understanding and less emphasis on practice. The lessons cover most of the key conceptual ideas in the "Number", "Algebra" and "Ratio, proportion and rates of change" content strands of the KS3 curriculum together with some related aspects of the "Geometry and measures" strand. In addition to the lessons outlined here, the programme provides short 'revisit' tasks that enable teachers to choose to either consolidate or extend ideas

Lesson Pairs		Teaching Year		National Curriculum KS3 Programme of Study
Multiplicative Reasoning	Algebra	Y7	Y8	
				Working mathematically
				Develop fluency
All	21			Consolidate their numerical and mathematical capability from key stage 2 and extend their understanding of the number system and place value to include decimals, fractions, powers and roots
All, 1, 2, 15	12, 21			Select and use appropriate calculation strategies to solve increasingly complex problems
1, 2, 15	All			Use algebra to generalise the structure of arithmetic, including to formulate mathematical relationships
-	All	×		Substitute values in expressions, rearrange and simplify expressions, and solve equations
7, 9, 16	All			Move freely between different numerical, algebraic, graphical and diagrammatic representations [for example, equivalent fractions, fractions and decimals, and equations and graphs]
-	All	×		Develop algebraic and graphical fluency, including understanding linear and simple quadratic functions
All	All			Use language and properties precisely to analyse numbers, algebraic expressions, 2-D and 3-D shapes, probability and statistics.

ICCAMS: Links to the National Curriculum

Multiplicative Reasoning	Algebra	Y7	Y8	National Curriculum KS3 Programme of Study
				Reason mathematically
All	All			Extend their understanding of the number system; make connections between number relationships, and their algebraic and graphical representations
All	5, 18, 21			Extend and formalise their knowledge of ratio and proportion in working with measures and geometry, and in formulating proportional relations algebraically
-	All	×		Identify variables and express relations between variables algebraically and graphically
-	All	×		Make and test conjectures about patterns and relationships; look for proofs or counter- examples
All	All			Begin to reason deductively in geometry, number and algebra, including using geometrical constructions
All	-		×	Interpret when the structure of a numerical problem requires additive, multiplicative or proportional reasoning
-	-	×	×	Explore what can and cannot be inferred in statistical and probabilistic settings, and begin to express their arguments formally.
				Solve problems
All	All			Develop their mathematical knowledge, in part through solving problems and evaluating the outcomes, including multi-step problems
All, 7, 16	All			Develop their use of formal mathematical knowledge to interpret and solve problems, including in financial mathematics
All	All, 3, 4, 13, 20, 21			Begin to model situations mathematically and express the results using a range of formal mathematical representations
All	All			Select appropriate concepts, methods and techniques to apply to unfamiliar and non- routine problems.

ICCAMS: Links to the National Curriculum

Multiplicative Reasoning	Algebra	Y7	Y8	National Curriculum KS3 Programme of Study
				Subject content
				Number
6, 7, 9, 16	-		×	Understand and use place value for decimals, measures and integers of any size
6, 9, 11, 16, 17	-		×	Order positive and negative integers, decimals and fractions; use the number line as a model for ordering of the real numbers; use the symbols =, ≠, <, >, ≤, ≥
9	-		×	Use the concepts and vocabulary of prime numbers, factors (or divisors), multiples, common factors, common multiples, highest common factor, lowest common multiple, prime factorisation, including using product notation and the unique factorisation property
1, 2, 15	-		×	Use the four operations, including formal written methods, applied to integers, decimals, proper and improper fractions, and mixed numbers, all both positive and negative
1, 2, 15	12			Use conventional notation for the priority of operations, including brackets, powers, roots and reciprocals
2, 15	-		×	Recognise and use relationships between operations including inverse operations
-	-	×	×	Use integer powers and associated real roots (square, cube and higher), recognise powers of 2, 3, 4, 5 and distinguish between exact representations of roots and their decimal approximations
-	-	×	×	Interpret and compare numbers in standard form $A \times 10^n$ $1 \leq A < 10$, where n is a positive or negative integer or zero
9	-		×	Work interchangeably with terminating decimals and their corresponding fractions (such as ...)
9, 16	-		×	Define percentage as 'number of parts per hundred', interpret percentages and percentage changes as a fraction or a decimal, interpret these multiplicatively, express one quantity as a percentage of another, compare two quantities using percentages, and work with percentages greater than 100%
7, 11, 16, 17, 24	-		×	Interpret fractions and percentages as operators
6, 7, 11, 16	-		×	Use standard units of mass, length, time, money and other measures, including with decimal quantities

ICCAMS: Links to the National Curriculum

Multiplicative Reasoning	Algebra	Y7	Y8	National Curriculum KS3 Programme of Study
				Number (continued)
7, 17	-		×	Round numbers and measures to an appropriate degree of accuracy [for example, to a number of decimal places or significant figures]
-	-	×	×	Use approximation through rounding to estimate answers and calculate possible resulting errors expressed using inequality notation $a < x \leq b$
-	-	×	×	Use a calculator and other technologies to calculate results accurately and then interpret them appropriately
-	-	×	×	Appreciate the infinite nature of the sets of integers, real and rational numbers.
				Algebra
-	3, 4, 14, 18, 20, 22	×		Use and interpret algebraic notation, including ...
-	All, 3, 4, 13, 18, 20	×		Substitute numerical values into formulae and expressions, including scientific formulae
-	All	×		Understand and use the concepts and vocabulary of expressions, equations, inequalities, terms and factors
-	18, 19, 20, 21, 23	×		Simplify and manipulate algebraic expressions to maintain equivalence by ...
-	4, 14, 18, 21	×		Understand and use standard mathematical formulae; rearrange formulae to change the subject
-	All, 3, 13, 14, 20, 22, 23	×		Model situations or procedures by translating them into algebraic expressions or formulae and by using graphs
-	18, 19, 21	×		Use algebraic methods to solve linear equations in one variable (including all forms that require rearrangement)
-	3, 13, 23	×		Work with coordinates in all four quadrants
-	3, 4, 5, 13, 20, 22, 23	×		Recognise, sketch and produce graphs of linear and quadratic functions of one variable with appropriate scaling, using equations in x and y and the Cartesian plane
-	All	×		Interpret mathematical relationships both algebraically and graphically
-	5, 20, 22	×		Reduce a given linear equation in two variables to the standard form $y = mx + c$; calculate and interpret gradients and intercepts of graphs of such linear equations numerically, graphically and algebraically

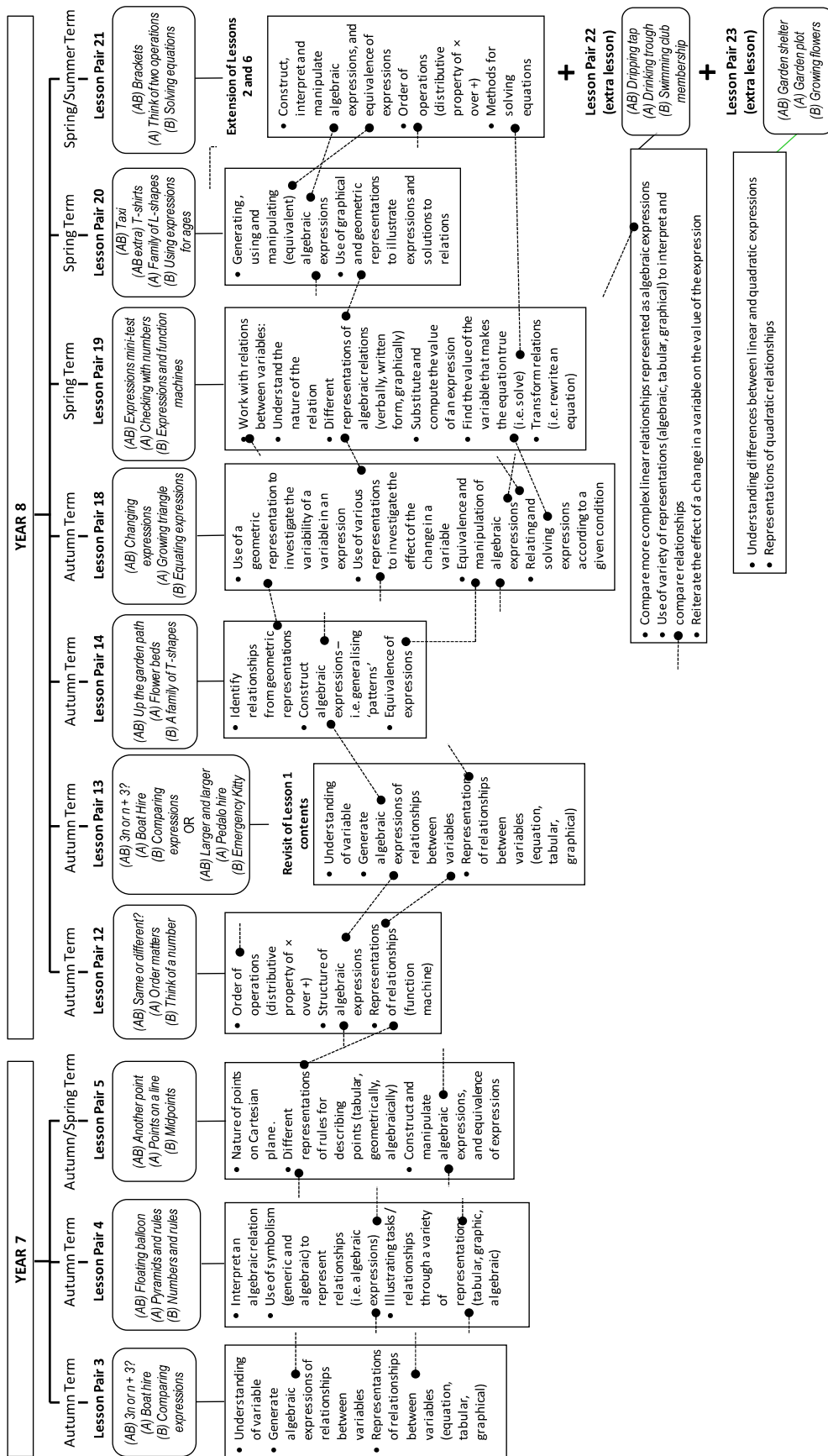
ICCAMS: Links to the National Curriculum

Multiplicative Reasoning	Algebra	Y7	Y8	National Curriculum KS3 Programme of Study
				Algebra (continued)
-	3, 13, 18, 20, 22	×		Use linear and quadratic graphs to estimate values of y for given values of x and vice versa and to find approximate solutions of simultaneous linear equations
-	5, 22, 23	×		Find approximate solutions to contextual problems from given graphs of a variety of functions, including piece-wise linear, exponential and reciprocal graphs
-	14, 20, 21	×		Generate terms of a sequence from either a term-to-term or a position-to-term rule
-	14, 19	×		Recognise arithmetic sequences and find the nth term
-	-	×	×	Recognise geometric sequences and appreciate other sequences ...
				Ratio, proportion and rates of change
6, 7, 11, 15	-		×	Change freely between related standard units [for example time, length, area, volume/capacity, mass]
6, 7, 24	-		×	Use scale factors, scale diagrams and maps
10, 16, 17	-		×	Express one quantity as a fraction of another, where the fraction is less than 1 and greater than 1
6, 7, 8, 11, 16	-		×	Use ratio notation, including reduction to simplest form
6, 7, 8, 11, 17	-		×	Divide a given quantity into two parts in a given part:part or part:whole ratio; express the division of a quantity into two parts as a ratio
6, 7, 8, 9, 10, 11, 15, 24	-		×	Understand that a multiplicative relationship between two quantities can be expressed as a ratio or a fraction
6, 7, 11, 17	-		×	Relate the language of ratios and the associated calculations to the arithmetic of fractions and to linear functions
7, 16, 17	-		×	Solve problems involving percentage change, including: percentage increase, decrease and original value problems and simple interest in financial mathematics
8, 11, 16, 24	-		×	Solve problems involving direct and inverse proportion, including graphical and algebraic representations
16	-		×	Use compound units such as speed, unit pricing and density to solve problems.

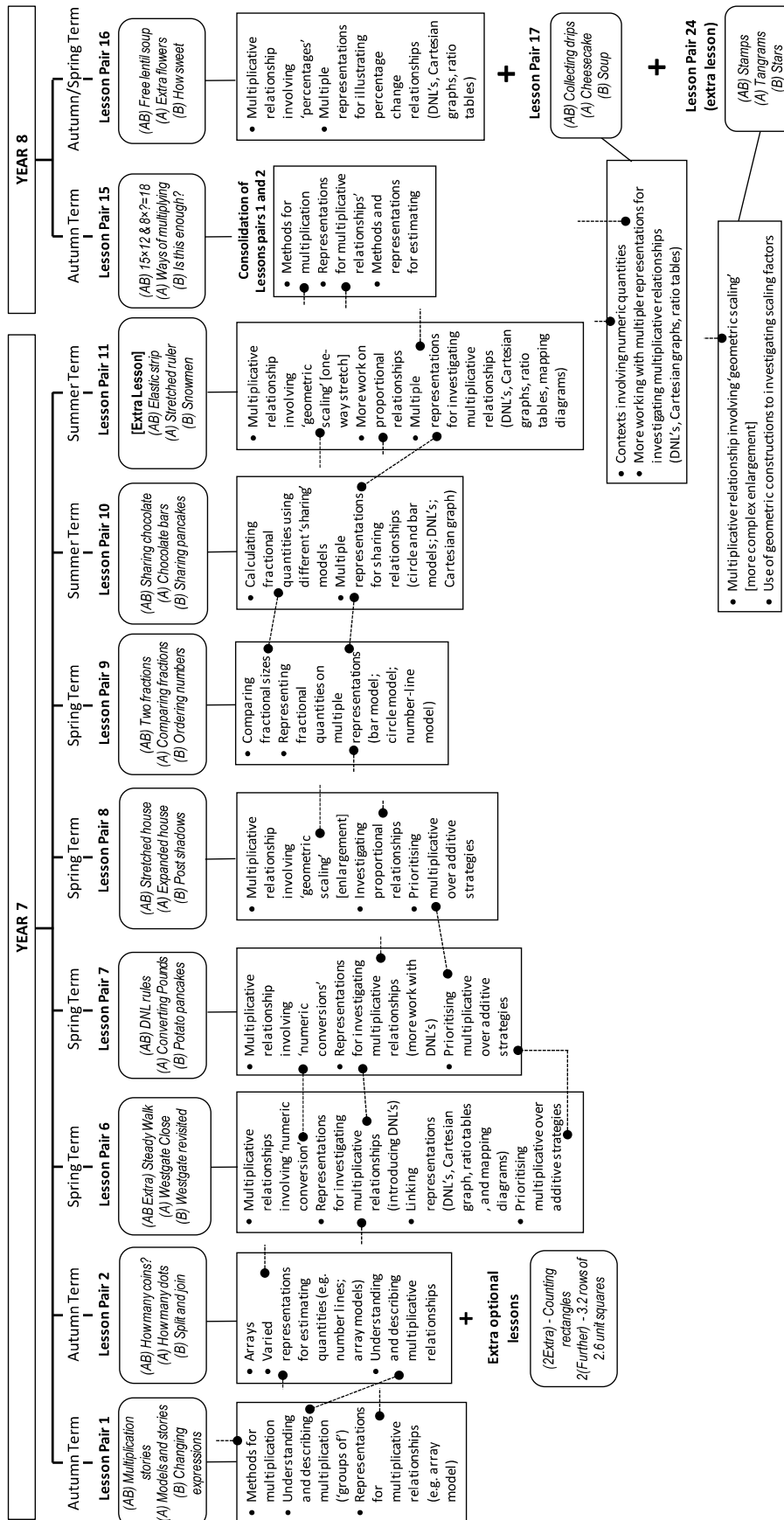
ICCAMS: Links to the National Curriculum

Multiplicative Reasoning	Algebra	Y7	Y8	National Curriculum KS3 Programme of Study
				Geometry and measures
-	14, 18, 20	×		Derive and apply formulae to calculate and solve problems involving: perimeter and area of triangles, parallelograms, trapezia, volume of cuboids (including cubes) and other prisms (including cylinders)
-	14, 20	×		Calculate and solve problems involving: perimeters of 2-D shapes (including circles), areas of circles and composite shapes
12	-		×	 construct similar shapes by enlargement, with and without coordinate grids
All	All			Interpret mathematical relationships both algebraically and geometrically.

Key Mathematical Content Map: Algebra Lessons



Key Mathematical Content Map: Multiplicative Reasoning Lessons



Options and extensions to the lessons

Year 7

Lesson pair 2: Lessons 2Extra and 2Further are optional lessons to be used if you judge your students need additional consolidation of the array / area models of multiplication. You may additionally wish to consolidate this work on models for multiplication using additional revisit tasks, see lesson materials for guidance.

If you teach Lesson 2Extra “Counting rectangles”, there is no requirement to teach Lesson 10B “Sharing pancakes”.

If you teach Lesson 2Further “3.2 rows of 2.6 unit squares”, there is no requirement to teach Lesson 10A “Sharing chocolate bars”.

Lesson Pair 11: This is an optional pair of lessons provided for extension or consolidation.

Year 8

Lesson Pair 13: You may wish to repeat the Boat Hire lesson pair (Lessons 3A and 3B) as an alternative to Lessons 13A and 13B. See the lesson materials for this and other alternative possibilities.

Lesson Pair 15: Mini-assessment 15AB Extra “ $8 \times ? = 18$ ” is optional. If used, it should be used in addition to the 15×12 task. See lesson materials for guidance.

Lesson Pair 17: You may wish to use some of earlier multiplicative reasoning lessons as an alternative to Lessons 13A and 13B (such as Lesson 3A, Lesson 4A or 8A). See the lesson materials for guidance.

Lesson Pair 20: The extra mini-assessment task, 20AB “T-shirts”, is provided as an alternative to Taxi! The task is very similar.

Year 8 Extensions

Lesson Pair 22: This is an optional pair of lessons building on and consolidating earlier algebra lessons.

Lesson Pair 23: This is an optional pair of lessons addressing quadratic expressions and relationships.

Lesson Pair 24: This is an optional pair of lessons building on and consolidating earlier multiplicative reasoning lessons and addressing enlargement.

Algebra and Multiplicative Reasoning

Mini-test

ICCAMS pre-test

ICCAMS pre-test Name Class Date

NO CALCULATORS Please show all your working/jottings

1. CELERY SOUP for 6 people

50 ml of cream

20 g of salt

30 g of butter

A small bunch of celery leaves finely chopped

← This is part of a recipe for Celery Soup. It is for 6 people.

How much cream is needed for 12 people?

How much salt is needed for 15 people?

How much butter is needed for 7 people?

2. Ben and Shirley work in a laundry where they iron shirts. They work at a steady pace but Ben is a bit slower than Shirley.

Write down the missing numbers:

When Ben has ironed 8 shirts, Shirley has ironed 9.

When Ben has ironed 16 shirts, Shirley has ironed

When Ben has ironed 20 shirts, Shirley has ironed

3. Each shoe print is 0.3 m long.

How long is the row of 4 shoe prints?

Look at these two numbers: 0.65 and 0.7

Which number is larger?

4. $24 \times 41805 = 1003320$.

Use this to write down the value of

24×41806 48×41805

5. a. If $u + v = 30$

$u + v + 4 =$

b. If $u + v = 30$

$3u + 3v =$

c. If $u + v = 30$

$u + 2v =$

Commentary

This is a short test which should be given to students at the beginning of Year 7. **Allow about 15 minutes.**

The test touches on some key ideas in algebra and multiplicative reasoning and should give you an indication of the variety of student responses in your class.

The items vary in difficulty and you should alert students to this: they shouldn't worry if some items appear strange or difficult, and they should have a go nonetheless. So for example, in the *Recipe* question (which is based on the CSMS Ratio test), some students might recognise that the situation is multiplicative where it involves doubling but might resort to an additive approach where the underlying multiplier is $\times 2.5$.

Question 3 (which is based on items from the CSMS Decimals test) probes students' understanding of place value and whether, for example, students ignore the decimal point when comparing decimal fractions.

Question 4 tests students' willingness to examine the structure of a numerical expression without evaluating the expression, while Question 5 (which is based on the CSMS Algebra test) looks at whether students can accept the lack of closure of an 'answer' like $30 + v$.

ICCAMS pre-test Name Class Date

NO CALCULATORS Please show all your working/jottings

1.

CELERY SOUP for 6 people50 ml of **cream**20 g of **salt**30 g of **butter**A small bunch of celery leaves
finely chopped← This is part of a recipe for Celery Soup.
It is for 6 people.

How much cream is needed for 12 people?

How much salt is needed for 15 people?

How much butter is needed for 7 people?

2. Ben and Shirley work in a laundry where they iron shirts.
They work at a steady pace but Ben is a bit slower than Shirley.

Write down the missing numbers:

When Ben has ironed 8 shirts, Shirley has ironed 9.

When Ben has ironed 16 shirts, Shirley has ironed

When Ben has ironed 20 shirts, Shirley has ironed

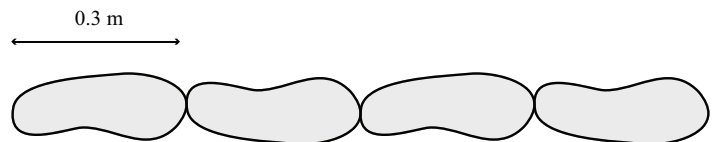
3. Each shoe print is 0.3 m long.

How long is the row
of 4 shoe prints?

Look at these two numbers:

0.65 and **0.7**

Which number is larger?



4. $24 \times 41805 = 1003320$.

Use this $\uparrow$ to write down the value of 24×41806 48×41805

5. a. If $u + v = 30$

$u + v + 4 = \dots\dots\dots$

- b. If $u + v = 30$

$3u + 3v = \dots\dots\dots$

- c. If $u + v = 30$

$u + 2v = \dots\dots\dots$

Mini-test

ICCAMS pre-test

Notes

Multiplicative Reasoning 1AB

Mini-assessment

1AB**Multiplication stories**Here's a story for $4 + 10$

A jar contains 4 red pens and 10 blue pens.

Invent another story for $4 + 10$

How many pens is this altogether?

Now invent a story for 4×10 **Commentary**

It is more difficult to come up with a multiplication story than an addition story, so give the class plenty of time to attempt the second item.

If a student comes up with a story like “Nick said what is 4×10 . I said it's 40”, say, “Yes, it's a story but we want a *situation* that involves multiplication”.

Beyond that, don't critique the multiplication stories, even if they don't fit.

Encourage students to come up with a variety of multiplication stories, even if they turn out to be very similar. Note that the focus is not on originality here. It is perfectly OK for students to use each other's stories and just adapt them slightly, especially if this indicates that they recognise a story-form that fits.

Keep a note of multiplication stories that *do* fit: you might want to refer to them in the subsequent pair of lessons.

Multiplicative Reasoning 1A

Lesson

1A**Models and stories****Here is an expression involving 12 and 3:**

$$12 \times 3$$

- Think of
- some ways of saying " 12×3 "
 - some ways of calculating 12×3
 - some diagrams that fit the expression
 - some stories that fit the expression.

Summary

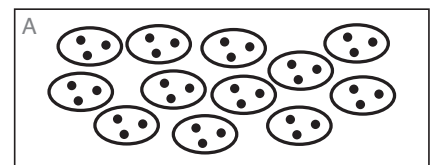
This lesson includes a strong assessment element designed to help you plan a *programme* of work on students' understanding of multiplication and on their skills and fluency.

Here we consider how diagrams and stories can be used to represent arithmetic operations, in particular multiplication. The aim is to help students recognise whether a situation is multiplicative, and to show that a given operation can be thought of in structurally different ways.

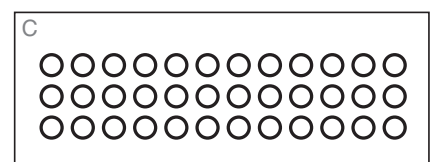
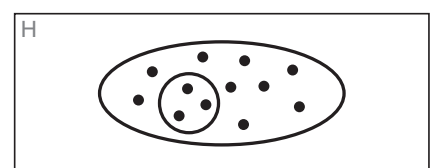
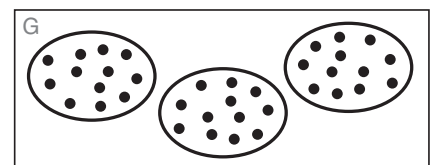
Lesson 1B provides further opportunities to use stories and diagrams, while lessons 2A and 2B look more closely at the array and include several *revisit* activities that can be used throughout the year. We focus on *ways* of multiplying and on estimation in the Year 8 lessons 15A and 15B.

Outline of the lesson

- Consider the expression 12×3 .
 - Present the above tasks. Let students work in small groups. Discuss students' verbalisations, methods, diagrams, stories.
- Interpret a diagram for 12×3 and a matching story.
 - Show Diagram A.
"We can think of this as '12 lots of 3'. Is it like any of our diagrams?"
 - Show this story.
"This fits the diagram. Is it like any of our stories? Write another that fits."
- Interpret another diagram for 12×3 and write a matching story.
 - Present Diagram G. "Here is another diagram for 12×3 . How could we say it in words? Is it like any of our diagrams?"
 - Write some stories that fit the diagram.
 - Compare the stories for Diagrams A and G.
- Determine expressions and write stories for other diagrams.
 - Present other diagrams, eg H and C.
(Choose from those on page 5.)
What expressions involving 3 and 12 could they represent?
 - Choose a diagram for 3×12 , eg C.
Write a story that fits.
- Draw diagrams for an expression.
 - Ask students to sketch some diagrams for 4×7 , say. Discuss the diagrams. Try to classify them into groups.
- Extension: Consider non-integer expressions.
 - Try to draw a diagram and to write a story for 4.1×7.2 , say.

**A story for 12x3**

Zak wants to learn to cook.
He has 3 lessons a week for 12 weeks.
How many lessons is this altogether?



Multiplicative Reasoning 1A

Lesson

1A

Overview

Mathematical ideas

The MR-1AB Mini-assessment will give you some indication of what meanings your students give to multiplication. This lesson focusses on meanings while Lesson 1B focusses more explicitly on the structure of multiplicative expressions.

These two lessons differ somewhat from our other lessons in that we see them as exemplars of lessons that you might want to give throughout the year, as a way of enhancing students' understanding of and fluency in multiplication.

The use of stories should help students realise that the two numerical elements in a multiplication are usually of a different kind (eg number of eggs and number of egg-boxes), in contrast to addition and subtraction.

The focus of the lesson is on multiplication but you might (during this and/or later lessons) want to take a similar look at the operations of addition, subtraction and division.

Students' mathematical experiences

Students have the opportunity to

- expand the range and variety of contexts, stories and diagrams that they can use to understand multiplicative relationships
- consider when contexts, stories and diagrams involve multiplicative relationships (and when they do not).

Assessment and feedback

The Mini-assessment activity should give you some sense of what meanings your students give to multiplication and the range and variety of stories they draw on.

Some students may generate very similar stories by, for example, changing "12 packets of 3 chocolate bars" to "12 boxes of 3 biscuits". Do allow them opportunity to do this, because even such simple changes enable students to expand the set of examples they can draw to understand multiplication.

In the lesson itself, you might find that some students struggle initially to invent multiplication stories, and/or feel that they have outgrown the need for diagrams to represent multiplication.

So it is worth being on the look-out for students who come up with good stories and diagrams during the first Stage of the lesson (or who did so during the Mini-assessment), so that you can share these with the class.

Key questions

What words do we have for " $\times$ "?

In what ways are these situations the same or different?

In what ways are these diagrams the same or different?

In what ways are these stories the same or different?

Adapting the lesson

It is worth visiting this kind of work on a regular basis. It is easy to change the task difficulty by varying the numbers in the expressions (eg by using different numbers such as 15×6 , or by using 3 figure numbers, or fractions or decimals) and it is useful to vary the structure of the expressions (eg by having a longer string of multiplications or by changing the operation to division).

Multiplicative Reasoning 1A

Lesson

1A**Outline of the lesson (annotated)**1. Consider the expression 12×3 .

- Present the above tasks; students work in small groups. Discuss students' verbalisations, diagrams, stories.

— Don't structure the work too much at this stage: allow plenty of time and note ideas on the board (in three columns, perhaps). If necessary, show students an example of a diagram and story.

Initially students may create stories that are very similar. For example, changing '3 bags of 12 oranges' to '3 bunches of 12 grapes'. This is fine - it shows students are beginning to engage with the underlying structure.

Sometimes the language students use will reveal the model of multiplication that they are thinking of. For example, "3 lots of 12" suggests repeated addition (or equal grouping). On the other hand "3 times 12" and "3 multiplied by 12" could indicate various models (see p 4).

Brackets notation can be used to indicate a particular form of multiplication. For example, $3(12)$ can be used to represent "3 lots of 12" or "12, 3 times". This notation might also help students make sense of more complex expressions, such as $a(b + c)$.

2. Interpret a diagram for 12×3 and a matching story.

- Show Diagram A.
"We can think of this as '12 lots of 3'."
Is it like any of our diagrams?"
- Show this story. "This story fits the diagram. Is it like any of our stories? Write another that fits."

— Our story matches the structure of the diagram but some students might prefer a more visual match, such as

Ringo bakes 12 buns, each containing 3 currants.

How many currants is this altogether?

3. Interpret another diagram for 12×3 and write a matching story.

- Present Diagram G. "Here is another diagram for 12×3 .
How could we say it in words?
Is it like any of our diagrams?"
- Write some stories that fit the diagram.
- Compare the stories for Diagrams A and G.

— For example, "3 lots of 12" or "12, 3 times".

— The corresponding story about Zak would be

Zak wants to learn to cook. He has 12 lessons a week for 3 weeks. How many lessons is this altogether?

4. Determine expressions and write stories for other diagrams.

- Present other diagrams, eg H and C. (Choose from those on page 5.)
What expressions involving 3 and 12 could they represent?
- Choose a diagram for 3×12 , eg C.
Write a story that fits.

Here are some examples for different diagrams:

C A sheet of stamps consists of 3 rows and 12 columns.
How many stamps are there altogether?

N How many 1m square tiles are needed to cover a 3m by 12m patio?

P Sue has written 12 postcards.
Carrie has written 3 times as many.
How many postcards has Carrie written?

W Kim has 3 coats and 12 scarves. How many different combinations of coat and scarf could she wear?

5. Draw diagrams for an expression.

- Ask students to sketch some diagrams for 4×7 , say.
Discuss the diagrams. Try to classify them into groups.

— Check to see whether the diagrams are structurally different or merely look different.

— For example,

- repeated addition • rate • array or area
- scaling • Cartesian product.

6. Extension: Consider non-integer expressions.

- Try to draw a diagram and to write a story for 4.1×7.2 , say.

— The most adaptable diagrams are those involving rate, area and scaling, in particular N, Q, T and V.

The most suitable stories involve continuous quantities, such as

What is the mass of a 4.1cm length of gold wire if a 1cm length has a mass of 7.2g?

Multiplicative Reasoning 1A

Lesson

1A

Background

Models of multiplication

Multiplication occurs in a large variety of contexts, some of which might be said to provide 'models' of multiplication. For example, Davis (2010) worked with a group of teachers who produced a list of 'realisations of multiplication' which included these items:

grouping process; repeated addition; times-ing; expanding; scaling; repeated measures; making area; making arrays; proportional increase; splitting; skip counting; transformations; stretching/compressing a number line.

Anghileri and Johnson (1992) identified six key 'aspects' of multiplication. We have modified these to obtain 5 different 'models' (below). Most models have variants and how well they work depends on whether the quantities are discrete (numbers of objects) or *continuous (length, mass, etc)* and on whether the latter are whole numbers or fractional.

- Repeated addition (or equal grouping or equal skips along the number line): The ring-tone on my phone consists of a sequence of 3 notes. If the phone rings 12 times, how many notes are played altogether?
The ring-tone on my phone takes 3 seconds to play. If the phone rings 12 times, for how long does it ring?
- Allocation/rate: A shop sells 12 kinds of soup. If you buy 3 tins of each, how many tins do you buy?
A shop sells 12 kinds of cloth. If you buy 3 metres of each, how many metres do you buy?
- Array (or area): Eggs are arranged on trays in 3 rows and 12 columns. How many eggs fit on a tray?
How many 1 m square paving stones are needed to cover a patio that is 12 m long and 3 m wide?
- Scale factor: Jake has 3 acorns, Meg has 12 times as many. How many acorns does Mary have?
Jake's model of a boat is 3 cm wide. Meg's model of the same boat is 12 times as big as Jake's. How wide is it?
- Cartesian product: Ice creams come in any one of 3 different sizes and any one of 12 different flavours. How many different choices are there?

Repeated addition and Allocation are likely to be the first models of multiplication that students meet and these will probably still dominate students' thinking in secondary school, even though these models are not well-suited to multiplication with rational numbers, and even though students meet the Array model quite early in their schooling.

The language of multiplication

The language that students use will not always reveal the model of multiplication that they are thinking of. For example, "3 times 12" might be used to indicate Repeated addition (either "3 lots of 12" or "3, 12 times") but it could also indicate one of the other models, eg Scaling ("3 times as much as 12").

It might be easier to discern an underlying model from students' diagrams and stories, though students might struggle initially to produce appropriate diagrams and stories, especially for multiplication.

Diagrams for different operations

A selection of diagrams for the numbers 3 and 12 and the operations $+$, $-$, $\times$, $\div$ are shown on the next page. With the possible exception of Allocation/rate, at least one diagram is included for each of the five aspects of multiplication discussed above.

Stories for different operations

Once upon a time, we* asked students to write stories for various binary expressions, including $84-28$, $84\div 28$ and 84×28 . Suitable stories were given by 78%, 41% and 31% of the students respectively, in a sample of just over one thousand 10-11 year olds.

Why is writing a multiplication story more difficult? To begin with, imagine that the numbers refer to everyday objects like apples, say. Then it is fairly straightforward to write a subtraction (or addition) story about 84 apples and 28 apples. It is even possible to write a division story such as the one below (albeit involving quotient rather than the more familiar idea of partitioning, ie sharing):

How many trays holding 28 apples are needed for 84 apples?

But how does one 'multiply' 84 apples and 28 apples? One could construct a Cartesian product story, such as this:

In how many ways could I choose one of 84 French apples and one of 28 English apples?

However this seems highly contrived and few students would be familiar enough with the Cartesian product to do this. For the other models to work, students need to realise that the two numbers have to represent different kinds of entities - they can't both represent apples. For example, one number could refer to apples and one to boxes, as in the rate model:

There are 28 apples in a box and 84 boxes.

Or one number could refer to apples while the other is a pure scalar, as in

There are 28 apples in your box and 84 times as many in mine.

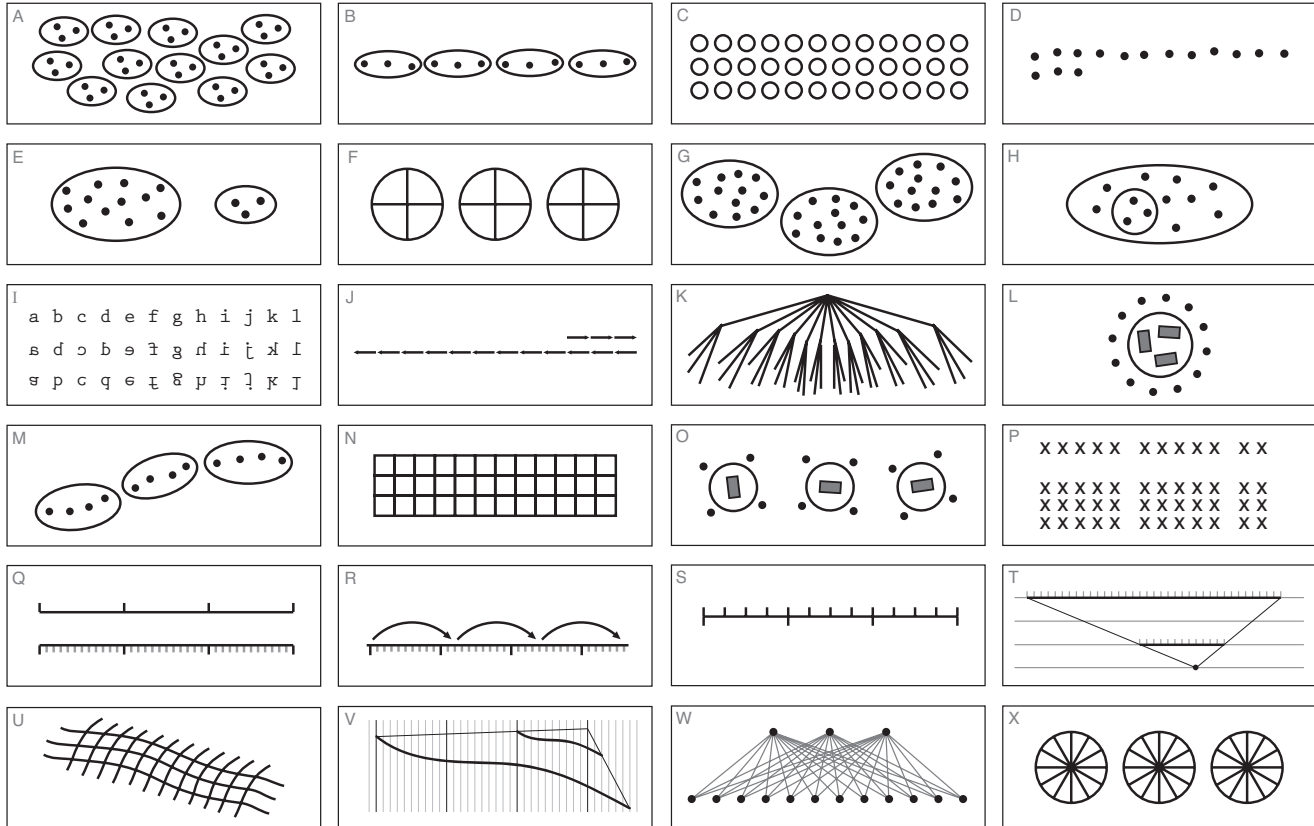
It is thus worth being on the look-out for students with good multiplication stories in the first Stage of the lesson, so that these can be shared with the rest of the class.

References

- Anghileri, J., & Johnson, D. C. (1992). Arithmetic operations on whole numbers: multiplication and division. In T. R. Post (Ed.), *Teaching mathematics in grades K-8: Research based methods* (Second ed., pp. 157-200). Boston, MA: Allyn and Bacon.
- Davis, B. (2010). Concept studies: designing settings for teachers' disciplinary knowledge. In Pinto, M.M.F. & Kawaski, T.F. (Eds.), *Proceedings of PME XXXIV Conference*, Vol 1, pp. 63-78. Belo Horizonte, Brazil: PME.
- *Brown, M. & Küchemann, D. (1977). Is it an 'add' Miss, Part 2. *Mathematics in School*, 6, 1, pp 9-10.

Multiplicative Reasoning 1A

Lesson

1A**Diagrams for arithmetic operations involving 3 and 12****Note**

Some of the diagrams are not easy to interpret and may provoke heated debate. It is worth stressing that we are not claiming that a given diagram can be read in only one way, merely that it is possible to find *an* interpretation involving an expression in 3 and 12 and one of the operations $+$, $-$, $\times$ and $\div$.

Clearly, the diagrams can be interpreted in many ways if we allow numbers other than 3 and 12. For example, Diagram B, which is intended to represent $12 \div 3$, can also be interpreted as $12 \div 4$ and 3×4 .

The diagrams can be read as binary arithmetic operations on 3 and 12 in the following ways, though there will be other, equally valid interpretations:

- A. 12×3 or 3×12 [repeated addition or grouping: 12 lots of 3 or $12(3)$]
 B. $12 \div 3$ [quotient or grouping: number of shares of 3 or groups of 3]
 C. 12×3 or 3×12 [rectangular array: 3 rows of 12 columns]
 D. $12 - 3$ [difference]
 E. $12 + 3$ [aggregating]
 F. $3 \div 12$ [partition or sharing: size of each of the 12 shares or groups]
 G. 12×3 or 3×12 [repeated addition or grouping: 3 lots of 12 or $3(12)$]
 H. $12 - 3$ [take away]
 I. 12×3 or 3×12 [Cartesian product: combining any one of 12 elements with any one of 3 transformations - identity, vertical reflection, horizontal reflection]
 J. $3 - 12$ [displacement: 3 units to the right, 12 to the left]
 K. 12×3 or 3×12 [tree diagram: allocation or rate]
 L. $3 \div 12$ [partition or sharing: size of each of the 12 shares or groups]
 M. $12 \div 3$ [partition or sharing: size of each of the 3 shares or groups]
 N. 12×3 or 3×12 [area: number of squares in a 3 units by 12 units rectangle]
 O. $3 \div 12$ [partition or sharing: size of each of the 12 shares or groups]
 P. 12×3 or 3×12 [scaling: 3 times as many]
 Q. 12×3 or 3×12 [scaling: 12 times as many]
 R. 12×3 or 3×12 [repeated addition: 3 lots of 12]
 S. $12 \div 3$ [partition or sharing: size of each of the 3 shares or groups]
 T. 12×3 or 3×12 [scaling: 3 times as many or as long]
 U. 12×3 or 3×12 [array or Cartesian product]
 V. 12×3 or 3×12 [scaling: 3 times as long]
 W. 12×3 or 3×12 [Cartesian product: all 3 dots joined to all 12 dots]
 X. $3 \div 12$ [partition or sharing: size of each of 12 shares or groups]

Lesson

1A

Notes

Multiplicative Reasoning 1B

Lesson

1B**Changing expressions**

Look at expression A ↓.

Imagine we add 1 to one of the numbers, so we get expression B ↓ or expression C ↓.

Which is larger, B or C?

A. 21×53

B. 22×53

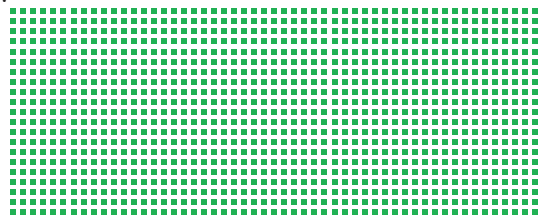
C. 21×54

Summary

In this lesson we focus on the structure of multiplicative expressions. We look at *differences* between expressions with the aim of developing what might be called an algebraic habit of mind, ie one where we begin to let go of thinking about the actual result of a calculation and consider instead the effect that individual elements in an expression have on that result. We make use of stories and diagrams, similar to those in Lesson MR 1A, to help students see the structure of the expressions.

Outline of the lesson

- Which expression is larger, B or C ?
 - Show the class the above task. Then say to the class, "Look at expressions B and C. *Don't calculate their values.* Hands up if you think B is bigger than C. Hands up if you think C is bigger than B."
- Compare expressions A and B; then compare A and C.
 - Say, "Tell me, *without calculating*, how much bigger than A is B".
 - Repeat for expressions A and C.
- Invent a scenario for expression A: 21×53 .
 - Ask the class to think of a 'real life' story that would require them to calculate 21×53 .
- Use the invented scenario to compare A, B and C.
 - Choose a fitting scenario from Stage 3. Apply it to B and compare A and B once more (still without calculating their individual values).
 - Apply the scenario to C and compare A and C.
 - Finally, compare all three expressions.
- Model and compare the expressions using a diagram.
 - Ask the class to invent some diagrams to represent 21×53 .
 - Choose a fitting diagram and discuss how it could be used to compare A, B and C, as in Stage 4 (above).
- Model and compare the expressions using an array.
 - Show this array. Discuss how it models expression A.
 - Ask students to imagine similar arrays for B and C. "How can we use the arrays to compare the expressions?"
 - Extension: Model the expressions 'algebraically', eg by writing B as $(21+1) \times 53$. "How does this help us compare B and A?"



Multiplicative Reasoning 1B

Lesson

1B

Overview

Mathematical ideas

This lesson extends students' understanding of multiplication by considering the structure of multiplicative expressions through examining the effect of adding 1 to either the multiplier or the multiplicand in the expression 21×53 . The focus is on comparing the different expressions rather than on calculating the answers, and we facilitate this by using a range of stories and diagrams to model the expressions.

The array model is a particularly powerful model for understanding the structure of multiplicative expressions. Many students will have met this model in primary school in the context of the grid model for multiplication or for partitioning numbers in multiplicative expressions. However, they may not have become fluent with using it to understand structure.

Students' mathematical experiences

Students have the opportunity to

- generate a range of stories
- adapt each other's stories
- discuss how the expressions may be represented using different diagrams
- consider arrays in some depth
- notice that 22×53 is 53 greater than 21×53 , whereas 21×54 is 21 greater.

The students may realise that

- the diagrams and stories can be used to explain how the expressions differ.

Key questions

Can you think of different ways of saying 22×53 ?

Can you represent that story using a diagram?

How can we use the stories (or diagrams or the arrays) to compare the expressions?

Assessment and feedback

During the early part of the lesson, observe the different ways in which students respond. Many students may think that the expressions B and C (22×53 and 21×54) are equal. Some may think that the value of each expression is 1 more than expression A (21×53).

Give students plenty of opportunities to 're-say' the expressions and to 're-tell' each others' explanations.

Some students may generate very similar stories by, for example, changing "21 packets of 53 biscuits" to "21 boxes of 53 crackers". Do allow them opportunity to do this, because even such simple changes enable students to expand the set of examples they can draw on to understand multiplication.

Look for students' stories that may be particularly appropriately modeled using arrays, such as "A cinema has 21 rows of 53 seats". If none are suggested, you may need to suggest one yourself.

Adapting the lesson

During this lesson, the students consider the algebraic structure of the expressions implicitly. With some classes or students, you may want to extend the lesson by considering the structure explicitly by forming the open expressions $(21 + 1) \times 53$ and $21 \times (53 + 1)$ and comparing them.

You can alter the task difficulty by varying the numbers in the expressions. It can be useful at a later point to introduce decimals or fractions (eg by changing 21×53 to 2.1×5.3), although students may need a lot of support to generate stories to represent this (eg "A car travels for 2.1 seconds at a speed of 5.3 metres per second").

Multiplicative Reasoning 1B

Lesson

1B**Outline of the lesson (annotated)**

1. Which expression is larger, B or C ?

- Show the class the above task. Then say to the class, "Look at expressions B and C.
Don't calculate their values.
Hands up if you think B is bigger than C.
Hands up if you think C is bigger than B."

— Later, as a way of assessing/consolidating students' understanding, you might want to use a simpler set of expressions, such as this:

A. 28×6 B. 29×6 C. 28×7

This could be done at the end of the lesson, or on another occasion.

2. Compare expressions A and B; then compare A and C.

- Say, "Tell me, *without calculating*, how much bigger than A is B".
- Repeat for expressions A and C.

— Ask one or two students for explanations, but don't scrutinise them at this stage.

3. Invent a scenario for expression A: 21×53 .

- Ask the class to think of a 'real life' story that would require them to calculate 21×53 .

— Reject stories that don't *model* the expression, such as "My teacher asked me to calculate 21×53 ".

Allow plenty of time, but if no one comes up with a story that *fits*, give an example like this:

"My Nan's school had 21 classes of 53 children".

Students might also find it helpful to use an every day term like 'lots' for 'multiply', as in '21 lots of 53'.

4. Use the invented scenario to compare A, B and C.

- Choose a fitting scenario from Stage 3. Apply it to B and compare A and B once more (still without calculating their individual values).
- Apply the scenario to C and compare A and C.
- Finally, compare all three expressions.

— For example,

'B means an extra class, so 53 extra children'

'C means an extra child per class, so 21 extra children'.

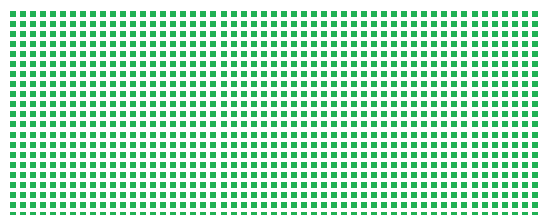
5. Model and compare the expressions using a diagram.

- Ask the class to invent some diagrams to represent 21×53 .
- Choose a fitting diagram and discuss how it could be used to compare A, B and C, as in Stage 4 (above).



6. Model and compare the expressions using an array.

- Show this array. Discuss how it models expression A.
- Ask students to imagine similar arrays for B and C. "How can we use the arrays to compare the expressions?"
- Extension: Model the expressions 'algebraically', eg by writing B as $(21+1) \times 53$. "How does this help us compare B and A?"



— Some students might simply want to close this down again - 'do the brackets first'. But some might see that this can be transformed into $21 \times 53 + 1 \times 53$.

Multiplicative Reasoning 1B

Lesson

1B

Background

Seeing structure rather than calculating

This lesson involves numerical expressions, but the focus is on their structure rather than their actual value.

This shift in focus may seem strange at first. When we have asked students to compare expressions like the ones in this lesson, they commonly try to calculate their value, rather than consider how the numbers interact. This seems part of a broader tendency to see mathematics as primarily about learning procedures for finding answers.

To counteract this tendency we stress that students should not evaluate the expressions in this lesson. We have also chosen expressions that are difficult to calculate mentally. Initially, this can lead to superficial responses, along the lines of 'One of the numbers has increased by 1 so the expression has increased by 1'. We thus try to give meaning to the expressions, which we do by building on the work of Lesson MR 1A, ie by using stories and diagrams to represent the expressions.

Stories for multiplication

As you may have found in Lesson MR 1A, students can find it difficult at first to construct multiplication stories, as in this attempt to write a story for 9×3 .

Lee had 9 and Jim had ~~three~~³
 Cholesterol 12 you multiply by them
 ow much do they have.

However, once a fitting story has emerged, perhaps with a helpful nudge from the teacher, it should be fairly easy for the class to compile a repertoire of multiplication stories (below) which can then be drawn on when needed.

It is likely that most of the stories will model repeated addition or equal grouping, rather than the other categories listed by Anghileri and Johnson (1992) [rate, array, scale

21×53
 21 sacks of potatoes, in each sack ^{there are} 53 potatoes
 21 rows of 53 soldiers standing
 there are 53 soldiers who have to walk 21 miles
 21 packets of biscuits with 53 inside
 21 classes with 53 people in each class

factor, Cartesian product]. However, such stories are perfectly fine for this lesson - we don't need to strive for diverse or highly original stories.

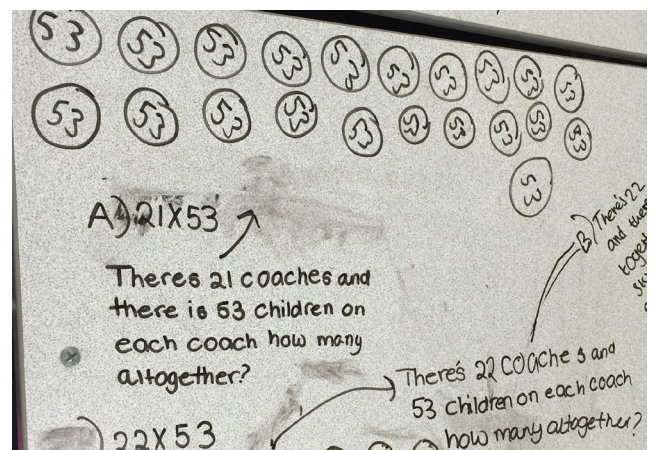
Diagrams for multiplication

Towards the end of the lesson we present the class with a 21 by 53 array, and use this to think about an array for the other two expressions.

The array provides a powerful model for multiplication, not least because it leads on to the area model and the use of rational numbers. However, we shouldn't assume that students can use arrays fluently, even though arrays are mentioned several times in the current *Mathematics programmes of study: key stages 1 and 2* document (DfE, 2013). One difficulty stems from the fact that the number of elements in a row is given by the number of columns, and vice versa. Also, while we might think of the array simply in terms of rows and columns, students might see it differently, as in this representation of 4×6 (which we would tend to see as 8×3).



It is important, therefore, that we give due weight to students' diagrams for multiplication (such as the one below), even though they might not be as powerful or efficient as the array seems to us.



We look more closely at the array in Lessons 2A and 2B.

Reference

Anghileri, J., & Johnson, D. C. (1992). Arithmetic operations on whole numbers: multiplication and division. In T. R. Post (Ed.), *Teaching mathematics in grades K-8: Research based methods* (Second ed., pp. 157-200). Boston, MA: Allyn and Bacon.

Lesson

1B

Notes

Multiplicative Reasoning 2AB

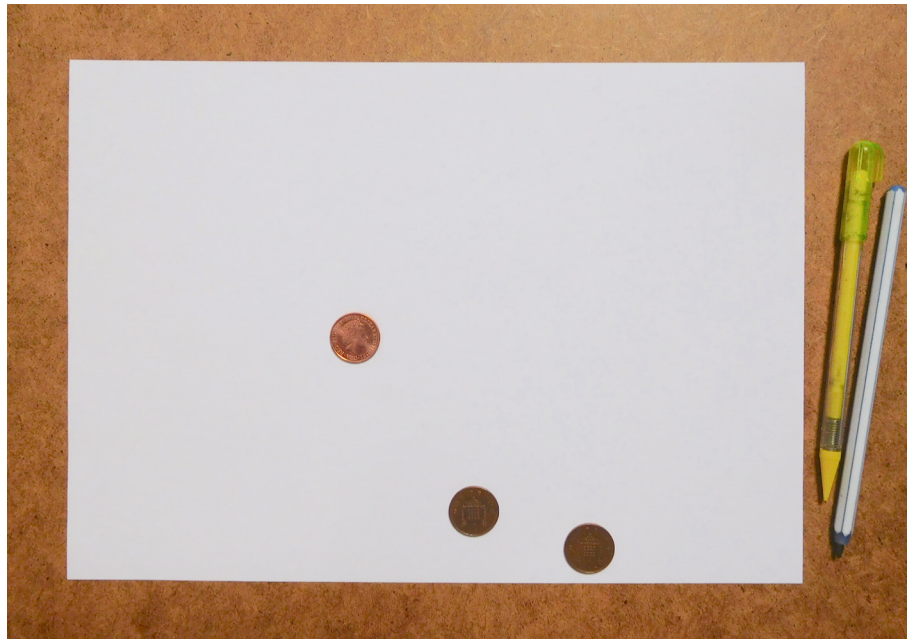
Mini-assessment

2AB

How many coins?

Roughly how many 1p coins can fit on a sheet of A4 paper?

[The coins can touch but they are not allowed to overlap.]



Commentary

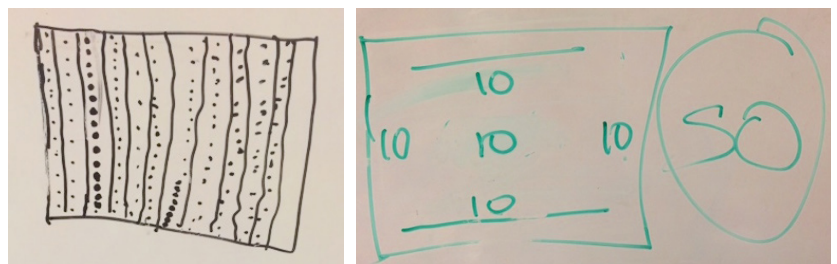
As with other Mini-assessments, this is designed to inform how you teach Lessons 2A and 2B. These lessons themselves have an explicit assessment focus and this Mini-assessment will give you some advance thinking about what to look for and the questions to ask.

The main point of interest here is whether students think of arranging the coins in an array. If so, do they analyse the array in terms of rows and columns, or do they estimate the number of coins more ‘globally’?

Where students do think of rows and columns, it is interesting to see how accurately they estimate the numbers of coins in each row and column, and whether they multiply these numbers or use repeated addition.

The degree to which students structure the arrangement of coins can vary quite considerably, as here (above, right).

The size of an A4 sheet of paper is 21.0 cm by 29.7 cm, or roughly 20 cm by 30 cm. A 1p coin has a diameter of about 2 cm, so the paper can be covered by about 10×15 coins (or £1.50!).



Multiplicative Reasoning 2A

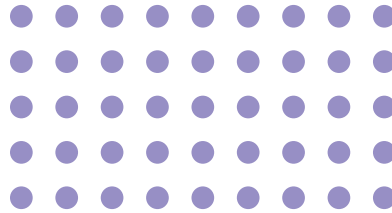
Lesson

2A**How many dots?**

Look at this array of dots. →

Show for just 1 second!

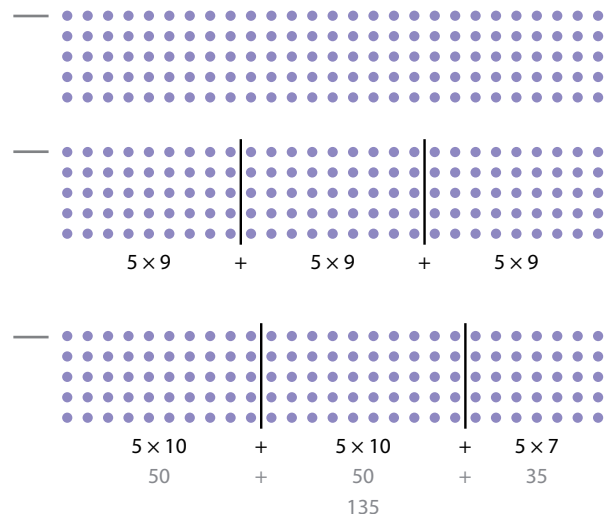
1. Roughly how many dots are there?
2. How could we find the exact number of dots without counting every dot?

**Summary**

Students might well know (but not fully understand) that the procedure for finding the number of elements in a rectangular array is to multiply 'length by width', as in the rule for finding the area of a rectangle. Here we look at why this works and how the 'row and column' structure of arrays can be used to model multiplication. We also consider simple ways to *partition* arrays, to ease the task of multiplying larger numbers.

Outline of the lesson

1. Find the number of dots in a small array (5×9).
 - "Get ready to look at this array Now!"
Show the 5×9 array very briefly (about 1 second).
 - **Hide the array again.** Now ask, "Roughly how many dots were there in the array?"
Note down some quick estimates on the board.
Then discuss students' ideas and methods in more detail, *with the array still hidden*.
 - "How could we find the exact number of dots in the array without counting every dot?"
Discuss. Then **show the array again**. Discuss further.
2. Find the number of dots in another array (4×12). [optional]
 - Repeat Stage 1 with another array (4×12).
[This can probably be done more quickly.]
3. Find the number of dots in a 'very wide' array (5×27).
 - Repeat Stage 1 with another array (5×27).
[Note that this is 3 times as wide as the original array.]
 - Ask, "How does this array compare to our first array?"
Tease out the idea that the array is 3 times as wide and show this by partitioning the array and by writing this expression: $5 \times 9 + 5 \times 9 + 5 \times 9$.
[Note: $9 \times 5 + 9 \times 5 + 9 \times 5$ is equally valid.]
 - Tease out other convenient ways of partitioning the array.
Mark the partitions on the array and write the resulting expressions,
for example: $5 \times 10 + 5 \times 10 + 5 \times 7$,
and $5 \times 20 + 5 \times 7$
and perhaps $5 \times 30 - 5 \times 3$.
4. Arrange a given number of dots (145) in an array.
 - Say, "I have arranged 145 dots in an array.
How many rows and columns could it have?"
 - Try other numbers of dots, chosen by you or the class.

**145**

Multiplicative Reasoning 2A

Lesson

2A

Overview

Mathematical ideas

This lesson extends students' understanding of multiplication by using arrays to consider the structure of multiplicative expressions. We emphasise that the number of elements in the array can be calculated by multiplying the number of dots in each row by the number of dots in each column. We then also consider how arrays can be partitioned. Here we are addressing the distributive law, as in $5 \times 27 = 5 \times (20 + 7) = 5 \times 20 + 5 \times 7$, or more formally $a(b + c) \equiv ab + ac$. Additionally, we tackle the associative law more implicitly, as in $5 \times 27 = 5 \times (9 \times 3) = (5 \times 9) \times 3$.

Students' mathematical experiences

Students have the opportunity to consider

- The array as a model for multiplicative expressions.

Students should begin to realise

- the array can be used to explain how multiplicative expression may be 'simplified' by partitioning the multiplier and/or the multiplicand
- that multiplication is distributive over addition (although this realisation may be implicit):
 $a(b + c) \equiv ab + ac$.

Key questions

How can you find the exact number of dots without counting every single dot?

How does this array compare to our first array?

How many rows and columns could an array with 145 dots have?

Assessment and feedback

One aim of this lesson is to informally assess the students' skills and fluency with the array as a model for multiplication and with partitioning more generally.

To what extent can students explain the multiplicative structure of the array by describing multiplication in terms of rows and columns?

Do any students make an explicit link to associativity by, for example, commenting that $5 \times 9 + 5 \times 9 + 5 \times 9$ is equivalent to $(5 \times 9) \times 3$?

Do any students use repeated addition? If so, can they (or other students) make a link between repeated addition and the array?

Consider both the language that they use and how fluently they are able to split, or partition, the array. You may decide that you need to revisit this task using different sized arrays with some or all of the students in your class.

Adapting the lesson

You can alter the task difficulty by varying the numbers in the array.

You may extend this task using the optional *Extra* and *Further* Lessons 2E (*Counting rectangles*) and 2F (*3.2 rows of 2.6 unit squares*). Building on Lesson 2B (*Split and join*), in these lessons we generalise from the array to an area model of multiplication. The area model is a powerful way of considering multiplicative expressions involving decimals and/or fractions, but will also be useful in modeling algebraic expressions involving multiplicative relationships, as in Algebra Lesson 6A (*Row of Tiles*).

Multiplicative Reasoning 2A

Lesson

2A

Outline of the lesson (annotated)

1. Find the number of dots in a small array (5×9).
 - “Get ready to look at this array Now!”
Show the 5×9 array very briefly (about 1 second).
 - **Hide the array again.** Now ask,
“Roughly how many dots were there in the array?”
Note down some quick estimates on the board.
Then discuss students’ ideas and methods in more detail,
with the array still hidden.
 - “How could we find the exact number of dots in the array
without counting every dot?”
Discuss. Then **show the array again.** Discuss further.

Students might start talking about the ‘size’ of the array, eg its *width* and *height*, or the number of *rows* and *columns*.

Be prepared to spend quite a long time on this. You might find that you need to clarify what we mean by *row* (a horizontal line of dots) and *column* (a vertical line of dots).

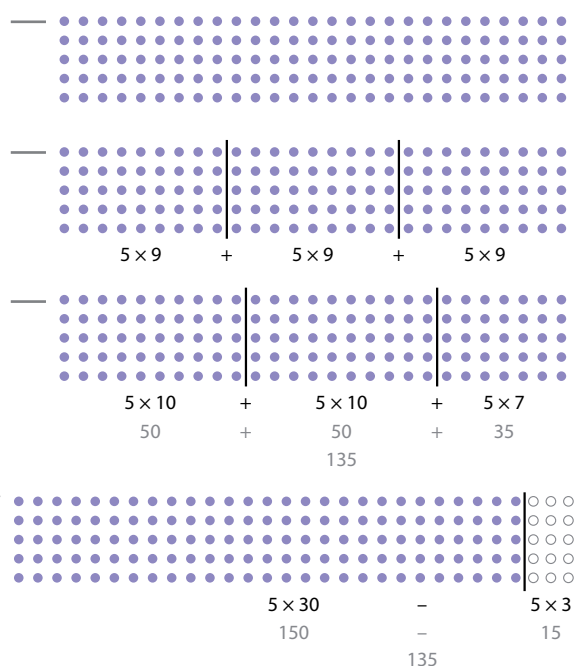
[Note: it is easy to get confused by the fact that the *number* of dots in a row is equal to the number of columns, and vice versa.]

Make use of everyday language like '5 rows of 9 dots' or simply '5 lots of 9' and use this to help show that we can find the total by multiplying (5×9) or by using repeated addition ($9+9+9+9+9$).

Also, talk of ‘9 columns of 5 dots’ and ‘9 lots of 5’.

2. Find the number of dots in another array (4×12). [optional]
 - Repeat Stage 1 with another array (4×12).
[This can probably be done more quickly.]

3. Find the number of dots in a ‘very wide’ array (5×27).
- Repeat Stage 1 with another array (5×27).
[Note that this is 3 times as wide as the original array.]
 - Ask, “How does this array compare to our first array?”.
Tease out the idea that the array is 3 times as wide and show this by partitioning the array and by writing this expression:
 $5 \times 9 + 5 \times 9 + 5 \times 9$. [Note: $9 \times 5 + 9 \times 5 + 9 \times 5$ is equally valid.]
 - Tease out other convenient ways of partitioning the array.
Mark the partitions on the array and write the resulting expressions, for example: $5 \times 10 + 5 \times 10 + 5 \times 7$, and $5 \times 10 + 5 \times 7$ and perhaps $5 \times 30 - 5 \times 3$.



4. Arrange a given number of dots (145) in an array.
 - Say, “I have arranged 145 dots in an array. How many rows and columns could it have?”
 - Try other numbers of dots, chosen by you or the class.

The 5×27 array consists of 135 dots. Students might notice that adding two columns of 5 dots gives us an array for 145 dots (ie 5×29).

You might want to choose other ‘related’ numbers like 155 (5×31) or 186 (6×31) or a ‘fresh’ number like 119 (for which there are just 4 possible arrays: 1×119 , 119×1 , 7×17 , 17×7).

Multiplicative Reasoning 2A

Lesson

2A

Background

The array model of multiplication

This pair of lessons is designed to help students consolidate their understanding of the rectangular array as a model for multiplication, and, in the second lesson of the pair, to help them see how the array leads to the area model.

The first model that students are likely to construct of multiplication is repeated addition (or equal grouping). The array model is closely related to this but provides a more structured view that would seem to be well-suited to revealing some of the laws of arithmetic that multiplication obeys. For example, a 5×13 array, when turned through 90° , becomes a 13×5 array, thus demonstrating the commutative law. Further, a 5×13 array can be neatly split into, say, a 5×10 and a 5×3 array, thus demonstrating that multiplication is distributive over addition, or, perhaps more to the point, showing how long multiplication works.

In the current *Mathematics Programmes of study* document for key stages 1 and 2 (DfE, 2013), the array is mentioned five times, though not tied to years, and in a manner that is quite vague, as in this first reference:

Previously, in the *Supplement of examples* in the *National Numeracy Strategy* pack (DfEE, 1999), the array is again

Statutory requirements

Pupils should be taught to:

- solve one-step problems involving multiplication and division, by calculating the answer using concrete objects, pictorial representations and arrays with the support of the teacher.

Notes and guidance (non-statutory)

Through grouping and sharing small quantities, pupils begin to understand: multiplication and division; doubling numbers and quantities; and finding simple fractions of objects, numbers and quantities.

They make connections between arrays, number patterns, and counting in twos, fives and tens.

mentioned just five times, and all on just one page, under the heading *Understanding multiplication* for Year 2 and Year 3. The first and most explicit reference is this:

It thus seems likely that students will meet the array quite early in their school lives. However, there is also the strong

Understand multiplication as:

- **repeated addition:** for example, 5 added together 3 times is $5 + 5 + 5$, or 3 lots of 5, or 3 times 5, or 5×3 (or 3×5).
- **describing an array:** for example,

$$\begin{array}{|c|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} 4 \times 2 = 8$$

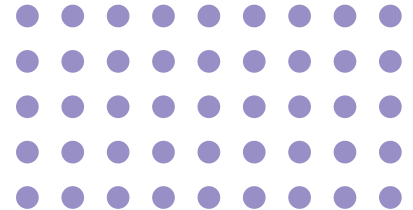
$$2 \times 4 = 8$$

possibility that some students (alongside their teachers perhaps) will see the array as just a temporary staging post from which they rapidly move on, certainly by the time they reach secondary school and put away childish things.

We regard the array as an important model of multiplication, and though we would want students' thinking to go beyond the array, we see it as a model that provides vital support and to which students can continually return as the need arises.

Making sense of the array

We have found that there are several aspects of the array that can cause confusion, even though these features might seem trivial. For example, in the process of counting the number of dots in a row, we are simultaneously counting the number of columns. Also, to find the dimensions of this 5×9 array, we count one of the dots twice, which might seem strange and unsettling to some students. Further, while one person might call this a '5 by 9' array (perhaps on the basis that there are 5 rows, each with 9 dots), it is equally valid to describe it as '9 by 5' (9 columns of 5 dots). Moreover, while one person might interpret 5×9 as '5 lots of 9', another person (or the same person on another occasion) might interpret it as '5, 9 times'. Yet both people might read the expression in the same way, as '5 times 9'. These different interpretations can make communication difficult, even though they might all be equivalent.



Revisits

To help consolidate students' understanding of the array, we have devised several follow-up activities which are shown in the Revisits sections. The first set involve arrays of dots similar to the ones in this lesson, which are used to examine changes in expressions in the manner of Lesson 1B (MR 1B). We then look at the possible number of arrays into which different numbers of dots can be arranged. Next, we look at arrays and sub-arrays of stamps, along with activities involving multiplicative patterns based on the familiar dot patterns on a die.

Finally, we provide an optional lesson that makes explicit that an array can be composed of a set of identical *strips*. This lesson is highly structured.

References

DfEE (1999). *The National Numeracy Strategy, Supplement of Examples*, page 47. Cambridge University Press. <http://www.edu.dudley.gov.uk/numeracy/nns/Y1,2,3%20-%20Calculation.pdf>

DfE (2013). *Mathematics programmes of study: key stages 1 and 2*. https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/335158/PRIMARY_national_curriculum_-_Mathematics_220714.pdf

Multiplicative Reasoning 2B

Lesson

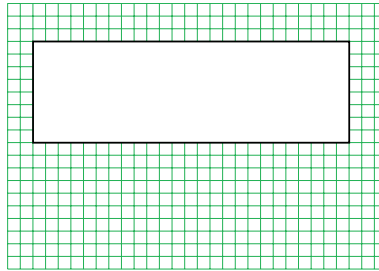
2B

Split and join

Here is a sheet of A4 paper, →
which is covered in 1 cm squares.

Show for just 1 second!

Roughly how many 1 cm squares
are there underneath the white card?



Summary

In this lesson we use the area of a rectangle, *measured in unit squares*, as a model for multiplication. We look for ways of splitting the rectangle into equal strips and re-joining the strips to make a new rectangle whose area is easier to calculate.

Outline of the lesson

- Find the number of 1 cm squares covered by the card.

- Introduce the task (above).
“Get ready to look at the sheet and card Now!”
Show the above diagram very briefly (about 1 second).
- Hide the diagram again.** Discuss students’ estimates.
- “How could we find the exact number of 1 cm squares underneath the white card?”
Show the diagram again. Find the exact number.

— The card measures 8 cm by 25 cm but don’t reveal this yet.

- Cut the card into equal horizontal strips and re-join.

- Ask the class to imagine cutting the card into 4 equal horizontal strips, and re-joining the strips to make a thin, wide rectangle.
- “What are the dimensions of each strip?”
“What are the dimensions of the new rectangle?”
“Calculate how many 1 cm squares it covers.”



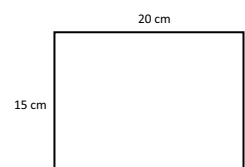
- Cut the card into equal vertical strips and re-join.

- Ask the class to imagine cutting the card into 5 equal vertical strips, and joining the strips again to make a thin, tall rectangle.
- “What are the dimensions of each strip?”
“What are the dimensions of the new rectangle?”
“Calculate how many 1 cm squares it covers.”



- Split and re-join a 15 cm by 20 cm rectangular card.

- “How many 1 cm squares are covered by a 15 cm by 20 cm rectangle?”
“Let’s check this by making a new rectangle that is also easy to calculate.”
- “How many ‘easy’ new rectangles can we find?”



- Split and re-join other rectangles.

- Think of some other rectangles. Which are easy to split and re-join?
- Challenge: “How can we change a 15 by 24 rectangle into a 10 by 36 ?”

Multiplicative Reasoning 2B

Lesson

2B

Overview

Mathematical ideas

Building on Lesson 2A, this lesson begins to generalise arrays to an area model of multiplication. We consider the area of a rectangle, measured in unit squares, and emphasise that the area, or the total number of unit squares, can be calculated by multiplying the number of unit squares in each row by the number of unit squares in each column. We then partition, or split, the rectangle in order to consider the associative law. By partitioning the $8\text{cm} \times 25\text{cm}$ rectangle into 5 equally sized $8\text{cm} \times 5\text{cm}$ rectangles and then re-arranging these into a $40\text{cm} \times 5\text{cm}$ rectangle, we are considering the following equivalences:

$$8 \times 25 = 8 \times (5 \times 5) \equiv (8 \times 5) \times 5 = 40 \times 5.$$

Students' mathematical experiences

Students have the opportunity to consider

- area as a model for multiplication and multiplicative expressions.

Students should begin to realise:

- That multiplication is associative (although this realisation may be implicit):

$$a \times (b \times c) \equiv (a \times b) \times c.$$

Assessment and feedback

Like Lesson 2A, this lesson has an explicit assessment focus.

Observe how different students respond to the problem and to each other's ideas.

- To what extent can students explain the multiplicative structure of the rectangle's area by describing multiplication in terms of rows and columns of unit squares?
- Are any students surprised that the area of (or number of unit squares covering) the re-arranged rectangle is unchanged?
- Do any students use repeated addition? If so, can they (or other students) make a link between repeated addition and the area of rectangle in terms of rows and columns of unit squares?

Consider how fluently they are able to split, or partition, the rectangle. You may decide that you need to revisit this task using different sized arrays with some of all of the students in your class.

Key questions

How can you find the exact area of the rectangle without counting every single unit square?

How does the area of the $40\text{cm} \times 5\text{cm}$ rectangle compare to the area of the $8\text{cm} \times 25\text{cm}$ rectangle?

Can you explain how you can work out the exact area of the rectangle?

Adapting the lesson

You can alter the task difficulty by varying the numbers in the array.

You may extend this task using the optional *Extra* and *Further* lessons 2E (*Counting rectangles*) and 2F (*3.2 rows of 2.6 unit squares*). In these lessons, we generalise from the array to an area model of multiplication. The area model is a powerful way of considering multiplicative expressions involving decimals and/or fractions, but will also be useful in modeling algebraic expressions involving multiplicative relationships, as in Algebra Lesson 6A (*Row of Tiles*).

Multiplicative Reasoning 2B

Lesson

2B**Outline of the lesson (annotated)**

1. Find the number of 1 cm squares covered by the card.

- Introduce the task (above).
“Get ready to look at the sheet and card Now!”
Show the above diagram very briefly (about 1 second).
- **Hide the diagram again.** Discuss students' estimates.
- “How could we find the exact number of 1 cm squares underneath the white card?”
Show the diagram again. Find the exact number.

— The card measures 8 cm by 25 cm but don't reveal this yet. When the diagram is shown again, students can determine this by counting squares along a row and column.

— “What's a quick way of finding the total number of squares covered by the card?” ($8 \times 25 = 200$)

Some students might use repeated addition, eg $25+25+25+25+25+25+25+25$, or $50+50+50+50$, or $100+100$. This is not necessarily efficient, but does show some insight into the structure of the array.

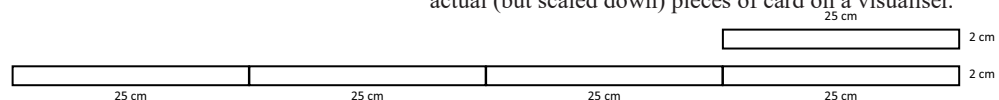
2. Cut the card into equal horizontal strips and re-join.

- Ask the class to imagine cutting the card into 4 equal horizontal strips, and re-joining the strips to make a thin, wide rectangle.
- “What are the dimensions of each strip?”
“What are the dimensions of the new rectangle?”
“Calculate how many 1 cm squares it covers.”



— Some students may have difficulty retaining all the information in this task and distinguishing things like the *number* of strips and the *height* of each strip.

Thus it might be worth *carefully* sketching and annotating a single strip and also the joined strips, as below. Or display actual (but scaled down) pieces of card on a visualiser.

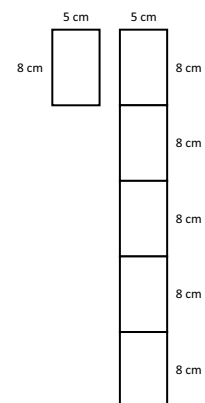


3. Cut the card into equal vertical strips and re-join.

- Ask the class to imagine cutting the card into 5 equal vertical strips, and joining the strips again to make a thin, tall rectangle.
- “What are the dimensions of each strip?”
“What are the dimensions of the new rectangle?”
“Calculate how many 1 cm squares it covers.”



— Again, it might be worth sketching and annotating a single strip and also the joined strips, as here:



— We have deliberately kept the arithmetic fairly simple here. It is worth emphasising that our prime interest is not the numerical answer but in finding ways to re-form the rectangle.

4. Split and re-join a 15 cm by 20 cm rectangular card.

- “How many 1 cm squares are covered by a 15 cm by 20 cm rectangle?”
“Let's check this by making a new rectangle that is also easy to calculate.”
- “How many ‘easy’ new rectangles can we find?”

For example, 5×60 , 3×100 , 30×10 , 60×5 .

For example, change the 15×24 rectangle to a 5×72 rectangle and then to a 10×36 rectangle.

Note also that

$15 \times 24 = 3 \times 5 \times 2 \times 12 = 5 \times 2 \times 3 \times 12 = 10 \times 36$,
and that $2/3$ of 15 = 10 and $3/2$ of 24 = 36.

5. Split and re-join other rectangles.

- Think of some other rectangles.
Which are easy to split and re-join?
- Challenge: “How can we change a 15 by 24 rectangle into a 10 by 36?”

Multiplicative Reasoning 2B

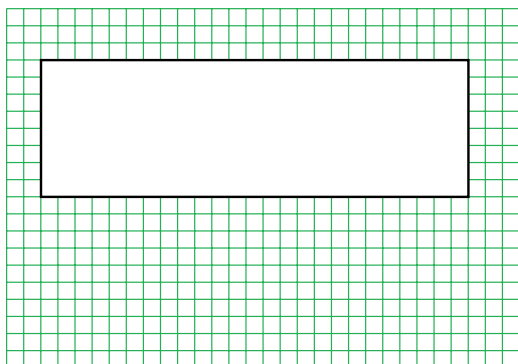
Lesson

2B

Background

The area model of multiplication

Here we shift from the array model of multiplication to the area-of-a-rectangle model. However, to ease the transition, we think of the rectangle as a piece of card covering part of a grid of unit squares. Also, we restrict ourselves in



this lesson to *whole numbers* of unit squares, although ultimately, the reason for using an area model is to deal with the multiplication of rational numbers.

Thus, in this lesson, we can imagine finding the area of the piece of card by counting the hidden squares, or, more efficiently, by thinking of the hidden squares as forming an array whose number can be found by counting the number of squares in a row (or column) and multiplying by the number of rows (or columns).

Most students seem to know that the number of hidden squares in this lesson's initial task can be found by multiplying the dimensions of the rectangle - in this case, 8×25 . However, when asked 'Why does this work?' students have sometimes struggled to give a satisfactory explanation. They have tended to see our question as asking about the *procedure* that they used to perform the multiplication (eg, long multiplication or the grid method). We thus have to be careful about assuming that students who have an appropriate procedure for finding the area (or the number of hidden squares) already have a sound understanding of why it is valid. In turn it is worth continually linking the notion of 'area' to the notion of an underlying array of unit squares.

Multiplication and the associative law

The first lesson of this pair considers how partitioning can be used to transform a multiplication so that it involves 'easier' numbers (or more familiar number bonds). From a formal point of view, partitioning rests primarily on the associative law (though one wouldn't necessarily make this fact explicit to students). In this lesson we do something similar, but this time using a process of 'splitting and joining' that rests formally on the associative law.

We begin by transforming an 8 cm by 25 cm rectangle into a 4 cm by 100 cm rectangle by first splitting it into four 2 cm by 25 cm rectangles and then joining the pieces in a different way, to form a long thin rectangle.

This can be expressed as follows and involves the associative law, then the commutative law and then the associative law again:

$$8 \times 25 = (4 \times 2) \times 25 = 4 \times (2 \times 25) = 4 \times (25 \times 2) = (4 \times 25) \times 2 = 100 \times 2.$$

Revisits

We have restricted ourselves to the use of whole numbers in this lesson. However, it is worth at some stage applying the area model to the multiplication of rational numbers. A simple first step might be to split our 8 cm by 25 cm rectangle vertically into two (or perhaps 4 or 8) equal parts, giving two 8 cm by $12\frac{1}{2}$ cm rectangles, and then one 16 cm by $12\frac{1}{2}$ cm rectangle. Does this still give an area of 200 unit squares? Can we still think of these unit squares as forming an array? When does the array model begin to break down?

On the next page we present the outline of an *Extra* (optional) lesson that focusses on the area model of multiplication (Lesson 2E: *Counting rectangles*), while again still maintaining a link to the array model. In this lesson we use rectangles other than the unit square as our unit of area measurement. This is followed by a *Further* (optional) lesson (Lesson 2F: *3.2 rows of 2.6 unit squares*).

Multiplicative Reasoning 2E

Lesson

2Extra

Counting rectangles

Imagine we have lots of pieces of card, exactly like this one. →



1. Roughly how many cards are needed to cover this → sheet of paper?

Show for just 1 second!

2. How could we find the exact number of cards, without covering the whole sheet?



Summary

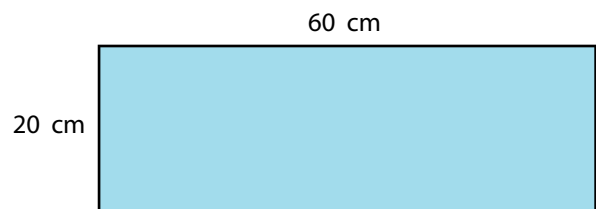
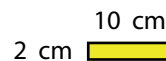
In this lesson we shift from the array model to the area model of multiplication. To help draw out the meaning of area, and how it can be measured efficiently using multiplication, we use various (rectangular) shapes as our unit of measurement, rather than starting with the unit square, and we arrange the rectangles in rows/columns.

Outline of the lesson

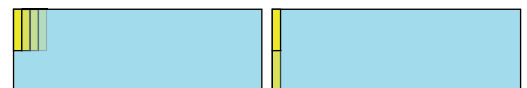
1. Estimate the number of cards.
 - Introduce Task 1 (above).
“Get ready to look at the sheet of paper Now!”
Show the ‘sheet of paper’ very briefly (about 1 second).
 - **Hide the ‘sheet’ again.** Discuss students’ estimates.
 - “How could we find the exact number of cards needed without covering the whole sheet?”
Discuss. **Show the ‘sheet’ again.** Discuss further.

— Students might start talking about the ‘size’ of the ‘sheet of paper’, eg its *width* and *height*; or they might relate it to the size of the card and consider the number of cards that could be laid out in a *row* and/or in a *column*.

2. Find the exact number of cards needed.
 - Tell the class that each card measure 2 cm by 10 cm and the sheet measures 20 cm by 60 cm.
 - Discuss how to use this information to find the exact number of cards needed to cover the sheet.
Allow several students to explain their ideas.
Allow plenty of time.
Try to ensure that these key questions are addressed:
 1. How many cards would there be in a row? (6)
 2. How many cards would there be in a column? (10)
 3. So how many cards altogether? (10 rows of 6 = 60)



3. Find the exact number of cards again.
 - Imagine orienting the cards ‘vertically’.
How could we cover the sheet now?
 1. How many cards would there be in a row? (30)
 2. How many cards would there be in a column? (2)
 3. So how many cards altogether? (2 rows of 30 = 60)



— You might want to ask students to interpret this:
 $20 \times 60 \div 20 = 60$.

4. Cover the ‘sheet’ with 4 cm by 15 cm cards.
 - Discuss how the 20 cm by 60 sheet could be covered by cards measuring 4 cm by 15 cm. *Start by briefly showing the old and new cards and asking for a quick estimate.*
 - How does this new number of cards relate to the previous number?
 - Consider other sizes of card (eg 5 cm by 6 cm) or of paper.



— Here we get 5 row of 4 cards = 20 cards,
or $1\frac{1}{3}$ rows of 15 cards = 20 cards. (Is $1\frac{1}{3}$ allowed?)
Why are there now 20 cards rather than 60?

Lesson

2E

Notes

Multiplicative Reasoning 2F

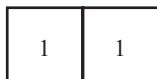
Lesson

2F**3.2 rows of 2.6 unit squares**

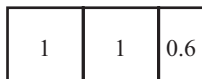
This is a
unit square:



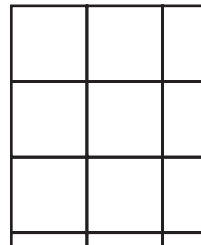
This is a row of
2 unit squares:



This is a row of
2.6 unit squares:



Here are 3.2 rows of
2.6 unit squares:



How many unit squares are there altogether in 3.2 rows of 2.6 unit squares?

Summary

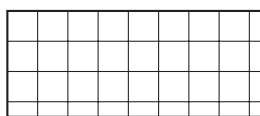
In this lesson we focus on the area model for multiplication and apply it to decimal numbers. Initially, our numbers represent *number of unit squares in a row*, and *number of rows*. However, this meaning is a bit strained for fractional numbers of squares and rows, and we effect a shift in meaning, to thinking of both numbers as distances along orthogonal number lines, whose product is an area.

Outline of the lesson

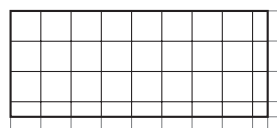
- Find the number of unit squares in 3.2 rows of 2.6 unit squares.
 - Ask for some quick estimates (eg, 'more than 6', 'about 3 rows of 3 squares, which is 9 squares').
 - Ask students (in small groups) to solve the task *geometrically*, ie by finding the size of each of the 12 squares/rectangles in the diagram.
 - Discuss their results. Compare them to the calculation 3.2×2.6 .

- Compare 3.5×8.5 with 3×8 and 4×9 .

- Ask students to estimate 3.5×8.5 *numerically*.
Is it nearer 3×8 or 4×9 , or exactly in the middle?
- Ask students to evaluate 3.5×8.5 *geometrically*, by sketching a 3.5 by 8.5 rectangle, like this drawing

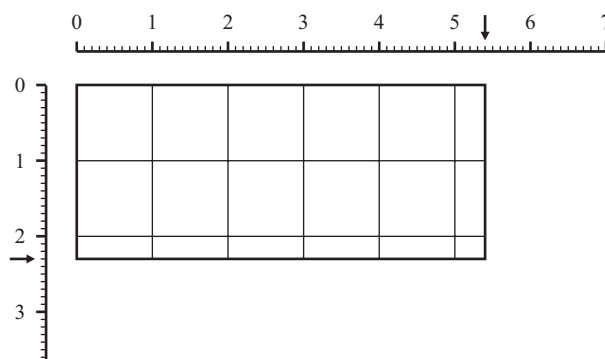


or this:



- Variants on 2.3×5.4 .

- Present this diagram.
Ask for a quick estimate for 2.3×5.4 .
- Ask students to imagine changing the rectangle by moving one or both arrows. Pose questions like these:
 - How much bigger than 2.3×5.4 is 2.3×6.4 ?
 - How much bigger than 2.3×5.4 is 2.3×5.5 ?
 - Which is larger, 2.3×6.4 or 3.3×5.4 ?
 (Note that we are only interested in the *change* in value here, not the values themselves.)



- Compare methods.

- Present a multiplication like 7.2×3.8 .
Discuss different ways of performing the calculation.
- Use the area model to check students' answers.

Multiplicative Reasoning 2F

Lesson

2F

Overview

Mathematical ideas

The ideas in this lesson hinge on the fact that multiplication is distributive over addition: $a \times (b + c) = a \times b + a \times c$.

This law doesn't work the other way round!

Addition is not distributive over multiplication: $a + (b \times c) \neq (a + b) \times (a + c)$.

The lesson provides experience of using and expanding brackets. The open expression $(100+10) \times 3$ can be expanded to $100 \times 3 + 10 \times 3$ and the algebraic expression $3(N + 10)$ can be expanded to $3N + 30$.

Students' mathematical experiences

Students may discover some of the following

- when multiplication and addition are combined, the order matters
- multiplication is distributive over addition
- switching the order of the function machine $\times 4$ then $+50$, changes the output by $50 \times 4 - 50$, ie by $50 \times (4 - 1)$
- switching the order of $\times 3$ then $+10$, changes the output by $10 \times 3 - 10$, ie by $10 \times (3 - 1)$
- we can use open expressions to describe function machines. For example, $\times 3$ then $+10$ can be written as $100 \times 3 + 10$ for an input of 100, or as $3N + 10$ for an input of N
- open expressions can reveal structure
- a Cartesian graph can reveal properties of functions/expressions and raises the issue of continuity: do non-whole number inputs work?
- it can be useful to try special cases, like an input of 0.

Assessment and feedback

Our interest is in rules-with-reasons, so give students plenty of chances to air their ideas (and don't evaluate or probe these too quickly).

Continually listen out for the nature of students' ideas so you can

- discern whether students are struggling with the arithmetic
- discern whether students are focussed too much on an empirical approach
- decide when (or whether) to move to the next step in the lesson.

If you find students are struggling with the comparison in Part 2 of the lesson, you could see whether another visual representation helps, this time for $100 \times 3 + 10$ and $(100+10) \times 3$:

Key questions

Is there a rule for the constant difference?

How can I change 'this' open expression with brackets to an open expression without brackets?

Can we use the open expressions to show/explain the constant-difference rule?

Adapting the lesson

The lesson provides plenty of opportunities for students to come up with, and hone, their explanations; however, this will require continual judgement as to when to develop specific explanations further, and when to put these on hold in order to explore the work further by using open expressions and Cartesian graphs.

Some students might struggle with some of the arithmetic, eg with calculating $(100+50) \times 4$, or, later in the lesson, they might balk at this open expression. However, since we are not really interested in the arithmetic, we can turn this to our advantage. We can make $(100+50) \times 4$ more accessible by writing it as $100+50 + 100+50 + 100+50 + 100+50$, or simply as 100 50 100 50 100 50 100 50; this can then be compared to $100 \times 4 + 50$ by writing it as 100 100 100 100 50.

Multiplicative Reasoning 2F

Lesson

2F**Outline of the lesson (annotated)**

- Find the number of unit squares in 3.2 rows of 2.6 unit squares.

- Ask for some quick estimates (eg, 'more than 6', 'about 3 rows of 3 squares, which is 9 squares').
- Ask students (in small groups) to solve the task *geometrically*, ie by finding the size of each of the 12 squares/rectangles in the diagram.

— There are 6 whole unit squares, three 0.6 unit squares, two 0.2 unit squares and one 0.12 unit squares. Some students might have difficulties with the latter piece and it might be worth drawing a diagram of this sort:



- Discuss their results. Compare them to the calculation 3.2×2.6 .

— You could simply use a calculator for 3.2×2.6 . Alternatively, if students use some kind of algorithm which breaks the calculation into several steps (eg the grid method or long multiplication), you might want to relate the results of each step with features of the diagram.

Other students might use ideas from Lesson 3A and change 3.2×2.6 to, say, 0.8×10.4 to, say, $8 \times 1.04 = 8.32$.

- Compare 3.5×8.5 with 3×8 and 4×9 .

- Ask students to estimate 3.5×8.5 *numerically*. Is it nearer 3×8 or 4×9 , or exactly in the middle?

— Some students will assume that 3.5×8.5 is midway between 3×8 and 4×9 . It is salutary to realise it is not (even though it is close). Others might evaluate 3.5×8.5 as $3 \times 8 + 0.5 \times 0.5$ and so conclude it is very close to 3×8 .

Some students might extend ideas from Lesson 3A and double *both* numbers (to 'get rid' of the decimals) and then divide by 4: $3.5 \times 8.5 = 7 \times 17 \div 4$

- Ask students to evaluate 3.5×8.5 *geometrically*, by sketching a 3.5 by 8.5 rectangle.

— You could also look at this *algebraically*:

$$\begin{aligned} 3 \times 8 &= 24 \\ 3.5 \times 8.5 &= (3 + 0.5)(8 + 0.5) = 24 + 5.5 + 0.25 \\ 4 \times 9 &= (3 + 1)(8 + 1) = 24 + 11 + 1 \end{aligned}$$

- Variants on 2.3×5.4 .

- Present this diagram. Ask for a quick estimate for 2.3×5.4 .

— Here the focus is not on the value of 2.3×5.4 itself, but on *changes* in value. In turn this focusses attention on the strength of the area model, which is dealing with the fractional parts of a multiplication.

In the diagram, the numbers are represented as distances along orthogonal number lines. However, allow students to continue to refer to 'number of units squares' and 'number of rows (or columns)' if they want to.

Students will tend to think that expressions like 2.3×6.4 and 3.3×5.4 are equivalent. The diagram shows very clearly that they are not.

- Ask students to imagine changing the rectangle by moving one or both arrows. Pose questions like these:

How much bigger than 2.3×5.4 is 2.3×6.4 ?

How much bigger than 2.3×5.4 is 2.3×5.5 ?

Which is larger, 2.3×6.4 or 3.3×5.4 ?

(Note that we are only interested in the *change* in value here, not the values themselves.)

— Again, you might want to include an algebraic approach and bring out the distributive property of multiplication. For example, when comparing 2.3×5.4 with 2.3×6.4 , we can write the latter as $2.3 \times (5.4 + 1)$, which is $2.3 \times 5.4 + 2.3$.

- Compare methods.

- Present a multiplication like 7.2×3.8 . Discuss different ways of performing the calculation.

— This would include the use of various algorithms *and* the use of various models (eg, skips along the number line, the double number line, equivalent rectangles).

- Use the area model to check students' answers.

— Again, you might want to make explicit links between the steps in the algorithms and features of the area model.

Multiplicative Reasoning 2F

Lesson

2F

Background

Algebraic structure and the laws of arithmetic

In this lesson we examine how a pair of operations (multiplication and addition) interact. Order matters for this pair because multiplication is distributive over addition:

$(100+10) \times 3 = 100 \times 3 + 10 \times 3$ rather than $100 \times 3 + 10$.

Similarly

$(x + b) \times a$ is equivalent to $ax + ab$ rather than $ax + b$.

Put more concretely, if a collection of objects is multiplied by 4, say, then each element in the collection is multiplied by 4; but if 4 is added to a collection, 4 is not added to each element, just to the collection as a whole.

Explanation and proof

In this lesson we compare outputs of the form $ax + b$ and $(x + b)a$.

Their difference is $b(a - 1)$, which can of course be proved algebraically:

$(x + b)a - (ax + b) = ax + ab - ax - b = ab - b = b(a - 1)$.

We move tentatively *towards* such a proof in this lesson, but suggest adopting a numerical approach to start with, and to gradually look for rules-with-reasons. The balance of the approach you use will depend on the class and on students' responses during the lesson.

In the case of the outputs for the $\times 3, +10$ and $+10, \times 3$ function machines, where the difference is always 20, students might give a rule/reason like this initially:

In the first machine you add one 10. In the second machine you add 3 lots of 10, so you have added $(3 - 1)$ extra 10s.

Later, students might come up with something like this:

For an input of 100,
the first machine has an output of $100 \times 3 + 10$;
the second has an output of $(100+10) \times 3$ or $100 \times 3 + 10 \times 3$.
The difference is $10 \times 3 - 10$, which is $10 \times (3-1)$.

Or

for an input of N ,
the first machine has an output of $3N + 10$;
the second has an output of $3(N+10)$ which is $3N + 3 \times 10$.
The difference is $(3-1) \times 10$.

Generalised number and variable

We are working more in the realm of generalised number than variable in this lesson. We don't consider a systematic set of inputs (although that would be a perfectly viable strategy).

Take the function machines $\times 3, +10$ and $+10, \times 3$.

Here we are not primarily interested in the input (which is multiplied by 3 in both machines). Though we have to keep the input in mind, our focus is on the fact that in the first machine we add 10 while in the second we add 10×3 .

Thus, we are trying to adopt a *generic* approach. Though we are considering specific numerical inputs and outputs, we want students to focus on the *structure* of multiplication and addition, not just on their numerical results. We hope students will see the *general in the particular*, especially when we use large (albeit simple) numbers.

[An alternative, more *dynamic*, approach involving the notion of variable, would be to note what happens when the input *changes*. In contrast to Boat Hire, say, here the two outputs will always change in identical ways (namely by 3 times the change in the input). This means there is a constant difference between the two outputs. This can be seen nicely on the Cartesian graph.]

Students create algebraic expressions and use brackets

In the lesson we make use of the quasi-variable 100, in open expressions like $100 \times 3 + 10$, and move towards the creation of algebraic expressions like $3N + 10$ and $3(N+10)$. How far you take this and how much weight you give it will depend on the class and the feedback you get from them. The aim is that students will begin to see that algebraic expressions help reveal structure and allow one to prove rules.

In developing the algebraic expressions, you will need to decide how closely they should mirror the function machines from which they arise and how far you move towards conventional algebraic notation. Thus, for the function machine $\times 3, +10$ you might move from the algebraic expression $N \times 3 + 10$ to $3 \times N + 10$ to $3N + 10$. The work provides a good opportunity to use brackets and to gain an understanding of how they can be expanded.

Students use and interpret Cartesian graphs

The use of the Cartesian axes, to represent the inputs and outputs for a pair of function machines, is potentially very rich.

There is an opportunity here to re-focus on the notion of variable: for the function machines $\times 3, +10$ and $+10, \times 3$, if the input changes by a specific amount, the output changes by 3 times that amount. This of course is represented by the slope of the two straight lines.

The constant difference between corresponding outputs for the two machines is shown by the fact that the lines are parallel and that one is 20 units above the other. It is also interesting that the y-intercept is 20 for one of the lines. Why is that?

The graphs provide another opportunity to discuss continuity. In the main part of the lesson, the emphasis has been on whole-number inputs (unless students have come up with other inputs). What about decimals (and negative numbers)?

Lesson

2F

Notes

Algebra 1AB

Mini-assessment

3AB

Which is larger, $3n$ or $n + 3$?

Commentary

The aim of this Mini-assessment is to see what approaches students use to compare algebraic expressions.

- Do students understand the algebraic notation?
- Do they focus on the operations ('multiplication makes bigger') ?
- Do they evaluate the expressions for specific values of n ?
- Do they respond to the fact that we don't know the value of n ?
- Do they realise that the difference between the expressions might change as n varies?

Use the Mini-assessment a few days before teaching the two lessons.

Algebra 1A

Lesson

3A**Boat hire**

Olaf is spending the day at a lake.

He wants to hire a rowing boat for some of the time.

Freya's Boat Hire charges £5 per hour.

Polly's Boat Hire charges £10 plus £1 per hour.

Whose boat should Olaf choose?

Summary

In this lesson, the boat hire problem is used to explore the two algebraic relationships underlying Freya's and Polly's different hire charges. A variety of representations are used to express the relationships:

- everyday language
- algebraic expressions
- tables of values
- points on a Cartesian graph.

Outline of the lesson

1. Display the *Boat hire* problem.
 - Ask students for their immediate responses.
 - Ask students to consider the problem further in small groups.
 - Collect numerical data on the total cost for various numbers of hours (and listen to students' arguments and conclusions - but don't pursue these at this stage).
 - Represent the data
 - 'randomly' on the board
 - in a (randomly ordered) table
 - in an ordered table (try to prompt the need for this, rather than simply produce such a table).
2. Ask students to represent the hire-rules algebraically.
 - For example, as expressions, eg $5a$ and $10 + a$
 - Or as relations, eg $b = 5a$, $b = 10 + a$.
3. Ask students to represent the data as points on a standard (Cartesian) graph.
 - Ask questions like "Are Freya's and Polly's charges ever equal?"
4. Discuss the links between the various representations and between them and the story.

For 1 hour
Freya is cheaper
£5 v £11

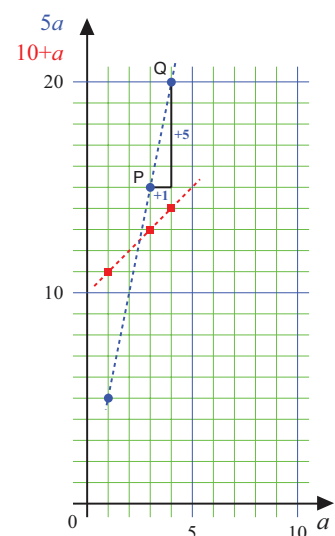
Try 3:
£15 > £13

4 hours:
£20 and £14
so Polly cheaper

£50 and £20
for 10 hours

Hours	Freya £	Polly £
1	5	11
4	20	14
10	50	20
3	15	13

Hours	Freya £	Polly £
1	5	11
3	15	13
4	20	14
10	50	20



Algebra 1A

Lesson

3A**Overview****Mathematical ideas**

When we work with equations, we often think of a letter as representing a single number as yet unknown. Here, we are working with relations between two variables (the number of hours and the charge) and we think of the letters as representing a set of numbers.

A Cartesian graph is a particularly powerful way of representing variables, since it allows us to represent a range of values simultaneously.

Students' mathematical experiences

Students might discover some of the following

- for some values of a , Freya's hire charge ($5a$) is larger than Polly's ($10+a$), but for others it is smaller
- when $a = 2.5$ the expressions are equal
- if a increases by 1, then $5a$ increases by 5, but $10+a$ only increases by 1
- each set of points on the graph forms a pattern: each lies on a straight line.

Students might discuss

- different slopes and how these relate to the hourly charges
- continuity, ie whether some or all points on the line fit the relationship.

Some students may want to change the scales of the axes. Discourage them from doing so. The expressions have been deliberately chosen to be represented on an equally-scaled graph.

Assessment and feedback

Be flexible over the organisation and timing of the lesson. Some teachers have taught this lesson over two periods.

Choose some students to contribute to a subsequent discussion. Some (less confident?) students' contributions may be more coherent if you "rehearse" with them beforehand: "That's a great idea. I'm going to ask you to explain that to the class. Let's have a go at preparing what you'll say."

Allow students time to generate algebraic expressions, but if they really struggle you may want to provide the expressions for them.

Some students may have difficulties constructing a Cartesian graph. Observe the students and decide whether you need to spend some time with either a group or the whole class teaching these skills.

You may need to prompt or challenge some students to consider whether the charges can ever be equal.

Key questions

When is Freya (or Polly) cheaper?

How could we record this more systematically?

What happens to the cost as the number of hours increases?

What if Olaf hired a boat for 1.5 hours?

Adapting the lesson

You might want to adopt a different context for a subsequent lesson. For example, the price of an ice cream cone for different numbers of scoops, or the yearly cost of belonging to a swimming club (based on a membership fee and a cost for each visit). Choose the numbers carefully - for example, keep the multipliers small if you want to graph the relationships, and think about where the values coincide - do you want this to occur for a simple whole number (4 ice cream scoops, say), or something more obscure (3.4 scoops, say)?

Algebra 1A

Lesson

3A**Outline of the lesson (annotated)**1. Display the *Boat hire* problem.

- Ask students for their immediate responses.

Students will tend to say that one or other hire firm is cheaper. Let them voice their opinions but don't discuss their reasons at this stage. (Their decisions may be based on one or two numerical examples, or they may have focussed on the lower hourly rate, in the case of Polly, or the absence of a down-payment in the case of Freya.)

The differences of opinion should make the task more engaging for students.

- Ask students to consider the problem further in small groups.

This allows students to think more carefully about their and other students' ideas.

- Collect numerical data on the total cost for various numbers of hours (and listen to students' arguments and conclusions - but don't pursue these at this stage).

With the students' help, start to organise the data in a more systematic way, culminating in ordered tables.

Some students might already have developed a coherent understanding of the problem, involving the idea that the choice of hire company depends on how long one wants the boat for. But, keep it simple and allow students time to consider how and why the two expressions differ.

- Represent the data
 - 'randomly' on the board
 - in a (randomly ordered) table
 - in an ordered table (try to prompt the need for this, rather than simply produce such a table).

Ordered tables allow one to bring out the fact that increasing the hire time by one hour, say, increases the hire charge by £5 (Freya) or £1 (Polly). This can of course be related back to the original story, but also later in the lesson to the algebraic and graphical representations of the relationship.

2. Ask students to represent the hire-rules algebraically.

- For example, as expressions, eg $5a$ and $10 + a$
- Or as relations, eg $b = 5a$, $b = 10 + a$.

If students find this difficult, then present the expressions to them, without too much ado. Or state the expressions in words, eg " $5 \times$ number of hours".

3. Ask students to represent the data as points on a standard (Cartesian) graph.

- Ask questions like "Are Freya's and Polly's charges ever equal?"

It is more illuminating to use axes with the same scale, even though the resulting line for Freya is then very steep. To avoid getting bogged down in the issues of choosing and drawing suitable axes (which are of course very important), you might want to give students a blank version of the graph with the axes already drawn and numbered.

4. Discuss the links between the various representations and between them and the story.

Algebra 1A

Lesson

3A**Background****Students develop their understanding of 'variable'**

The kind of variable we are learning about (be it called ' a ' or 'the number of hours') has these properties:

- it is a number;
- it can take on lots of different values;
- as it changes in a systematic way, the 'dependent variable' (in this case 'the total hire charge') may also change in a systematic way.

Specifically, students might discover that

- for 'largish' values, such as $a=8$, $5a$ is larger than $10+a$
- for some 'small' values, like $a=1$, $5a$ is smaller than $10+a$
- there is a particular value, $a=2.5$, where the two expressions are equal
- if a is increased by a specific amount, then $10+a$ increases by the same amount, whereas $5a$ increases by 5 times that amount.

Students develop their ability to 'read' various representations

Each of the properties of variable mentioned above can be seen in the three representations (tables, expressions, graph). For example,

- if a value of a in a table goes up by 1, then the corresponding values of $5a$ have a difference of 5 (see the second table, right)
- if two points representing $(a, 5a)$ are 1 horizontal unit apart on a graph, they will be 5 vertical units apart (see points P and Q on the graph, below right)
- the two sets of points on the graph form a pattern (in this case, they lie on two straight lines), and so help students get a general sense of the relationships, ie a sense of what happens for a range of values rather than just isolated numerical cases
- the patterns (ie the straight lines) also suggest that there is a systematic relationship between a and $5a$, and between a and $10+a$ and that any further boat-hire points that we might calculate will also lie on the corresponding line
- the two lines cross, so there is a strong incentive to find the common point [in this case, $(2.5, 12.5)$] and to consider what it might mean in the present boat-hire context.

Students create algebraic expressions

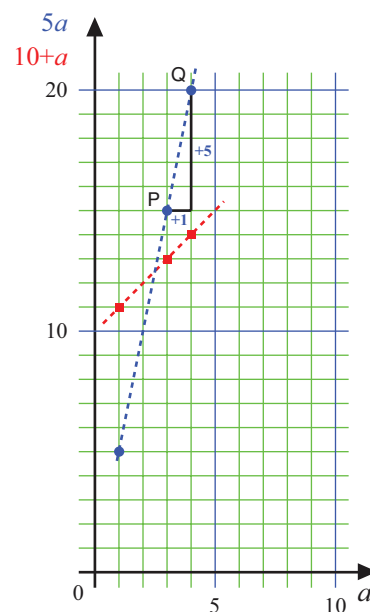
The problem can be solved perfectly well without algebraic symbolisation; on the other hand, the expressions do provide a convenient shorthand and the context should help make them meaningful to the students. It also allows us to refer explicitly to variables.

Students engage with the notion of continuity

The graph, in particular, prompts the question of 'intermediate' values: given that each set of points lies on a line, what about some other points on that line: do they fit the algebraic relationship, and do they fit the boat-hire story? And do 'all' points on the line fit the relationship?

Many hire firms charge in whole numbers of hours (so a boat used for 2 hours 15 mins, say, would be charged for 3 hours). This results in a step function, represented on a Cartesian graph by a set of horizontal lines.

not ordered		ordered	
a	$5a$	a	$5a$
1	5	1	5
4	20	3	15
3	15	4	20



Algebra 3B

Lesson

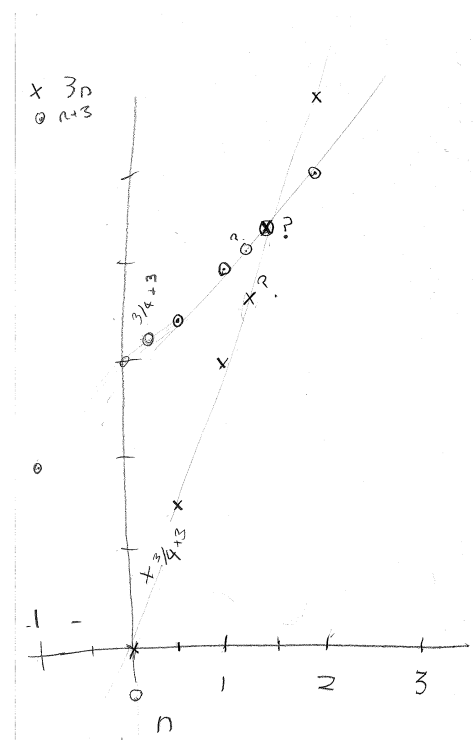
3B**Comparing expressions****Which is larger, $3n$ or $n + 3$?****Summary**

The students compare two algebraic expressions as in *Boat hire*. However, the problem is set in a 'pure' mathematical context, and some students may find this more difficult.

As in *Boat hire* a variety of representations are used to compare the two expressions. You should place more emphasis on continuity and extending n to include rational and negative numbers.

Outline of the lesson

1. Display the problem, and, if necessary, remind the students that they have looked at this problem before.
 - Discuss which of the two expressions is bigger: $3n$ or $n + 3$, allowing time and space for students to express different ideas.
 - Prompt alternatives, if necessary: Does anyone think that $n + 3$ is always bigger?
 - Encourage correct and incorrect explanations, recording them on the board.
 - Ask students to consider the problem further in pairs.
2. Collect numerical data on the board. As in the Boat Hire lesson, record these initially in a random layout.
 - Encourage the students themselves to suggest using a table, then an ordered table of values.
 - Can $3n$ and $n + 3$ ever be the same?
3. Put a standard pair of Cartesian axes on the board.
 - Ask students to represent some of the values they've come up with by placing a few of the points on the axes.
 - You may want to ask students, in pairs, to construct their own graphs
 - Can we draw a line through the points?
4. Discuss and make links between the various representations and between them and the algebra.
 - Ask students to use the graph to predict pairs of numbers that satisfy each relation, then check, recording these in the table.
 - Ask: Can you use the algebraic relations to predict points on each graph? Are you right?
5. Homework: Investigate similar problems.
 - Say to students: Make up your own "Which is bigger?" problem and show how it could be solved.



Algebra 1B

Lesson

3B

Overview

Mathematical ideas

A straight-line graph, such as $y = 3n$ or $y = n + 3$, is a set of points that all satisfy a particular relation.

The different tools used in this lesson (Cartesian graphs, symbolic algebra, tables of values and everyday language) are different ways of representing the same algebraic relationships.

Students' mathematical experiences

Allow students plenty of time to make, discuss and correct mistakes, such as abbreviating $n + 3$ to $n3$.

The students should ...

- revisit and extend the mathematical ideas they encountered in *Boat hire*
- justify, discuss and check their ideas
- investigate whether points on the Cartesian graph satisfy the relation expressed in symbolic algebra or in everyday language, and vice versa
- discuss the differences (and similarities) between $3n$, $n3$ and $n + 3$ in everyday language and in symbolic algebra.

The students should discover ...

- sometimes $3n$ is greater and sometimes $n + 3$ is greater, depending on the value of n
- when $n = 1.5$, the two expressions are equal
- as n increases by 1, then $3n$ increase by 3, but $n + 3$ only increases by 1.

Assessment and feedback

This lesson revisits the starter activity to this pair of lessons. Look back at your notes to decide what areas to focus on.

Choose 3 students to assess during the lesson. Talk to each at least once during the lesson and observe them working.

Some students may argue that

- $3n$ is greater, because “multiplication always makes things bigger”
- $n + 3$ can be abbreviated to $n3$ or $3n$ and, hence, that the two expressions are exactly the same.

Encourage all these ideas and allow time for students to explore them using the different representations.

The relationships can be extended to fractions, decimals and negative numbers. However, some students may have difficulty multiplying (or adding) such numbers. Discuss the different methods that students use, as a class.

Key questions

Why do you think that ...?

What do you notice about the graph?

Is there a point where $3n$ is equal to $n + 3$?

Can you predict what would happen for $n = \frac{1}{2}$, $n = -1$, ...?

Adapting the lesson

Think up other expressions for students to compare (or select expressions that students have come up with). Keep them fairly simple. You might want to control for what kind of value of the unknown (small whole number, large whole number, fraction?) the values of the expressions coincide. And you might want to vary the operations (eg compare $x+14$ and $100-x$, or $x+14$ and $100 \div x$).

Algebra 1B

Lesson

3B**Outline of the lesson (annotated)**

1. Display the problem, and, if necessary, remind the students that they have looked at this problem before.
 - Discuss which of the two expressions is bigger: $3n$ or $n + 3$, allowing time and space for students to express different ideas.
 - Prompt alternatives, if necessary: Does anyone think that $n + 3$ is always bigger?
 - Encourage correct and incorrect explanations, recording them on the board.
 - Ask students to consider the problem further in pairs.

———— The problem is related to this question from the CSMS Algebra test: Which is larger, $2n$ or $n + 2$? Nationally, only 4% of Y8 students answered this item correctly. However, the task is easier when worked on collectively, in a classroom setting.

2. Collect numerical data on the board. As in the Boat Hire lesson, record these initially in a random layout.
 - Encourage the students themselves to suggest using a table, then an ordered table of values.
 - Can $3n$ and $n + 3$ ever be the same?

———— Some students may not immediately recognize $3n$ as $3 \times n$. Encourage the students to clarify the syntax issue for themselves by talking about the expressions.

3. Put a standard pair of Cartesian axes on the board.
 - Ask students to represent some of the values they've come up with by placing a few of the points on the axes.
 - You may want to ask students, in pairs, to construct their own graphs
 - Can we draw a line through the points?

———— The expressions are equal when $n = 1.5$. Students will need some time to consider and discuss the multiplication (and addition) of fractions and decimals. Encourage class sharing and discussion of the students' ideas and methods - resist simply reminding them of previously encountered formal methods.

4. Discuss and make links between the various representations and between them and the algebra.
 - Ask students to use the graph to predict pairs of numbers that satisfy each relation, then check, recording these in the table.
 - Ask students to use the algebraic relations to predict
Ask: Can you use the algebraic relations to predict points on each graph? Are you right?

———— Allow students plenty of time to explore whether the points on the straight line do actually satisfy the relation.

5. Homework: Investigate similar problems.
 - Say to students: Make up your own "Which is bigger?" problem and show how it could be solved.

———— The students themselves are likely to suggest expressions of the form kn and $n + k$, $4n$ and $n + 4$, $5n$ and $n + 5$ etc. You may want to suggest expressions involving different letters or large numbers such as $4x$ and $x + 100$. Prepare these in advance and then adapt / use as appropriate during the lesson.

Algebra 1B

Lesson

3B**Background****Assessment**

This lesson is an ideal opportunity for formative assessment and feedback. However, giving thoughtful feedback to the whole class can be very time consuming. One way of dealing with this is to assess the work of a few students in depth.

Look at the students' classwork together with this homework task: Make up your own "Which is bigger problem" and show how it could be solved.

Consider the issues raised for teaching the next lessons. For example:

- How do the students understand the key ideas of variable and the use of algebraic symbols and a Cartesian graph?
- Are there any related issues in their understanding [such as the multiplication of fractions]?

Comments to students could include one thing they have understood and one thing they need to do. A possible comment for student F in the video might be:

It was good that you noticed that the points for $3n$ and $n + 3$ seem to lie in straight lines. Can you test this with some other values for n ? ... 2 ... 1.4?

Using a number line to work out the fraction multiplications was a good idea. Compare your answers with someone else too

An interview with 3 students

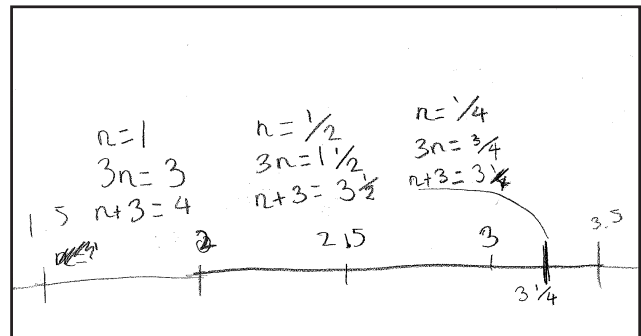
We made a video recording of an interview with three Year 8 students on this task.

You might want to watch the following extracts, which are available on the ICCAMS website: www.iccams-maths.org

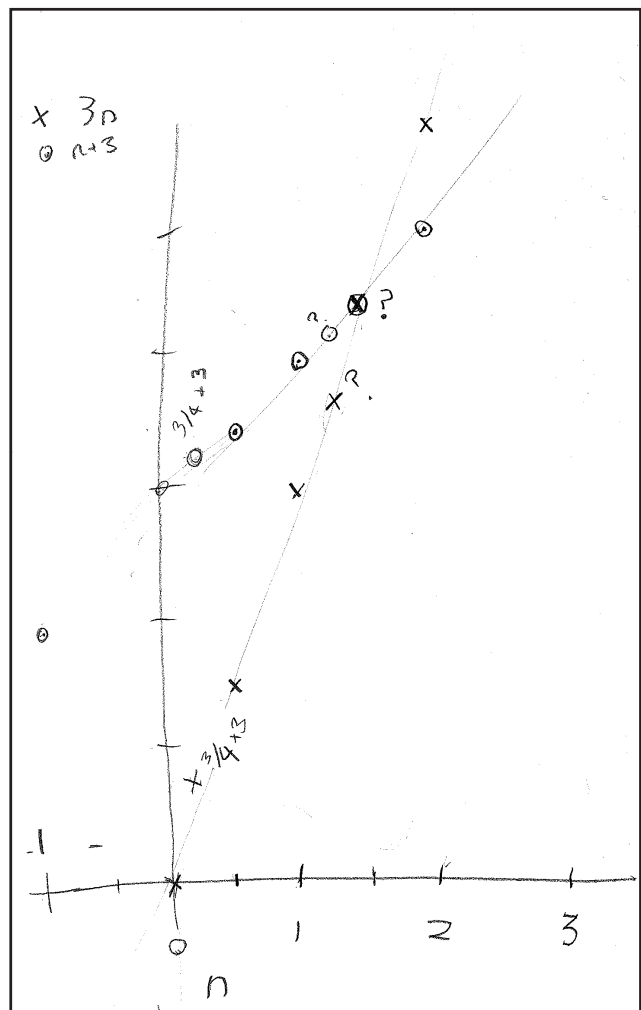
Extract A [28.00-34.18]: One student makes a generalization involving whole numbers ("3n is bigger if n equals more than 1, like 2, 3, 4, 5 and 6, and n + 3 is bigger if the number's less than 1, like 1, a half, a quarter and an 8th, and 3n and n + 3 are the same if n equals infinity") and they begin using Cartesian axes.

Extract B [35.20-40.20]: They make predictions, suggest that the points seem to lie in a line and consider the values of $3n$ and $n + 3$ when $n = 0$.

The figure below shows where the students have listed values of the expression for $n=1$, $n=\frac{1}{2}$ and $n=\frac{1}{4}$, and where they have marked one of the values on a number line.



On the page below, the axes and some of the points were drawn by one of the interviewers. The lines through the points were drawn, in a very confident and fluent way, by one of the students, who also added further points.



Lesson

3B

Notes

Algebra 2AB

Mini-assessment

4AB**Floating balloon**

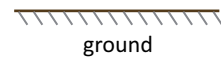
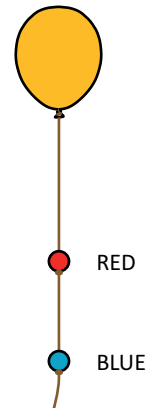
A RED bead and a BLUE bead are tied to a helium-filled toy balloon.

This equation is about the heights (in cm) of the beads above the ground:

$$w = v - 30$$

- 1 a. What do v and w stand for?
- b. What does the “30” tell us?
- c. Find some values for v and w .

- 2 Write the equation as $v = \dots\dots\dots$.

**Commentary**

The aim of this Mini-assessment is to explore how students interpret a simple algebraic relation.

Can students match the elements in the relation to the context?

Do students realise that the relation is about *numbers*?

Do students realise that the relation is about a *general* situation, ie it that applies to *varying* heights of the balloon, rather than just to specific distances in the given ‘static’ picture?

Does finding numerical values for the variables help students make sense of the situation?

Can students transpose the equation? What methods do they use?

Notes:

The diagram is quite schematic so students might take a while to make sense of it. For example, they might not realise that the shaded line represents the ‘ground’. Or they might take ‘height’ to mean distance from a bead to the balloon.

Some might think the beads can slide up and down on the string - how could one modify the equation to allow for this?

Algebra 2A

Lesson

4A**Pyramids and rules**

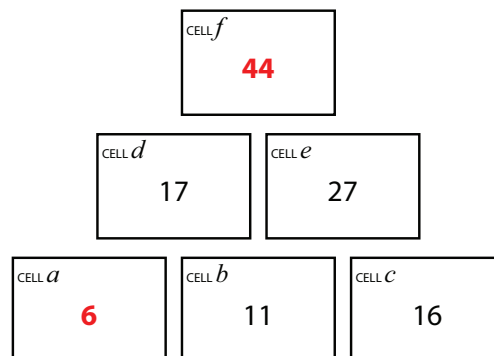
The numbers in the cells in the bottom row go up in 5s.

For the other cells, each number is the result of adding the numbers in the two cells beneath it.

Think of a number, eg 6.

Put it in cell *a*.

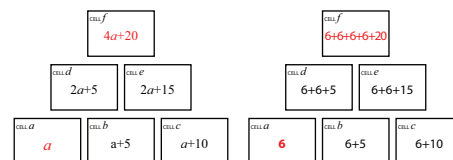
Find the number in cell *f*.

**Summary**

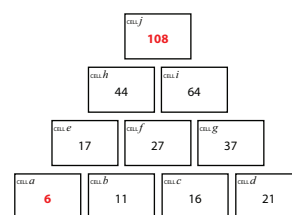
Here we use the familiar context of ‘pyramids’ to find rules for determining the number in a particular cell in terms of the number in some other cell. By using a task with lots of steps it becomes difficult to keep track of what is going on. This provides an incentive to record the steps and ultimately to simplify them. Students are encouraged to use algebraic notation, alongside a generic approach that tracks the numbers used, but without losing touch with more basic approaches.

Outline of the lesson

- Find a rule for the number in cell *f*.
 - Go through the example above.
 - Now ask students to use their own starting number for cell *a* and to find the resulting number in cell *f*.
 - Ask a student to tell you their starting number. Instantly tell them the number in cell *f* [by secretly using the rule $f = 4(a + 5)$, or $f = 4a + 20$]. Repeat a few times.
 - Challenge the class to find a rule for *f*.
- Justify the rule for the number in cell *f*.
 - Some students may simply *spot* the rule $f = 4(a + 5)$ (or an equivalent). Encourage students to give *reasons* for the rules. Let students express their reasons verbally, numerically or symbolically.
 - Encourage the use of a symbolic approach. But do this alongside a generic approach where the operations are left fully or partly ‘open’ to show the structure.



- Modify the task in some way. Find a new rule.
 - For example, change the ‘adding number’ in the bottom row from 5 to 10. Find the new rule for *f* in terms of *a*.
Or find a rule for *f* in terms of *a* for ‘any’ adding number.
Or find a rule for *a* in terms of *f*.
Or find the value of *a* for a specific value of *f*, say 100 [eg by trying to solve the equation $4(a+5)=100$].
Or increase the number of cells.



Algebra 2A

Lesson

4A

Overview

Mathematical ideas

Pyramids, or similar arrangements, appear in the *Framework Maths* and the *Maths Frameworking* textbooks (as well as in the National Strategy *Framework* document for Key Stage 3). However, they are often used solely as vehicles for practising algebraic manipulation, with the inherent structure ignored (an elegant exception is Malcolm Swan's 'Pyramid' task in the *Improving Learning in Mathematics* materials). Thus an opportunity is lost to motivate students to search for rules and to see the utility of algebraic symbolisation.

Here students experience the utility of symbolising for keeping track of a complex task, for revealing its structure and for simplifying the structure. Thus, students use algebraic and generic symbolisation to record the steps in a complex task and to simplify these steps. In turn this allows students to derive rules between 'starting' and 'finishing' numbers (independent and dependent variables) and to justify rules that they have already spotted (induced). The meaning of the symbolisation is maintained by relating it to everyday language and to calculations that the students have performed.

Students' mathematical experiences

The students should experience that

- open expressions can convey meaning/structure
- open expressions can be written in different but equivalent ways
- they can cope with a complex set of symbols and variable: $a, b, c, d, \dots$

Assessment and feedback

During the lesson, try to observe which of the students are

- working primarily on the basis of results from numerical calculations
- working numerically but discerning structure (eg through the use of open expressions such as 6, $6+5$, $6+5+5$, and $6+6+5$, $6+5+6+5+5$, etc)
- working symbolically and discerning or not discerning structure.

Use this to decide how to facilitate the discussion. It is important to use all these approaches (1) to enable all students to engage and (2) to encourage students to make links back and forth between ideas expressed numerically, symbolically and in every day language.

The *Floating balloon* Mini-assessment should give you an idea of how *willing* and *able* students are to interpret an algebraic relation. As with the current pair of lessons, the activity is set in a meaningful context, though it is not as purposeful.

Key questions

How does the starting number (eg 6) affect ('ripple through') the other cells?

How does the 'adding number' (eg 5) affect the other cells?

How many times does the starting number and the adding number appear in the top cell?

How does a change (eg, $+1$ or $\times 2$) to the starting number (or to the adding number) affect the number in cell f ?

Adapting the lesson

We have already suggested changing the 'adding number', the number of rows, or the direction of the relationship (eg $f \rightarrow a$ instead of $a \rightarrow f$). You could also change the rule for the bottom row (eg $\times 5$ instead of $+5$, or a 2-step rule such as $\times 2 + 1$, etc). Or you could introduce a 'submerged' row (see the *Iceberg* starter).

You could decide to have *two* independent variables, as in this example:

Choose any number for cell a and any number for cell b . Make up a rule for cell c (eg $c = a + b$). Find a rule for cell f .

- Which number, a or b , has the greater influence on f ? Why?
- For what values of a and b is f equal to 24 (say)? Graph the relationship between a and b .

Algebra 2A

Lesson

4A**Outline of the lesson (annotated)**1. Find a rule for the number in cell f .

- Go through the example above.
- Now ask students to use their own starting number for cell a and to find the resulting number in cell f .
- Ask a student to tell you their starting number. Instantly tell them the number in cell f [by secretly using the rule $f = 4(a + 5)$, or $f = 4a + 20$]. Repeat a few times.
- Challenge the class to find a rule for f .

Students can be quite amazed by this and thus keen to find out how it is done.

It can help to use an extreme starting number, like 100. Students are likely to express the rule verbally rather than symbolically. This is fine at this stage.

Initially, students might spot that f is 4 times b . This is fine, but ask "So how is f related to a ?"

Try to ensure that both versions of the rule arise:
 $f = 4(a + 5)$ [or $+5$ then $\times 4$], and
 $f = 4a + 20$ [or $\times 4$ then $+20$].

A verbal explanation might go something like this:

The number in cell a also appears in cells b and c ;
 so it appears twice in d and twice in e ,
 and so it appears four times in f ...

This is a promising start since it is looking at the structure.

For the original example, with 6 in cell a and an 'adding number' of 5, a (fully open) generic approach would be to write $6+5$ for cell b , $6+5+5$ for c , $6+6+5$ for d , $6+6+5+5+5$ for e , and $6+6+6+6+5+5+5+5$ for f .

Cell f 'simplifies' to $(4 \times 6) + (4 \times 5)$ or $4(6+5)$.

3. Modify the task in some way. Find a new rule.

- For example, change the 'adding number' in the bottom row from 5 to 10. Find the new rule for f in terms of a .

Ask, "What happens if the starting number is negative or a decimal? Will the rule still work?"

Or find a rule for f in terms of a for 'any' adding number.

Let us call the 'adding number' k , say. We then get $f = 4(a+k)$, etc.

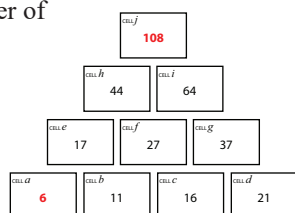
Or find a rule for a in terms of f .

The inverse function can be written as $a = f \div 4 - k$ or as $a = (f - 4k) \div 4$. "Are they equivalent?"

Or find the value of a for a specific value of f , say 100 [eg by trying to solve the equation $4(a+5)=100$].

The equation could be solved by trial and improvement, matching, cover up, balancing ...

Or increase the number of cells.



You might want to name the cells 'diagonally', like this:

CELL j
 CELL f CELL i
 CELL c CELL e CELL h
 CELL a CELL b CELL d CELL g

Here we don't have to re-name cells when we add more rows (which we do 'diagonally').

Algebra 2A

Lesson

4A

Background

Engagement

Our experience suggests that many students find the 'pyramid' task highly engaging. This might be because

- initially they are intrigued by the facility with which the teacher can determine the number in top cell
- they then enjoy spotting a rule for themselves, and find it surprising that a consistent rule should emerge after some fairly lengthy arithmetic
- there is satisfaction in being able to work successfully with lots of letters (as in the cell names).

Note too, that the task is engaging to students despite the fact that it is not set in a 'relevant', 'real life' context. What is important here is that the task makes sense to students and that the search for a rule is meaningful to them.

Symbolising

The task is designed to bring out the utility of using algebraic and generic symbols to represent a situation. However, it is important also to let students draw on the results of calculations and to express what they see in every day language (be it about the numbers in individual cells, or about the rules between numbers in different cells). This is important for two reasons: to engage all the students in the class and to help students make connections back and forth between the various forms of representation/symbolisation so that the use of symbols remains meaningful.

Manipulating symbols

The task is also designed to bring out the utility of *manipulating* symbols, be it algebraically or generically, as here:

if the number in cell a is 6, then the number in cell f is $6+6+6+6+5+5+5+5$ which is $4 \times 6 + 4 \times 5$ or $4 \times (6+5)$, etc;

if the number in cell a is a , then the number in cell f is $a+a+a+a+5+5+5+5$ which is $4 \times a + 4 \times 5$ or $4 \times (a+5)$, etc.

This is very powerful, since it allows us to summarise or 'simplify' a string of steps and to derive or verify rules. However, this should not be rushed. Rather, here too it is important to continue to draw on students' numerical work and everyday language to help them understand the syntax. Syntactical rules such as 'gathering like terms' appear easy but they're not!

Abandoning meaning

A powerful feature of algebra is that one can temporarily abandon the meaning of symbols (or at least the specific meaning they have in a particular task) while manipulating them. However, this is not our focus here. Quite the contrary. Abandoning meaning is only viable if one can return to the symbols' meaning and when one can use this or some more general meaning to check, at any time, that a particular manipulation is valid. Thus our aim here is to help students develop meaning for symbols and for how they can be manipulated.

The Floating balloon Mini-assessment

It is important that students should be able to make sense of abstract expressions and relations like $x - 30$ and $y = x - 30$. However, we have

deliberately set this activity in a context to help students give meaning to the symbols. Also the context should help students to see that the symbols stand for numbers and that these can vary (unless the balloon gets stuck to the ceiling...) and so the relation is general.

Students can have difficulty translating between words and algebraic symbols.

For example, some students might read $w = v - 30$ as 'v is 30 less than w'. It is helpful to give the symbols numerical values to counter this error.

In the case of the current lesson, we start with numerical values and so the error is less likely to occur. However, it is quite common in Lesson 5B, where students are prone to translate "Ed is 5 years older than Daisy" as $e + 5 =$

d or $e = d - 5$ (where e and d represent Ed's and Daisy's ages in years). Such an error is even more common for a sentence like "There are 7 days in a week". It is tempting to shorten this to $7d = w$, say, which is fine if d and w are intended to be abbreviations of *day* and *week* (ie the letters represent objects). However, it doesn't work if the letters are meant to represent *numbers* of days and weeks. Here, we know that when the number of weeks, w , is 4, say, then the number of days, d , is 28. Plainly, $7d = w$ is not true for these values.



Algebra 2B

Lesson

4B**Numbers and rules**

1. Think of a rule for mapping A.

Find lots of rules.

Are any of the rules 'the same'?

2. Find a rule that works for mapping A *and* mapping B.

Can you find different rules?

Can you find rules that *look* different but are really the same?

A. $4 \rightarrow 30$

B. $8 \rightarrow 50$

Summary

Here we look at a variety of rules for mappings in a purely numerical context. We consider whether rules are equivalent using both an empirical approach (For any input, do the rules always seem to give the same output?) and an algebraic approach (Can we transform one rule into another, using the laws of arithmetic?). We also try to develop a feel for general rules, by comparing them 'qualitatively', eg by considering which rule gives the smallest or largest output for an input of, say, 100, or 1, or 0.

The focus is on linear rules (ie of the form $n \rightarrow an + b$) but you could extend this to other kinds of rules.

Outline of the lesson1. Find rules for mapping A: $4 \rightarrow 30$.

- Write $4 \rightarrow 30$ on the board. Ask the class to come up with rules that map 4 onto 30.
- Write students' rules on the board as they come up.
Ensure you get one-step and two-step rules [eg $+26$ and $\times 2, +22$].
Ensure you get some equivalent rules [eg $\times 2, +22$ and $+11, \times 2$].
- Ask whether any of the rules are 'the same' (equivalent).
Discuss ways of determining this: encourage empirical and algebraic approaches.
For the latter, encourage the use of symbolisation
[eg $n \rightarrow 2n + 22$ for $\times 2, +22$ and $n \rightarrow 2(n+11)$ for $+11, \times 2$].

2. Compare rules qualitatively.

- Choose some of the rules from Step 1.
Ask "Which rule gives the largest output for an input of 100?", etc.
Try to compare the rules without calculating precise outputs.

3. Find a rule that fits $4 \rightarrow 30$ *and* $8 \rightarrow 50$.

- Ask students to think about the rules from Step 1.
Which also fit mapping B?
Students are likely to come up with $\times 5, +10$ and $+2, \times 5$
[ie $n \rightarrow 5n + 10$ and $n \rightarrow 5(n+2)$].
- Are they the same or different? How can we tell?

4. Graphing rules.

- Imagine a straight line through the points with Cartesian coordinates (4, 30) and (8, 50).
How many straight lines are there? What is its equation?

Algebra 2B

Lesson

4B**Overview****Mathematical ideas**

In many of the lessons we work with rules (or functions) that grow out of particular contexts (eg hire charges, tile patterns). Here we are working in a purely numerical context, so the rules are restricted only by the number of input/output pairs that they are required to fit.

In practice, it is likely that most students will only come up with *linear* rules and we suggest you focus mainly (or indeed only) on such rules (whether or not you make this explicit). An infinite number of linear rules will fit a single input/output pair, but only one such rule (and its equivalent forms) will fit two pairs. We examine this by representing the rules as Cartesian graphs.

Students' mathematical experiences

The students should discover ...

- that there are lots of rules (in fact an infinite number) that fit a particular input/output pair of numbers (like 4, 30)
- that there are lots of rules of the form $\times P, +Q$ (and $+Q, \times P$) that fit a particular input/output pair (in fact an infinite number of rules, if we don't restrict P and Q to positive whole numbers)
- that we can find rules that look different but are equivalent
- that there is only one rule of the form $\times P, +Q$ that fits two input/output pairs of numbers.

Key questions

Can you find another rule for this input/output pair of numbers?

Can you find another rule for these two input/output pairs?

Can you find another input/output pair for this rule?

What would the graph look like for this rule?

How can you be sure that two rules are really the same?

Assessment and feedback

You might want to assess students' arithmetic fluency by asking questions like:

My rule for $4 \rightarrow 30$ is $\times 12, -R$.

What is R?

My rule for $4 \rightarrow 30$ is $-2, \times R$.

What is R?

You could ask students to devise their own mappings for Step 3, such that finding the rule is easy or hard.

It could be interesting to know what sense students make (spontaneously and with prompts) of 'graphing rules' (using Cartesian axes). Thus you might take one of the (linear) rules from Step 1 of the lesson and ask,

"Could I graph this rule?"

"What would it look like?"

If students are bemused (eg they have no idea what might be meant by 'graphing a rule'), prompt them by asking for input/output pairs in addition to (4, 30) that fit the rule.

"If we think of the pairs as coordinates of points, where would they go?"

"If we plotted lots of points, what pattern would we get?"

"How steep?"

"Where does it cross the axes?"

Adapting the lesson

You might want to make students aware of non-linear rules.

You might want to devote more time to Step 4 (Graphing rules), for example by first thinking about (or sketching or plotting) the graphs of some of the rules for just $4 \rightarrow 30$ (whether or not they also fit $8 \rightarrow 50$).

Algebra 2B

Lesson

4B**Outline of the lesson (annotated)**1. Find rules for mapping A: $4 \rightarrow 30$.

- Write $4 \rightarrow 30$ on the board. Ask the class to come up with rules that map 4 onto 30.
- Write the rules on the board as they come up. Ensure you get one-step and two-step rules [eg $+26$ and $\times 2$, $+22$]. Ensure you get some equivalent rules [eg $\times 2$, $+22$ and $+11$, $\times 2$].
- Ask whether any of the rules are 'the same' (equivalent). Discuss ways of determining this: encourage empirical and algebraic approaches. For the latter, encourage the use of algebraic symbolisation [eg $n \rightarrow 2n + 22$ for $\times 2$, $+22$ and $n \rightarrow 2(n+11)$ for $+11$, $\times 2$].

—— You might want to encourage the use of subtraction and division [eg $\times 8$, -2 or $\div 2$, $+28$] and perhaps other operations like squaring [eg square, $+14$].

You might also want to encourage rules with more than two steps [eg $+8$, $\times 2$, $+6$]. "How could we simplify this?"

—— Make sure you devote sufficient time to translating the function machine notation into algebraic notation.

For example, $+11$, $\times 2$ can be written as $(n + 11) \times 2$
or $(n + 11)2$
or $2 \times (n + 11)$
or $2(n + 11)$
or $2(11 + n)$, etc.

2. Compare rules qualitatively.

- Choose some of the rules from Step 1. Ask "Which rule gives the largest output for an input of 100?", etc. Try to compare the rules without calculating precise outputs.

—— For example, consider the two possible rules $\times 3$, $+18$ and $\times 7$, $+2$ that fit mapping A. When the input is 'small', $+18$ will tend to dominate; when the input is 'large', $\times 7$ will tend to dominate.

3. Find a rule that fits $4 \rightarrow 30$ and $8 \rightarrow 50$.

- Ask students to think about the rules from Step 1. Which also fit mapping B? Students are likely to come up with $\times 5$, $+10$ and $+2$, $\times 5$ [ie $n \rightarrow 5n + 10$ and $n \rightarrow 5(n+2)$].
- Are they the same or different? How can you tell?

—— Students are likely to think only of linear rules (though probably without being aware of this). They might well be surprised that there is only one "times and add" rule (ie linear rule) that fits both mappings. The rule is $\times 5$, $+10$ [ie $n \rightarrow 5n + 10$], along with equivalent versions such as $+2$, $\times 5$ [ie $n \rightarrow 5(n+2)$].

There is an infinite number of non-linear rules that fit, eg $n \rightarrow n^2 - 2n + 22$. You might want to consider such rules with some students.

4. Graphing rules.

- Imagine a straight line through the points with Cartesian coordinates (4, 30) and (8, 50). How many straight lines are there? What is its equation?

Algebra 2B

Lesson

4B**Background****Equivalent rules**

This lesson provides plenty of opportunity to consider equivalent rules and expressions. Perhaps the most central are those involving the associative law. Thus, for example, $n \rightarrow 5n + 10$ is equivalent to $n \rightarrow 5(n+2)$.

However, there are other equivalences that are worth airing, some involving the associative and commutative laws, some involving conventions in notation.

Thus, for $n \rightarrow 5n + 10$, we could also have:

$$n \rightarrow 10 + 5n$$

$$n \rightarrow 10 + 5 \times n$$

$$n \rightarrow 10 + n \times 5$$

$$n \rightarrow 10 + n5, \text{ etc.};$$

and variants like these:

$$n \rightarrow 5n + 8 + 2$$

$$n \rightarrow 2 + 5n + 8, \text{ etc.}$$

Graphing rules

It can be useful for students to devote time to *imagining* the graphs of rules (before or instead of sketching them or getting distracted by the technicalities of drawing them). For this, you could go back to one of the (linear) rules from Step 1 of the lesson.

For the rule for the two mappings (or two input/output pairs) in Step 3, students may be surprised that there is only one (linear) rule that fits both mappings.

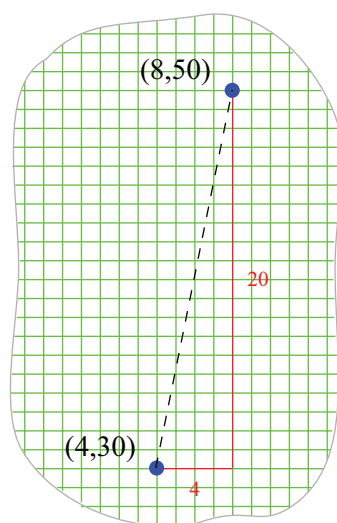
A linear rule is equivalent to a “times and add” rule. Here, as the input changes, so the change in output depends only on the multiplier.

For our two mappings, as the input changes by 4 (ie from 4 to 8), the output changes by 5 times as much (from 30 to 50). Thus (!) we need a $\times 5$ rule.

If we represent the two mappings by points on a Cartesian graph, then we can get from one point to the other by repeatedly going ‘1 across, 5 up’. The total amount up is 5 times the total amount across.

There is (of course!) only one straight line through the two Cartesian points (though an infinite number of curves). We only need one rule to describe the straight line.

Some students might find it instructive to plot points for a non-linear rule that fits the two input/output pairs, such as $y = x^2 - 2x + 22$ [for $x = 0$ to 9, say].

**Symbol sense**

This lesson also provides plenty of opportunity for students further to develop their symbol sense, particularly in Steps 2 and 4. What effect do the various terms of a rule have on its value, for small inputs, large inputs, and as the input changes?

Consider the rule $n \rightarrow 5n + 10$, say. This can be thought about in numerical terms, and qualitatively (eg, “As the input increases, the output changes a lot, but is fairly small for small inputs”) or quantitatively (eg, “As the input increases by 1, the output goes up by 5; when the input is zero, the output is 10”). It can also be thought of graphically (eg, “The graph is quite steep and cuts the y axis quite close to the origin”, or “The graph has a gradient of 5 and cuts the y axis at 10”).

Symbol sense is likely to develop best when students have the opportunity to *compare* expressions or rules, and you might want to select rules which are similar in some respects and different in others. For example, they could have the same multiplier but different constant term, or same constant term but one rule ‘goes up’ and one ‘goes down’; or similar constant term, and both go up (for positive inputs) but one does so at a changing rate, as here: $n \rightarrow 10n + 8$ and $n \rightarrow n^2 + 7$.

Lesson
4B

Notes

Algebra 3AB

Mini-assessment

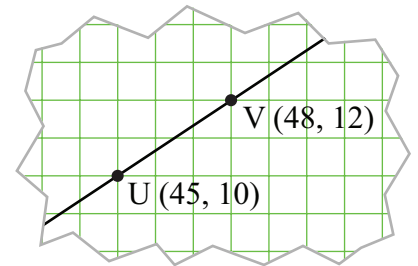
5AB**Another point**

Point U has coordinates (45, 10).

Point V has coordinates (48, 12).

Think of another point on the line through U and V.

What are its coordinates?

**Commentary**

The aims of this Mini-assessment are to get a sense of

- how readily students see a line as a set of points
- how readily they determine other points on the line (and what methods they use).

Can students find other points?

What sort of points do they find (eg 'distant' points, or points with fractional coordinates)?

What methods do they use (eg a 'step by step' approach, or a general rule)?

Algebra 3A

Lesson

5A**Points on a line**

The diagram shows a straight line drawn through points P and Q.

P has coordinates (10, 100) and

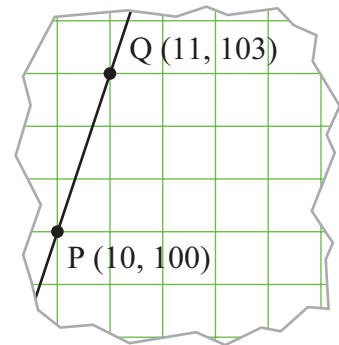
Q has coordinates (11, 103).

Another point A also lies on the line.

Its x -coordinate is 5.

Find the value of its y -coordinate.

Find different ways of getting the answer.

**Summary**

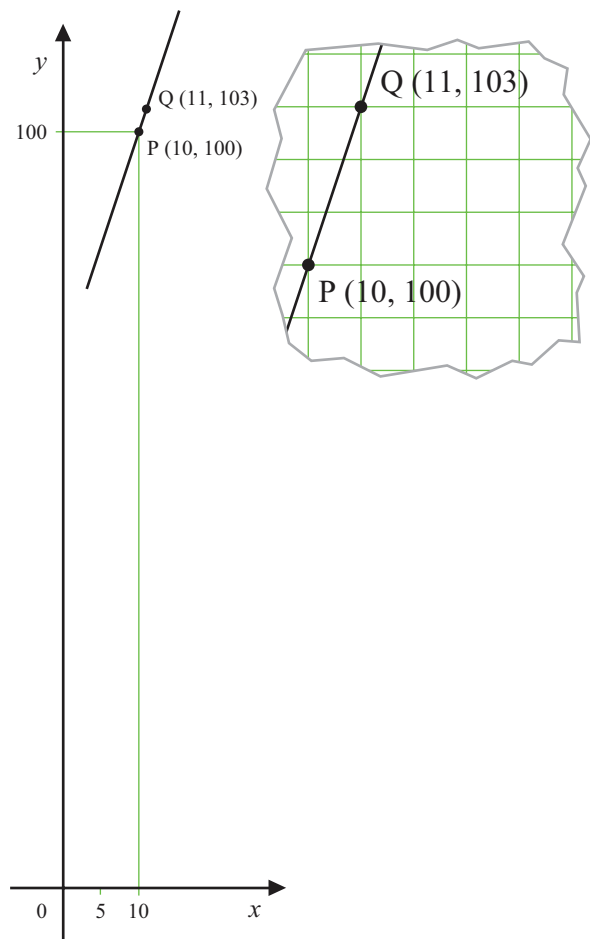
In this lesson, we start with (part of) a Cartesian graph, and consider various ways in which points on the graph can be found: geometrically, with a table, and algebraically.

For the algebraic approach we are looking for a *functional* relationship, ie a rule that maps x onto y (rather than a *scalar* rule that maps one value of x onto another value of x , and does the same for corresponding values of y). To focus students on this, we introduce a context at this stage, involving a cookery course.

Outline of the lesson

1. Present the task to the class. If need be, show (or describe) this more detailed diagram (right).
 - Ask for some immediate responses.
 - Explain that we are looking for a variety of methods and ask students to work on the task in small groups.
2. Ask students for their answers and explanations.
 - Listen to a variety of responses but focus on the non-algebraic ones initially.
 - Shift attention to finding a functional relationship: "If we know the x -coordinate, is there a rule for working out the y -coordinate?"
3. Introduce the Cookery Course context.
 - "I have joined an evening class to learn Spanish cooking. The graph we've been looking at is about the cost. Thus 10 lessons would cost me £100, and 11 lessons would cost me £103, etc. If I tell you how many lessons I've attended, what is the rule for working out the cost?"
 - Discuss ways of representing the rule.
4. Possible extension: modify the original graph.

For example, imagine moving Q up 1 unit, to (11, 104).
What happens to A?
What happens to our rule?



Algebra 3A

Lesson

5A**Overview****Mathematical ideas**

Points on a Cartesian graph represent pairs of numbers. Thus, the points can be used to model a numerical context involving two variables, eg the cost of attending various numbers of lessons on a cookery course.

For points on a straight line, it is possible to find fairly simple rules for the coordinates. For example, if one point is '3 units down and 1 unit to the left' from another point, then a further move of '3 down, 1 to the left' will locate another point on the line. In particular, it is possible to find a *functional relationship* between the x and y coordinates.

Students' mathematical experiences

Students may discover some of the following

- when we know the coordinates of two points on a straight line, there are ways of finding the coordinates of other points on the line
- one can hop along a line in regular steps
- it is possible to find a function rule for all points on a line, that maps the x -coordinate of any point onto its y -coordinate
- coordinates needn't just describe points in space; they can also refer to variables like 'number of lessons' and 'total cost of lessons'

Key questions

We can get from Q to P by going "3 down, 1 to the left" lots of times. Could we use fewer steps?

How can we use the coordinates of points P and Q to find the y -coordinate of point A?

If we know the x -coordinate (of a point on the line), is there a rule for working out the y -coordinate?

If I tell you how many cookery lessons I've attended, what is the rule for working out the cost?

Assessment and feedback

When introducing the task, check whether students can visualise the axes of the graph and the approximate position of point A.

Check that students remember the conventions for Cartesian coordinates (eg the first coordinate measures 'across' and is called the x -coordinate).

During the initial group work, try to pick up on the range of methods students are using, so that you can orchestrate the order in which students report on their methods and so you can give initial support if students are really stuck.

Look out for students who adopt a purely 'photographic' approach, and for those who adopt a too narrow numerical approach (eg, P is (10, 100) so the rule is $y = 10x$).

Are students aware that a functional rule for the coordinates of points on the line, is one that works for every point on the line?

Adapting the lesson

If lots of students are stuck, the class might need some prompts. One way of doing this would be to ask someone who has a fruitful method, to reveal their starting-off point to the class (eg, "I worked out that a point with coordinates (9, 97) is on the line...").

The Cookery Course context should make the task more accessible to some students, so you need to judge when to introduce it. However, don't introduce it immediately - see what sense students make of the pure setting first.

Algebra 3A

Lesson

5A**Outline of the lesson (annotated)**

1. Present the task to the class. If need be, show (or describe) the more detailed diagram.

- Ask for some immediate responses.
- Explain that we are looking for a variety of methods and ask students to work on the task in small groups.

— On its own, the small ‘patch’ of graph is fairly abstract. It might help students to show the approximate location of the axes and origin, and perhaps also of the point A.

— Students’ immediate responses may indicate whether they have grasped the basic setting of the task.

2. Ask students for their answers and explanations.

- Listen to a variety of responses but focus on the non-algebraic ones initially.

— One, geometric, approach is to ‘step down’ the graph, using the observation that Q to P is ‘3 down, 1 to the left’. So the next point down has coordinates (9, 97), then (8, 94), and so on till (5, 85).

Some students might list these coordinate pairs in a table. And some might curtail this by arguing ‘Point A is 5 to the left of P, so it is 5×3 down’.

If you have sketched the position of A, some students might try to read its coordinates from the sketch, ie take the sketch too literally.

- Shift attention to finding a functional relationship: “If we know the x -coordinate, is there a rule for working out the y -coordinate?”

— Some students might have difficulty distinguishing perfectly good rules like ‘3 down for every 1 to the left’ from the kind of rule we are asking for here, namely a function rule.

Some students might not appreciate that the rule has to work for every coordinate pair; for example, given (10, 100), they might go for a $\times 10$ rule, which of course does not work for Q and which leads to an answer of (5, 50) for the coordinates of A.

3. Introduce the Cookery Course context.

- “I have joined an evening class to learn Spanish cooking. The graph we’ve been looking at is about the cost. Thus 10 lessons would cost me £100, and 11 lessons would cost me £103, etc. If I tell you how many lessons I’ve attended, what is the rule for working out the cost?”
- Discuss ways of representing the rule.

— The cookery context is introduced to focus students’ attention on the function rule.

If students are stuck you might want to give them a clue:
I had to pay a fee to join the course and then pay for each lesson I attended.

— The rule can be expressed in a variety of ways, eg as a function machine ($\times 3, +70$), as a mapping ($x \rightarrow 3x+70$), as a function ($f(x) = 3x + 70$), or as an equation ($y = 3x + 70$).

4. Possible extension: modify the original graph.

For example, imagine moving Q up 1 unit, to (11, 104).
What happens to A?
What happens to our rule?

— Changing the situation in this kind of way might help students to generalise what’s going on.

You might also want to ask about where the original line crosses the y axis, or address issues of continuity: “What happens if I stay for only half a lesson?”

Algebra 3A

Lesson

5A**Background****The nature and purpose of Cartesian graphs**

In contrast to earlier lessons, this one *starts* with a Cartesian graph (or at least a small ‘patch’ of one). One strength of a graph is, of course, its visual quality, which can make it seem very tangible (as well as allowing us to represent lots of information simultaneously).

However, this very property can lead us to forget that we use graphs to represent relations that are not necessarily visual at all, as with the context introduced during this lesson, of a cookery course (with the relation between ‘number of lessons attended’ and the ‘total cost’).

For many students, graphs don’t seem to be about anything: not only do they often lack a context, but students often don’t realise that a graph (in this case a simple straight line) is really a collection of points (and often an infinite set of points), and that there is a relationship between the two coordinates of all these points, and so the line is a representation of this relationship.

In this lesson we start with such a ‘pure’ graph, which may induce this sense of being adrift. However, in the course of the lesson students are reminded

- that the given line represents more than just the two points used to define it
- that the coordinates of points on the line obey rules
- that the points can be used to represent a relationship between variables from a ‘real’ context such as the Cookery Course (see comment, right).

Rules, functional relationships and notation

In this lesson students get the opportunity to encounter a variety of rules to describe the coordinates of points on our particular straight line: informal, grounded rules like ‘if you go 1 to the right you have to go up 3 to stay on the line’, or more general ‘scalar’ rules like this: ‘if you go e units to the right, you have to go up $3e$ units to stay on the line’. The later emphasis in the lesson is on finding a *functional* rule, ie a rule that maps the x -coordinate onto the y -coordinate, for any point on our given line. Here there is an opportunity to consider different notation for the rule, eg

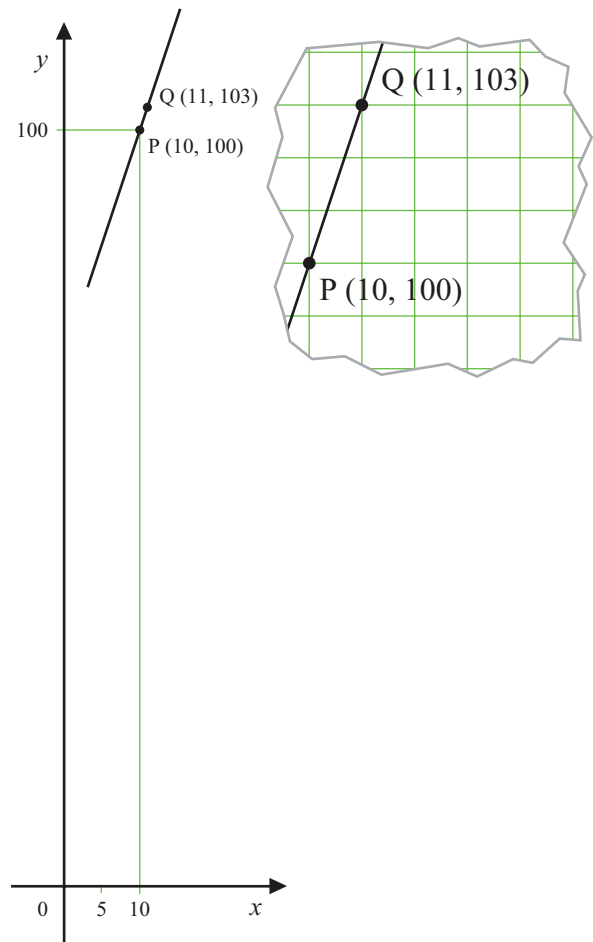
function machine notation: $\times 3, +70$

mapping notation: $x \rightarrow 3x + 70$

function notation: $f(x) = 3x + 70$

or equation notation: $y = 3x + 70$.

Of course, the function in our example is commonly called a *linear* function, as it describes the coordinates of points on a straight line. Such a relation is of the form $y = ax + b$ (or $y = mx + c$), and, as we know, the value of a gives the gradient of the straight line and the value of b gives the y -intercept. However, you may not want to pursue these aspects of the functional relationship here.



Consider what the features of this graph represent in the context of the Cookery Course. In particular, consider the meaning conveyed by points P and Q, by the slope of the line, and by the y -intercept.

Note on linear and affine functions

Functions of the form $f(x) = ax + b$, are called *affine* functions by some writers. They reserve the term *linear* function for functions of the form $f(x) = ax$. The latter have these two important properties, which are key features of proportion relationships: $f(kx) = kf(x)$ and $f(x + y) = f(x) + f(y)$.

Thus for example, if I travel at a constant speed and it takes me 4 seconds to cover 30m, then in 5×4 seconds I cover 5×30 m, and in 4 seconds + 20 seconds I cover $30\text{m} + 150\text{m}$.

Note that this does not work in the Cookery Course context:

If 10 lessons cost me £100,
20 lessons don’t cost £200;
and $10+11$ lessons don’t cost £100 + £103.

Algebra 3B

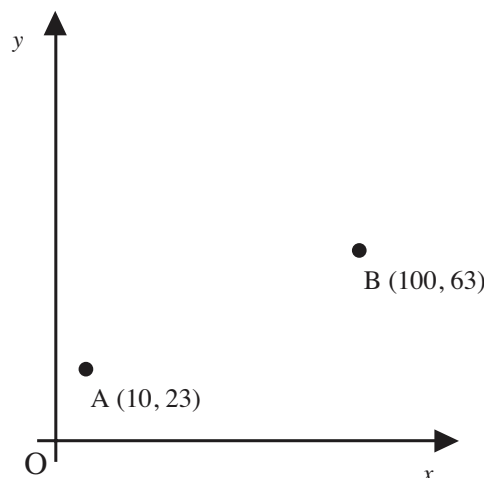
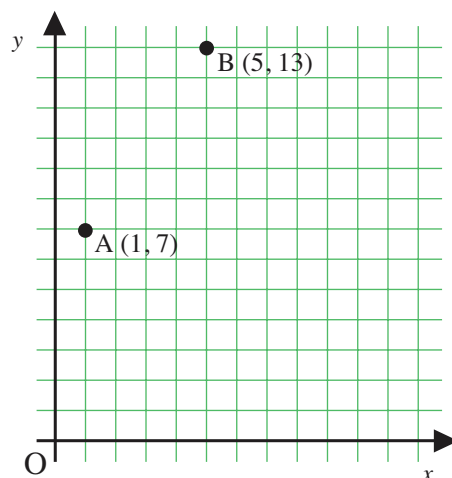
Lesson

5B**Midpoints**

Choose one of these two diagrams.

P is the midpoint of the straight line AB.

Find the coordinates of P.

**Summary**

In this lesson, we consider points on a Cartesian graph, and use the given coordinates of points to find the coordinates of other geometrically defined points, such as midpoints or vertices of a square.

We look for general rules for the coordinates of such points, and for functional relationships between the x and y coordinates of sets of points.

Outline of the lesson

1. Present the task. Ask the class which diagram they want to use.
 - Ask students to work on the task in small groups.
Ask students to explain how they got their answer.
2. Make up a related task, for example:
 - Find the midpoint of AP.
 - B is the midpoint of AC. Find the coordinates of C.
 - AB is the side of a square. Find the other two vertices.
 - AB is the diagonal of a square. Find the other two vertices.
 - Imagine moving B 10 units to the right. What happens to P?
 - Ask students to solve the task and explain their answers.
 - Ask students to make up related tasks.
Choose one for the class to solve.
3. Find a rule.
 - Choose a midpoint task that the class has tackled.
Ask students to come up with a general rule for finding the coordinates of the midpoint, given the coordinates of the endpoints.
 - Look for an alternative (version of the) rule.
What if A has coordinates (e, f) and B has coordinates (m, n) ?
How can we express the rule?
How can we show that the alternative versions are equivalent?
 - Find a function rule $x \rightarrow y$ that fits the three points.

Algebra 3B

Lesson

5B**Overview****Mathematical ideas**

Here we use the geometric properties afforded by a Cartesian graph (ie properties of 2D Euclidean geometry) to find the coordinates of points, in particular for the point midway between two given points. One method is to consider the mean of the given x coordinates (and the given y coordinates); another

to use a 'vector' approach, ie to consider the move necessary to $\frac{m+e}{2}$ $e + \frac{m-e}{2}$ $m - \frac{m-e}{2}$ get from one given point to halfway to the other given point. Some may use a 'trial and improvement' approach to work their way to the midpoint. The work can easily be extended to other geometric situations, such as vertices of a square.

The lesson provides another incentive for constructing and manipulating algebraic expressions, for example to show that expressions like these are equivalent (for x coordinates e and m):

Students' mathematical experiences

Students may discover some aspects of the following

- geometric properties can be used to *visualise* the position of a point, but also to *calculate* the precise values of its coordinates
- it is possible to make problems easier or harder
- there are different (but equivalent) calculation rules for finding midpoints (and other points)
- the rules can be described in words, with numbers acting as quasi-variables, or with letters standing for numbers
- algebraic expressions can be manipulated to show whether they are equivalent
- for some families of points (eg points on a straight line) we can find a rule connecting each point's x and y coordinates.

Assessment and feedback

It is important that students express a range of rules and methods. Observe the students working in groups to identify these different approaches.

You can increase the proportion of students answering questions by using 'wait-time' after asking a question. Wait for 3 to 5 seconds before choosing a student to respond.

Students can give feedback to each other. For example, you might suggest that a student tries to improve another student's explanation, or asks another student directly to repeat their response: "Billy, you could ask Sian to explain their idea again".

Key questions

How could we make this problem easier/harder?

Is there a rule that works for any numbers (coordinates)?

How can we express the rule?

How can we change 'this' open expression into 'that' open expression?

Adapting the lesson

There is plenty of scope for you (and the students) to adapt the lesson by modifying the tasks used, and by inventing related tasks. There is also scope to decide how to describe the rules students come up with, for example whether to use verbal descriptions, or quasi-variables or create full-blown algebraic expressions.

In discussing how to find the coordinates of midpoints, it might help to discuss ways of finding the mean of two numbers. In the case of a pair of numbers like 5 and 12, or 30 and 100, one might simply sum the numbers and divide by two. However, for the pair 237 and 243, it might be simpler to halve the difference and add this to 237 (or subtract it from 243). Line diagrams (representing heights, say) might help here too.

Algebra 3B

Lesson

5B**Outline of the lesson (annotated)**

1. Present the task. Ask the class which diagram they want to use.
 - Ask students to work on the task in small groups. Ask students to explain how they got their answer.

— You might want to make this decision yourself, but it can be useful for students to evaluate a task ‘from a distance’ rather than get stuck in and not see the wood for the trees...

— We are interested in methods as much as answers, but we will look at methods more closely in Part 3 when we try to find general rules.

2. Make up a related task, for example:
 - Find the midpoint of AP.
 - B is the midpoint of AC. Find the coordinates of C.
 - AB is the side of a square. Find the other two vertices.
 - AB is the diagonal of a square. Find the other two vertices.
 - Imagine moving B 10 units to the right. What happens to P?
 - Ask students to solve the task and explain their answers.
 - Ask students to make up related tasks. Choose one for the class to solve.

— The first version of the task (first diagram) can be solved visually rather than ‘analytically’. If the class used the first version, try to move them towards a task like the second one.

— You might challenge students to come up with an ‘easy’ task and a ‘difficult’ task (now, or for homework). It requires some insight to come up with effective ones.

3. Find a rule.
 - Choose a midpoint task that the class has tackled. Ask students to come up with a general rule for finding the coordinates of the midpoint, given the coordinates of the endpoints.
 - Look for an alternative (version of the) rule. What if A has coordinates (e, f) and B has coordinates (m, n) ? How can we express the rule?

Here we need a rule for the x coordinate and a separate (though similar) rule for the y coordinate.

The rule is rather different from our usual rules in that it involves *two* inputs (or ‘independent variables’) rather than one. We might want to call the rule a *formula*.

Students might express the rule verbally, or in terms of specific numbers (quasi-variables), or in terms of general coordinates.

— Consider the rule for the x coordinate of the midpoint. (The y coordinate can be found in a similar way.) One approach is to find the *mean* of the x coordinates of A and B: $x_p = (m + e)/2$. Another approach is to consider the movement (in the direction of the x axis) needed to get from A to B. We need to move half that amount (in that direction) to get to P: $x_p = e + (m - e)/2$.

— Here, it pays off to express the rules algebraically - although students might find it challenging to transform one expression into the other.

— Here we are back to a rule with one input (or independent variable), x , and instead of dealing just with x coordinates, or just with y coordinates, the rule maps x onto y . What other points fit the rule, in addition to A, B and P?

How can we show that the alternative versions are equivalent?

- Find a function rule $x \rightarrow y$ that fits the three points.

Algebra 3B

Lesson

5B**Background****Finding general rules**

Because of the geometric nature of Cartesian graphs, it is possible to solve some midpoint tasks in a *purely visual* way (for example, if students are provided with a graph showing a grid and if the given endpoints are not too far apart - and preferably an even number of units apart in both the x and y directions...).

In this lesson we want to make use of the visual and of students' geometric knowledge about midpoints, but move to an analytic approach whereby students find ways of *calculating* the midpoint's coordinates. To this end, it is worth modifying the tasks used (eg by not having a grid and by choosing points further apart) so that a purely visual approach is no longer adequate (though still providing useful support to the students' thinking - eg in terms of assessing whether a result looks reasonable).

Some students might want to start with a purely visual approach to generate a set of answers, and then adopt an empirical approach by looking for a rule that fits their numbers. This is a perfectly good fallback strategy, but initially at least it is another reason for moving away from versions of the midpoint task that can be solved in a purely visual way. We are, as usual, trying to find rules-with-reasons, ie rules based on properties, on structure.

Formulae and functions

In this lesson we are considering two kinds of rules. To find the x coordinate (and similarly the y coordinate) of the midpoint between two given points, we need a rule with *two* inputs, namely the x coordinates of the two given points. On the other hand, the rule that fits the two coordinates of each of our three points has a *single* input, the x coordinate (which is mapped on to the y coordinate by the rule). Though both rules are functional relationships, we commonly use the term 'formula' for the first kind of rule.

Algebraic expressions and brackets

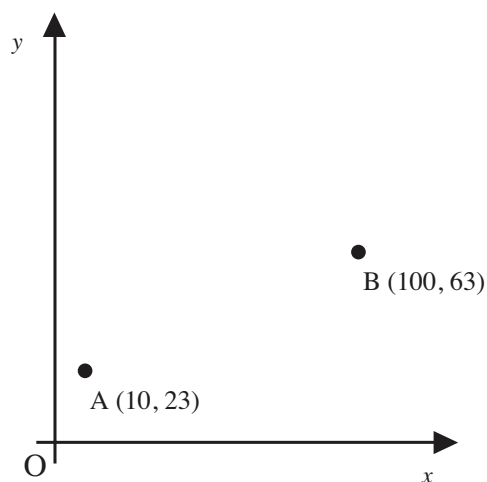
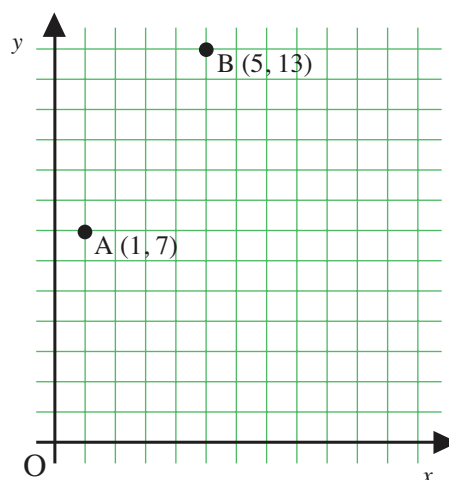
The lesson again provides an opportunity to create expressions (using quasi-variables or letters) that involve brackets. For example, using the second graph on page 1, where the given points are A (10, 23) and B (100, 63), the x coordinate of the midpoint can be expressed as $e + \frac{m - e}{2}$. Or, if the given coordinates are (e, f) and (m, n) , the x coordinate of the midpoint can be expressed as $e + (m - e)/2$.

Some students might represent the brackets with a fraction bar (right), and it is worth discussing this (more elegant?) alternative with the class.

Choosing appropriate coordinates for points A and B

This lesson is about finding the midpoint of a line segment AB. As mentioned in the preceding sections, it is worth thinking carefully about what coordinates to select for the endpoints A and B. We provide two configurations to choose from: A(1, 7) and B(5, 13) on a grid, or A(10, 23) and B(100, 63) on a plain background. Consider the first configuration: here the closeness of the points and the presence of the grid provide students with strong visual support for finding the midpoint. Further, the use of numbers that are all odd means that the midpoint has whole-number coordinates. You might want to vary some of these features, to provide more or less numerical and/or visual support, depending on the understandings of individual students or the class as a whole.

Note that there can be a tension here, between providing support that allows students to solve the task for a specific configuration, and allowing sufficient 'distance' for students to be able to see what is going on in general, ie to gain insight into the structure of the task.



Lesson

5B

Notes

Multiplicative Reasoning

Mini-test

Mini ratio test**(Versions A and B)**

Ask students to work through version A or B of this mini test, before starting on Lessons 2A and 2B. Distribute Versions A and B randomly to students in the class.

This can be done during another lesson, or for homework. **It should about 10 minutes.**

Name Class School Date
 Please show all your working/jottings

1. Ant is making a spicy soup for 11 people.
 He uses 25 ml of tabasco sauce.
 Bea is making the same soup for 33 people.
 How much tabasco sauce should she use?

2. These two Ls are exactly the same shape.
 How long is the grey curved line?

3. 6 Australian Dollars are worth 27 Argentine Pesos.
 What are 20 Australian Dollars worth?

MiniRatio A
 © ICCAMS 2002

Commentary

The aim of this mini test is to get a sense of the range of methods that students use to solve ratio tasks, and to see whether choice of method (and task difficulty) is influenced by context and the numbers in the task.

The test involves three very different contexts - a soup recipe, a geometric enlargement, and currency conversion.

The first two tasks can be solved by using a simple multiplier (of $\times 3$ and $\times 4$ respectively), whereas the third task involves the more complex multipliers $\times 3\frac{1}{3}$ or $\times 4\frac{1}{2}$. The scalar and functional relationships between the numbers are shown on the ratio tables on the next page.

Go through the students' responses. Make a brief record of the methods used and whether they were successful.

Look out for these methods (as applied to the Soup task on Version A):

- scaling (3 times as many people will need 3 times as much sauce)
- rated addition (11 people need 25 ml, so 22 need 50, so 33 need 75)
- the unitary method (11 people need 25 ml, so 1 person needs $25/11$ ml, so 33 people need 75 ml)
- constructing and solving an equation, eg $11/33 = 25/x$
- the rule of 3 (desired amount = $33 \times 25 \div 11$)
- an inappropriate addition strategy (22 extra people need 22 extra ml, making 47 ml in all).

Mini Ratio Test

Version A

Soup:

scalar relation: $\times 3$

functional rel: $\times 23/11$

Curved L:

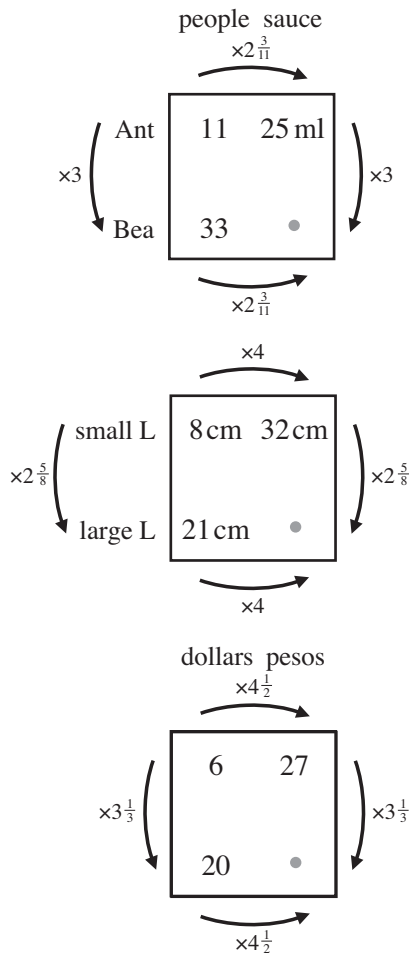
scalar relation: $\times 25/8$

functional rel: $\times 4$

Dollars:

scalar relation: $\times 31/3$

functional rel: $\times 41/2$



Ant is making a spicy soup for 11 people.

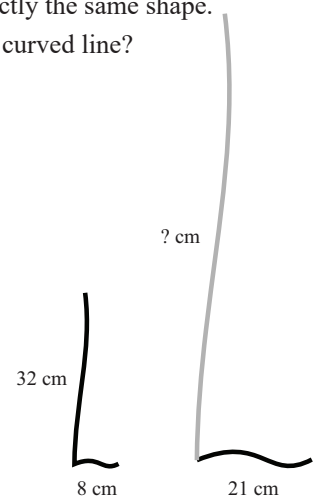
He uses 25 ml of tabasco sauce.

Bea is making the same soup for 33 people.

How much tabasco sauce should she use?

These two Ls are exactly the same shape.

How long is the grey curved line?



6 Australian Dollars are worth 27 Argentine Pesos.

What are 20 Australian Dollars worth?

Mini Ratio Test

Version B

Soup:

scalar relation: $\times 23/11$

functional rel: $\times 3$

Curved L:

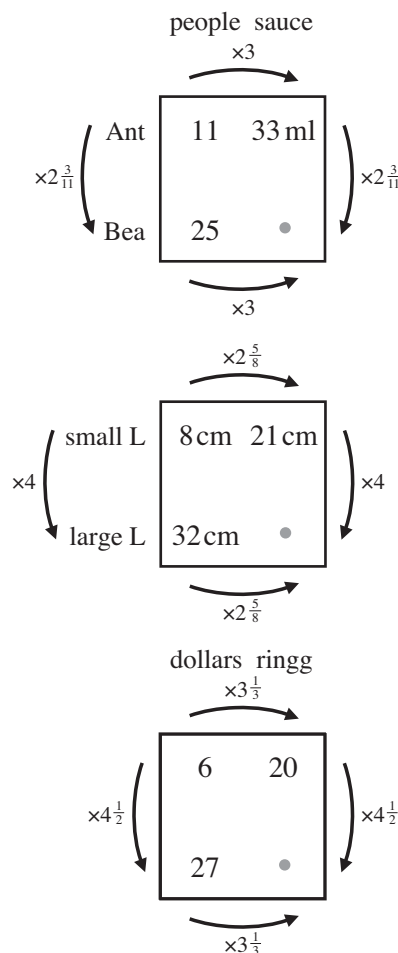
scalar relation: $\times 4$

functional rel: $\times 25/8$

Dollars:

scalar relation: $\times 41/2$

functional rel: $\times 31/3$



Ant is making a spicy soup for 11 people.

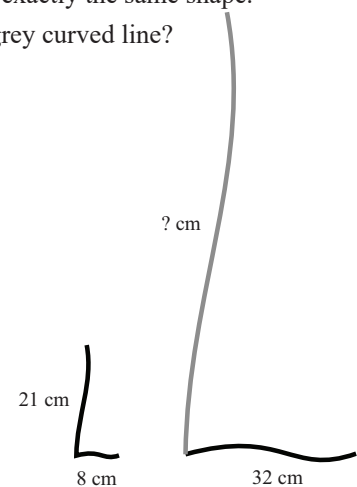
He uses 33 ml of tabasco sauce.

Bea is making the same soup for 25 people.

How much tabasco sauce should she use?

These two Ls are exactly the same shape.

How long is the grey curved line?



6 Australian Dollars are worth 20 Malaysian Ringgits.

What are 27 Australian Dollars worth?

Multiplicative Reasoning: Mini ratio test

Background

Mini Ratio Test data

We distributed items from the Mini Ratio Test, together with other such items, randomly to students within 29 classes, mostly Year 8, across a total of 14 schools. Each item was given to over 70 students.

The table below, left, shows the percentage of students (in these parallel samples of about 75 students) who answered the item correctly (or with just minor computation errors).

The table below, right, shows the scalar multiplier that could be used to solve the item.

Item	Version A	Version B	Item	Version A	Version B
Soup	91	51	Soup	$\times 3$	$\times 2\frac{3}{11}$
Curly L	36	75	Curly L	$\times 2\frac{5}{8}$	$\times 4$
Dollars	22	27	Dollars	$\times 3\frac{1}{3}$	$\times 4\frac{1}{2}$
Roughly correct responses (%)			Scalar multiplier for each item		

As can be seen, the currency conversion items (Dollars) were the hardest, presumably in large part because the numerical relations are more difficult. On the other hand, the geometric enlargement item (Curly L) was on average harder than the recipe item (Soup), even though the numerical relations are comparable. This suggests that students find it harder to see that the relations are multiplicative in a geometric context than in a recipe context. This is supported by the fact that a sizeable minority of students used the addition strategy (leading to the answer 45) on the Curly L item (22% and 8% respectively for Versions A and B), but very few did so on the Soup item (1% and 3% respectively for Versions A and B). We discuss this context issue further below.

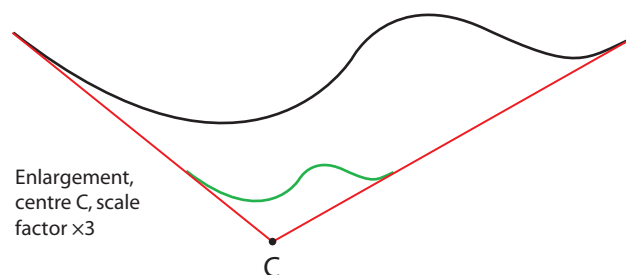
It is also interesting to note that the version of Soup and Curly L with the simpler scalar multiplier turned out to be substantially easier than the other version (ie with the more complex scalar multiplier but simpler functional multiplier). This suggests that the numerical relations that students construct or focus on is influenced by the context - they don't simply home-in on the simpler relation - and that they seem to prefer scalar to functional relations, eg relations that map people onto people and sauce onto sauce, rather than, say, people onto sauce.

Models of multiplication: scaling

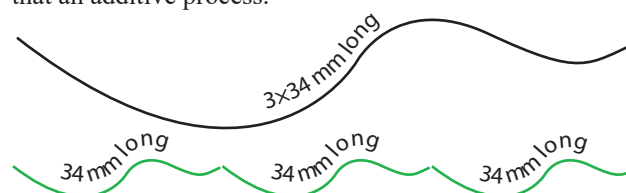
In the 2A and 2B lessons that follow this Starter we look at multiplication in terms of scaling. One of the first models of multiplication that students learn is repeated addition. This provides a secure basis for multiplying numbers when they are whole (it is difficult to repeat something 2 and a quarter times, say) and when they are small (so that there are not

too many additions to repeat). However, it also depends on the context. Repeated addition fits a story like "Every day I eat 3 packs of crisps. How many packs do I eat in 4 days?". But it does not fit "Crisps come in 3 pack sizes and 4 flavours. How many different types of pack are there?".

One context where scaling arises is geometric enlargement, as in the example below, where the small (green) S shape has been enlarged by a scale factor $\times 3$, with centre of enlargement C.



If the small S shape has a curved length of 34 mm, then the enlarged S shape will have a curved length of 3×34 mm. However, it is not helpful to attempt to 'add' three versions of the small S shape together to make the enlarged S shape (below): taken together, the three small S shapes look very different from the enlarged S shape, and it is not at all obvious from the diagram that their combined length is the same as the length of the enlarged S shape. (This kind of demonstration only works when the lines are straight, ie lie in one dimension.) Thus it is more appropriate to think of the larger S shape as resulting from a scaling process rather than an additive process.



It is worth noting that scaling does not apply only to geometric situations. We can scale other quantities, eg wages in China to wages in Japan, and conversions, like (number of) gallons to (number of) litres, pounds to Euros.

In lessons 2A and 2B we stay with a geometric context by considering the number of metres and feet represented by lines on a map. To keep things simple we restrict ourselves to straight lines, which opens up the possibility again of using additive strategies. However, the straight map-lines serve as a 'natural' introduction to the 'double number line' (DNL) which is an extremely useful (but also challenging) device for representing scaling, and which we consider further in later lessons, including for non-geometric contexts. Our prime interest in the DNL, here and in later lessons, is as a model *of* multiplication, rather than *for* multiplication - ie as a device to help us think about the nature of multiplication, rather than as a device for finding numerical answers.

miniRtest Name Class Date

Version A NO CALCULATORS Please show all your working/jottings

Ant is making a spicy soup for 11 people.
He uses 25 ml of tabasco sauce.

Bea is making the same soup for 33 people.
How much tabasco sauce should she use?

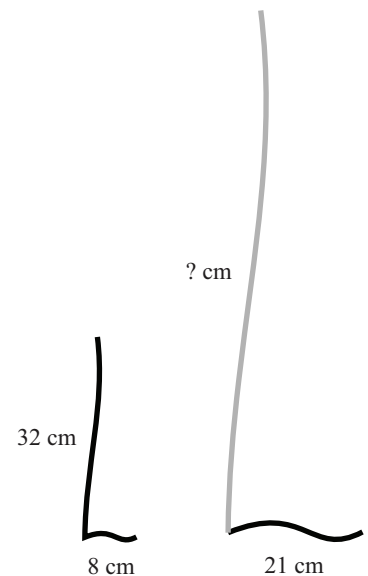
.....



These two Ls are exactly the same shape.

How long is the grey curved line?

.....



6 Australian Dollars are worth 27 Argentine Pesos.

What are 20 Australian Dollars worth?

.....



miniRtest

Name Class Date

Version B

NO CALCULATORS Please show all your working/jottings

Ant is making a spicy soup for 11 people.
He uses 33 ml of tabasco sauce.

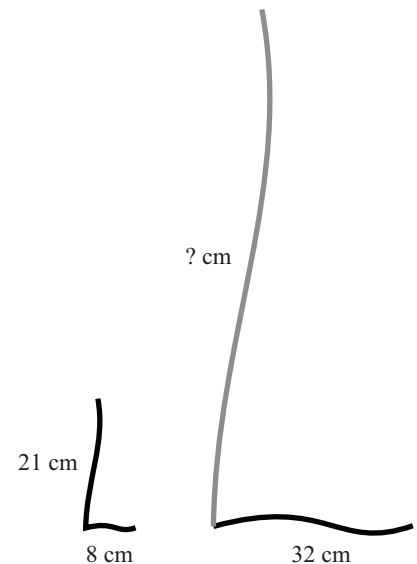
Bea is making the same soup for 25 people.
How much tabasco sauce should she use?

.....



These two Ls are exactly the same shape.
How long is the grey curved line?

.....



6 Australian Dollars are worth 20 Malaysian Ringgits.

What are 27 Australian Dollars worth?

.....



Mini-test

Notes

Multiplicative Reasoning 3AB

Mini-assessment

6AB**Steady walk**

Jay and Kim are walking across a field.

They both walk at a steady speed,
but Jay is slower than Kim.

Look at this information. →

Find the missing distance.

When Jay has walked **10** m, Kim has walked **12** m.

When Jay has walked **25** m, Kim has walked

Commentary

Data from the *Mini Ratio Test* suggests that students are likely to answer this task in a variety of ways.

Some might say that Kim has walked 27 m, using an additive argument of the sort, “Kim is always 2 m ahead of Jay”, or “Jay has walked another 15 m, so Kim will have walked another 15 m, making 27 m in all”.

Some might give the correct distance, 30 m, on the basis that Kim walks 1.2 times as far as Jay, or because Jay’s distance has increased by a factor $\times 2.5$, or by using some form of rated addition ($10 + 10 + \text{half of } 10 = 25$, $12 + 12 + \text{half of } 12 = 30$), or scaling (“When Jay has walked 5 m, Kim has walked 6 m, $5 \times 5 = 25$, $6 \times 5 = 30$ ”).

If the additive response, 27 m, is widespread you might want to follow with this simpler question:

How far has Kim walked when Jay has walked 40 m?

Reminder: you don’t have to resolve the task at this stage. Leave it as something for students to mull over and to return to at a later date.

Multiplicative Reasoning 3A

Lesson

6A

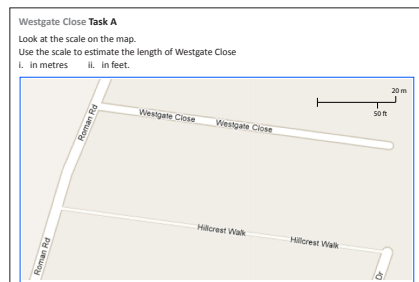
Westgate Close

Students estimate the length of Westgate Close, in metres and feet (Task A).

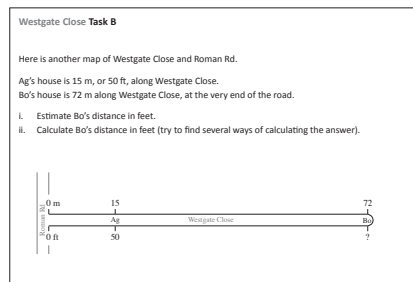
They then estimate and calculate two distances along Westgate Close (Task B and Task C).



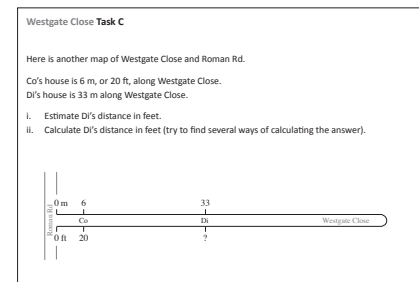
Task A



Task B



Task C



Summary

In this lesson we look at multiplication in terms of *scaling*, and model this with the Double Number Line (DNL).

We use the context of a map, with scales showing distances in metres and feet, to *introduce* the DNL model. At the same time we *use* the DNL model to convert metres to feet, and we relate this to the use of ratio tables.

Outline of the lesson

1. Task A: estimate the length of Westgate Close, in metres and feet.

- Ask for some quick estimates.
- Ask students to work on the task in small groups.
- Briefly discuss their methods and results.

2. Task B: change a 72 m distance on a map to feet.

- Ask students to work on the *estimate* in small groups.
- Briefly discuss their methods and results.
- Ask students to work on the *calculation* in small groups.
- Discuss their methods and results.

3. Task B: change 72 m to feet using a ratio table.

- Represent the information in Task B in a ratio table.
Discuss ways of operating on the numbers to find the required value in feet.



15	72
50	•

4. Task C: change a 33 m distance on a map to feet.

- Ask for some quick estimates.
- Represent the information in Task C in a ratio table.
Ask students to work on the *calculation* in small groups, using the map (DNL) and/or the ratio table.
- Discuss their methods and results.

6	33
20	•

5. Possible extension: revise Task C.

- Ask students to change the given distance, 33 m, to a different distance, to make the task easier or harder.

6. Possible extension, Task D: assess students' work.

- Ask students to evaluate some responses to Task B.

Multiplicative Reasoning 3A

Lesson

6A

Overview

Mathematical ideas

In this lesson we use a map to introduce students to the Double Number Line (DNL), and we use this and ratio tables to explore conversions between measurements involving metres and feet.

The tasks in this lesson are in some ways mathematically similar to the *Steady walk* Mini-assessment, though they look rather different. Also, the relations between the given numbers (eg $15 \times 3\frac{1}{3} = 50$) are at a similar level of complexity.

This suggests that some students might well construe the situation as additive – eg they might conclude that 72 m is the same as 107 ft, because 50 is 35 more than 15, and 72 plus 35 is 107. This lesson is designed to help students see that the situation is multiplicative and introduces them to a multiplicative model in the form of the DNL.

Students' mathematical experiences

Students

- estimate lengths
- talk about ratio problems and how to solve them
- use the DNL and ratio tables to model problems involving ratio.

Assessment and feedback

The Mini-assessment should help you anticipate the kinds of methods that will arise in the lesson - eg addition, rated addition, scaling, the unitary method. (You may also have met these methods from using the *Mini Ratio Test*.)

Find space in the lesson to make such methods explicit but don't at this stage feel that you have to try to resolve all the misconceptions that might come up.

Task D gives students an opportunity to reflect on different methods and ideas. It involves assessing work done by other students and thus provides practice in peer- and self-assessment.

You might also want to give students similar tasks to the ones in the lesson but involving a different ratio context - a recipe, for example. Do students find this easier or harder?

Key questions

How did you work that out?

We've got two answers and an explanation for both. How can we decide between them?

Adapting the lesson

The extension activity in Stage 5 asks students to revise Task C by choosing numbers that would make it easier or harder. This is a useful way for students to reflect on the nature of a task and methods of solution, and it provides insight for the teacher into the nature of students' understanding. You might want to adopt this strategy yourself, earlier in the lesson, if the given numbers prove to be too easy or hard.

You might also, here or in a later lesson, want to discuss how the DNL and ratio table can be used to model and solve the *Steady walk* Mini-assessment task, and/or tasks from the *Mini Ratio Test*.

Multiplicative Reasoning 3A

Lesson

6A**Outline of the lesson (annotated)**

1. Task A: estimate the length of Westgate Close, in metres and feet.

- Ask for some quick estimates.
- Ask students to work on the task in small groups.
- Briefly discuss their methods and results.

— This can be done quite effectively by eye, by marking off distances of approximately 15 m and distances of approximately 50 ft.

2. Task B: change a 72 m distance on a map to feet.

- Ask students to work on the *estimate* in small groups.
- Briefly discuss their methods and results.

— Again, this can be done by eye, or by using the idea that there are '50 ft for every 15 m', and that that 5×15 (or $15+15+15+15+15$) is a little over 72: so the distance in feet will be a bit less than 5×50 (or $50+50+50+50+50$).

- Ask students to work on the *calculation* in small groups.
- Discuss their methods and results.

— It is likely that some students will use the 'addition strategy' here, and thus get an answer of 107 ft rather than 240 ft (by arguing " $15+57=72$, so $50+57=107$ ", or " $15+35=50$, so $72+35=107$ ").

Ask the class to critique this. For example,

How does this answer compare to the earlier estimates?

How does it fit with the idea of '50 ft for every 15 m'?

How many ft would the method give for 16 m, or 1 m?

3. Task B: change 72 m to feet using a ratio table.

- Represent the information in Task B in a ratio table. Discuss ways of operating on the numbers to find the required value in feet.

— It is possible to find the desired number of feet (240) by using the multiplier $\times 3.\bar{3}$, which maps (the number of) metres onto (the number of) feet. Thus $15 \times 3.\bar{3} = 50$, and $72 \times 3.\bar{3} = 240$.

It is also possible to use the multiplier $\times 4.8$, which maps 15 m onto 72 m, and thus maps 50 ft onto $50 \text{ ft} \times 4.8 = 240 \text{ ft}$.

Challenge students to find these multipliers, if they don't arise spontaneously.

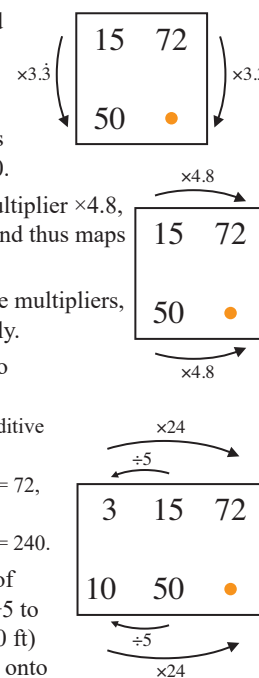
However, first allow students to report on their own ideas.

These might include successful additive strategies, eg 'rated addition':

$15 + 15 + 15 + 15 + 15 = \frac{1}{5}$ of 72, so similarly

$50 + 50 + 50 + 50 + 50 = \frac{1}{5}$ of 240.

Or they might include the use of intermediate values, eg using $\div 5$ to map (15 m, 50 ft) onto (3 m, 10 ft) and then using $\times 24$ to map this onto (72 m, 240 ft).



4. Task C: change a 33 m distance on a map to feet.

- Ask for some quick estimates.
- Represent the information in Task C in a ratio table. Ask students to work on the *calculation* in small groups, using the map (DNL) and/or the ratio table.
- Discuss their methods and results.

5. Possible extension: revise Task C.

- Ask students to change the given distance, 33 m, to a different distance, to make the task easier or harder.

6. Possible extension, Task D: assess students' work.

- Ask students to evaluate some responses to Task B.

— Try to make links between students' use of the double number line and students' use of the ratio table.

Ask students to evaluate these responses (see Task D). This could be set for homework.

$\begin{array}{r} \times 4 \\ + 12 \\ \hline 212 \end{array}$	$\begin{array}{r} m \quad f \\ 4.8 \times 15 = 50 \quad \times 4.8 \\ 72 = 240 \end{array}$
---	---

$$50 + 15 = 35 \quad 72 + 35 = 107 ?$$

Multiplicative Reasoning 3A

Lesson

6A

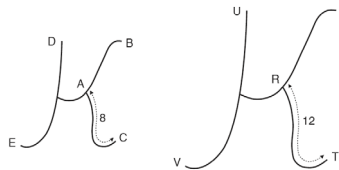
Background

Scaling in the context of conversions and enlargement

Though Westgate Close involves distances on a map, it does not involve geometric enlargement. We are simply converting a distance on a map measured in one unit to the same distance measured in another unit. We are not scaling a distance on a map to the equivalent distance in real life or onto a map drawn to a different scale.

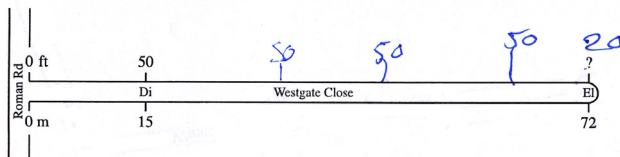
Thus the Westgate Close tasks are probably closer to the conversion items (Dollars) on the *Mini Ratio Test* than to the enlargement items (Curly L). As such, they are probably less demanding than the Curly K task shown below, where the scalar and functional multipliers are fractional ($\times 1\frac{1}{2}$ and $\times 1\frac{1}{8}$, respectively), as with the Westgate Close tasks. The item is from the CSMS Ratio test (Hart, 1981), and was answered correctly by just 14 % of a representative sample (N=309) of Year 8 students in 1976, and by a similar proportion (12%) of a representative sample (N=754) of Year 8 students in 2008.

These 2 letters are the same shape. One is larger than the other.
AC is 8 units. RT is 12 units.

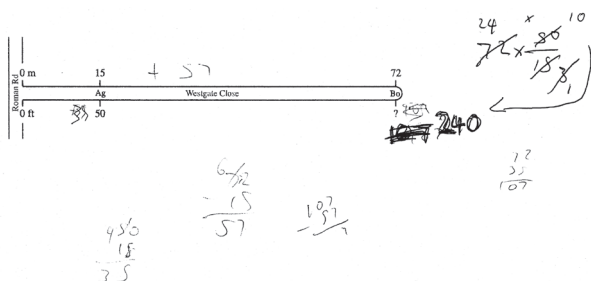


The curve AB is 9 units. How long is the curve RS?

One possible reason why the success rate on this item is so low is that the addition strategy response of 13 is quite close to the correct answer, 13.5. This is not so for Westgate Close. The response below (for an early version of Task B) was given by a student from a low attaining Year 8 group. He has not made any attempt to relate metres and feet but his estimate of 220ft for the length of Westgate Close is near to the correct distance (240ft) and is very different from the distance (107ft) produced by the addition strategy. Thus students who use the addition strategy here have a strong reason to think again.



The work below is by a student from a very high attaining Year 8 class. She first came up with the addition strategy



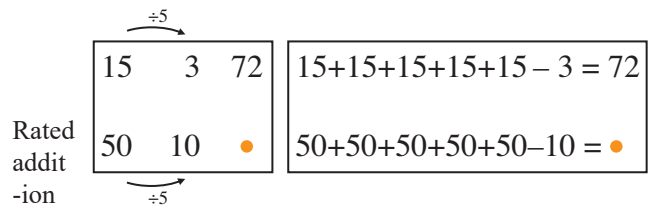
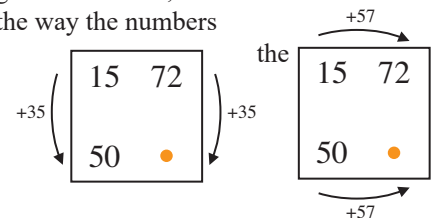
answer 107 (from $72+35$ and perhaps $50+57$), but subsequently arrived at the correct answer, 240, probably in the light of a whole-class discussion.

Note: In this lesson we are using the conversion “15 m corresponds to 50 feet”. This is not exact - the actual multiplier is closer to 3.28 or $3\frac{1}{4}$ than to $3\frac{1}{3}$.

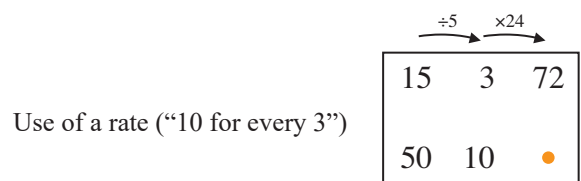
Ratio tables

A table can be very useful for organising information in a problem involving ratio, ie for showing how the value one is trying to find corresponds to the given information. It also provides a useful means for reflecting on and recording the operations that one might perform to find the missing value. Ratio tables are particularly easy to create for the Westgate Close tasks, since we can use a similar layout to the way the numbers are presented in given schematic maps.

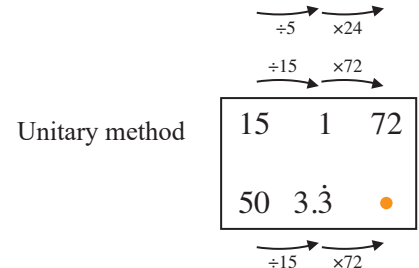
Addition strategy



Rated
addition

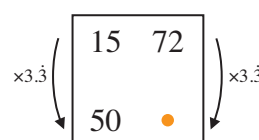


Use of a rate ("10 for every 3")

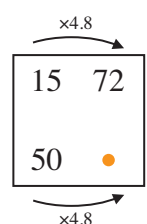


Unitary method

Functional multiplier



Scalar
multiplier



Multiplicative Reasoning **3A**

Lesson
6A

Tasks A, B, C, D (from the file 06A-MR-3A-tasks-ABCD.pdf)

Westgate Close **Task A**

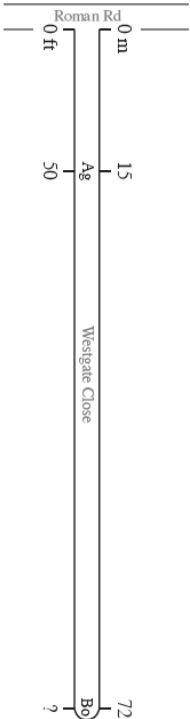
Look at the scale on the map.
Use the scale to estimate the length of Westgate Close
i. in metres ii. in feet.



Westgate Close **Task B**

Here is another map of Westgate Close and Roman Rd.
Ag's house is 15 m, or 50 ft, along Westgate Close.
Bo's house is 72 m along Westgate Close, at the very end of the road.

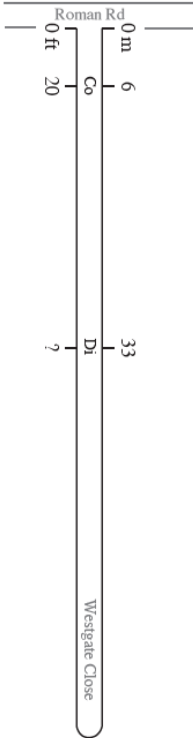
i. Estimate Bo's distance in feet.
ii. Calculate Bo's distance in feet (try to find several ways of calculating the answer).



Westgate Close **Task C**

Here is another map of Westgate Close and Roman Rd.
Co's house is 6 m, or 20 ft, along Westgate Close.
Di's house is 33 m along Westgate Close.

i. Estimate Di's distance in feet.
ii. Calculate Di's distance in feet (try to find several ways of calculating the answer).

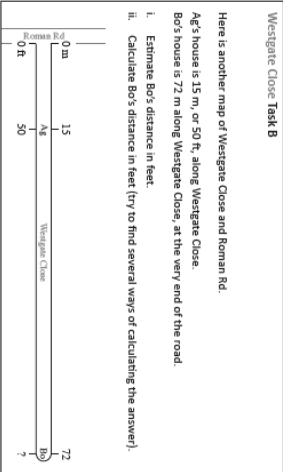


Westgate Close **Task D: Assessing students' work**

Here are three students' answers to Task B.

1. Look carefully at their work.
Try to explain the method you think they have used.

2. Write a helpful comment for each student.



Student A

$$\begin{array}{r} \times 4 \\ + 12 \\ \hline 212 \end{array}$$

Student B

$$\begin{array}{r} m \quad f \\ 4.8 \times 15 : 50 \times 4.8 \\ 72 : 216 \end{array}$$

Student C

$$50 + 15 = 35 \quad 72 + 35 = 107 ?$$

Lesson

6A

Notes

Multiplicative Reasoning 3B

Lesson

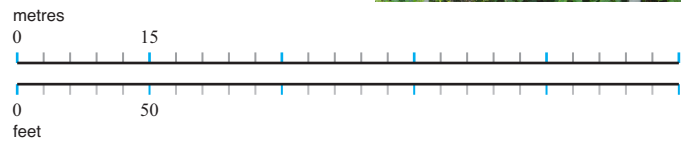
6B

Westgate Close revisited

These two lines show distances in metres and feet along Westgate Close.

Eric wants to convert 66 metres into feet.

He numbers the lines so he can read-off the answer.

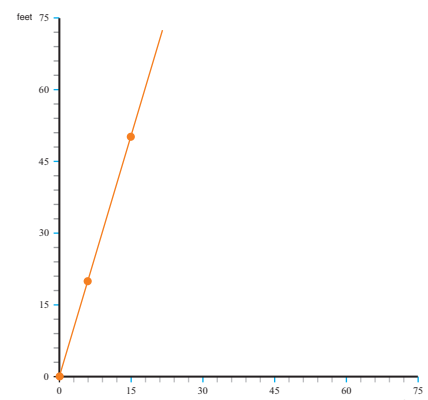
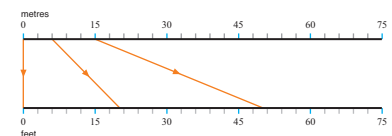


Summary

In this lesson, students consolidate ideas from Lesson 2A. They carefully number a double number line (DNL) for converting metres to feet (and feet to metres). They then create similar conversion devices using a mapping diagram and a Cartesian graph, and consider how multiplication is modelled by these different representations.

Outline of the lesson

- Discuss the above conversion task.
 - Discuss different ways of converting 66 m to ft.
 - Discuss Eric's method: How does he number the lines? Where is 66 m? What is the number underneath this?
- Students number a DNL to represent metres and feet.
 - Distribute the 6B-MR-3B Worksheet (see page 100). Point to the DNL.
 - Ask students to number the blue marks (only).
 - Discuss the numbers represented by the marks (blue and grey). How are the numbers spaced-out on each line? How big are the 'gaps' between adjacent marks? What is the relation between vertically aligned numbers?
 - Use the numbered DNL to go over Eric's method.
- Use the numbered double number line.
 - Locate the positions on the DNL of the numbers from Tasks B and C of Lesson 6A [ie 15 and 50, 72 and ?; 6 and 20, 33 and ?].
 - Discuss some other metres-feet conversions (of your or the students' choosing). Which numbers are easy? Which are more difficult?
- Draw a mapping diagram for metres and feet.
 - Use the mapping diagram on the worksheet. Draw arrows to represent (some of) the number-pairs from Stage 3 of the lesson.
 - Discuss the pattern made by the arrows - their slope, the space between them, whether they meet.
- Draw a Cartesian graph for metres and feet.
 - Use the Cartesian axes on the worksheet. Plot points for (some of) the previously considered number-pairs.
 - Discuss the pattern made by the dots.
- Compare the three representations.
 - How do various features of the representations correspond?
 - What are the strengths of the different representations?



Multiplicative Reasoning 3B

Lesson

6B

Overview

Mathematical ideas

In this lesson, students revisit the *Westgate Close* problem in order to consider the double number line (DNL) in more detail. Students focus on the linear nature of the number line scales and then compare the DNL with a mapping diagram and a Cartesian graph. All three representations provide models for thinking about multiplication (and for countering the 'addition strategy'). Also, the DNL and Cartesian graph provide instant 'ready reckoners' for reading-off conversions (of metres into feet), albeit reckoners that may not be very precise in practice.

Students' mathematical experiences

Students

- scrutinise linear scales
- use different representations to model a multiplicative situation
- compare representations.

Key questions

In what ways are the diagrams the same?
In what ways are the diagrams different?

Assessment and feedback

Do students realise that the DNL scales are both linear but numbered differently? Can they use the *Westgate Close* context to explain why?

Observe which, if any, of the diagrams help students see that the metres-feet conversion tasks are multiplicative rather than additive. Do they help students who used the 'addition strategy' on the Starter mini test?

Encourage students to describe the various representations in detail, and to read them 'dynamically'. What happens to the values (in metres and feet) as one moves steadily along the double number line (or along the straight line graph)? What changes, and what stays the same?

Adapting the lesson

Consider other conversions in everyday life or in science. For example, a ruler marked in cm and inches, or a measuring cylinder marked in pints and litres (what happens to the scales if the container tapers as with a measuring jug?). Look at some conversion charts and tables. [Note: in Lesson 7A-MR-4A we look at a currency conversion.]

What about temperature in °C and °F? Here the 0s don't line up [See page 99].

You might want discuss how the three representations can be used to model the items in the *Mini Ratio Test*. How can the models help us refute the 'addition strategy'?

Multiplicative Reasoning 3B

Lesson

6B**Outline of the lesson (annotated)**

1. Discuss the above conversion task.

- Discuss different ways of converting 66 m to ft.
- Discuss Eric's method: How does he number the lines? Where is 66 m? What is the number underneath this?

— We can 'skip' along the lines:

15,50 to 30,100 to 45,150 to 60,200 to 63,210 to 66,220.

Or we could find 3,10 and multiply by 22, or find 6,20 and multiply by 11.

Or we can find the multiplier $\times 3\frac{1}{3}$ than maps 15 onto 50 and apply this to 66.

2. Students number a DNL to represent metres and feet.

- Distribute the 6B-MR-3B Worksheet (see page 100). Point to the DNL.
- Ask students to number the blue marks (only).
- Discuss the numbers represented by the marks (blue and grey).
How are the numbers spaced-out on each line?
How big are the 'gaps' between adjacent marks?
What is the relation between vertically aligned numbers?
- Use the numbered DNL to go over Eric's method.

— This is to help students realise that the scales are linear but numbered differently: we are representing the fact that every 15 m interval is equivalent to 50 ft.

— Allow plenty of time for students to contemplate what is going on, here and in later stages of the lesson. Students might notice that

the numbers are evenly spaced (the scales are linear).
the gaps represent 3 m and 10 ft.number of metres $\times 3\frac{1}{3}$ = number of feet, or $x \rightarrow 3\frac{1}{3}x$.

3. Use the numbered double number line.

- Locate the positions on the DNL of the numbers from Tasks B and C of Lesson 6A [ie 15 and 50, 72 and ?; 6 and 20, 33 and ?].
- Discuss some other metres-feet conversions (of your or the students' choosing). Which numbers are easy? Which are more difficult?

— It is relatively straightforward to locate the 6 m mark and the corresponding 20 ft mark (for Task C), and to locate the 72 m and 33 m marks, and hence to estimate the corresponding values in feet (240 ft and 110 ft respectively).

You might want to challenge students to read-off conversions where neither value lies on a given mark. How could one use a calculator to make the conversions?

4. Draw a mapping diagram for metres and feet.

- Use the mapping diagram on the worksheet. Draw arrows to represent (some of) the number-pairs from Stage 3 of the lesson.
- Discuss the pattern made by the arrows - their slope, the space between them, whether they meet.

— It is not easy to draw accurate arrows for the given scales and markings. However, student should notice that the arrows splay out and get flatter and flatter.

We have drawn the two number lines $2\frac{1}{2}$ cm apart. It turns out that the arrows meet at a point 1 cm above the zero mark of the top number line. Why?!

5. Draw a Cartesian graph for metres and feet.

- Use the Cartesian axes on the worksheet. Plot points for (some of) the previously considered number-pairs.
- Discuss the pattern made by the dots.

6. Compare the three representations.

- How do various features of the representations correspond?
- What are the strengths of the different representations?

We have considered three diagrammatic ways to represent the scaling relationship of $\times 3\frac{1}{3}$ or $x \rightarrow 3\frac{1}{3}x$. On the DNL any number on the bottom scale is $3\frac{1}{3}$ times the number above it; on the mapping diagram any two arrows are $3\frac{1}{3}$ times as far apart at the head than the foot; on the Cartesian graph, the straight line has a gradient of $3\frac{1}{3}$.The scaling maps $0 \rightarrow 0$. On the DNL the zeros line up; on the mapping diagram the zeros are joined by an arrow; the Cartesian graph goes through the origin.

A strength of a mapping diagram is that one can choose any practicable scale (as long as it is linear and there is room on the page to record all the desired values). In contrast to the DNL, one does not have to work out the numbering system for the second scale, as it is the same as the first. However, one can't simply read-off conversions as one can with the DNL and the Cartesian graph. And for the latter one can use any two linear scales, and (in theory) just one plotted point, eg (15, 50), joined with a straight line to the origin.

Multiplicative Reasoning 3B

Lesson

6B

Background

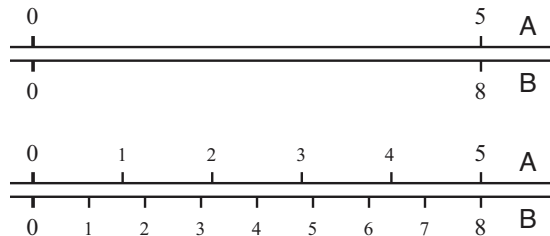
The double number line (DNL)

The DNL offers a neat way of representing multiplicative relations. We see it as a very useful model for *thinking* about multiplication (though it is sometimes less useful for actually *solving* multiplication problems).

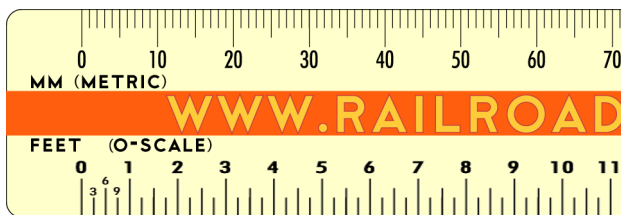
Consider the pair of lines A and B (first diagram, below), where 0 on line A is lined-up with 0 on line B and where 5 on A is lined up with 8 on B. We know that 8 is 1.6 times 5.

If we mark linear scales on each line, then any number on scale B will be 1.6 times the corresponding (lined-up) number on scale A (second diagram, below).

(In practice, drawing such linear scales accurately can be quite a challenge, and we would not always want students to do this - rather, we want them to appreciate the *idea* that the scales illustrate.)

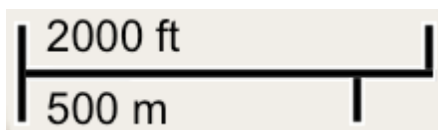


Scales on maps are well known examples of a double number line. There are two sorts. In one we are shown how distances on the actual map (measured in cm, say) correspond to distances on the object depicted by the map (measured in km, say). Such a scale is shown below, on a ruler used by model railway enthusiasts.



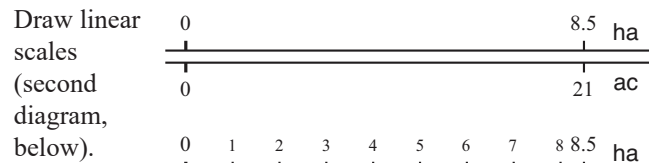
Scales of this sort are often also expressed numerically, as the ratio of a distance on the map (or ruler) to the corresponding distance in real life. In the case of the 0-scale used for model railways, this is commonly 1:43.5.

The other kind of map scale shows how distances represented on the map can be read in different units (for example, feet and metres). The example, right, is from Google Maps.

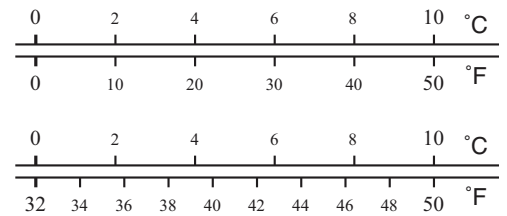


In effect, this kind of scale converts one unit to another. We can use such scales to represent other conversions, not just involving distance. For example, if we are told that 8.5 hectares is equivalent to 21 acres, we can construct a conversion scale as follows:

Draw two parallel lines to represent hectares and acres, and line-up 0 with 0 and 8.5 with 21 (first diagram, below).



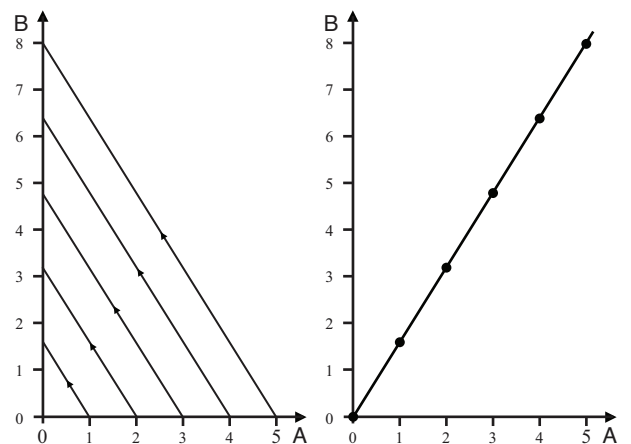
However, we have to be careful. Say we know that 10°C is the same as 50°F; we can't create a conversion scale by simply lining up 0 with 0 and 10 with 50 (first diagram, below). The equivalent of 0°C is 32°F, not 0°F (as illustrated in the second diagram, below). Does this diagram work?



In each of these two examples, the scales on the two number lines

have been different. We could keep the scales the same, but we would then have to show how the numbers on the scales correspond, for example by using arrows. Such a diagram is usually called a mapping diagram. The diagram below is for our first example, ie for the $\times 1.6$ mapping.

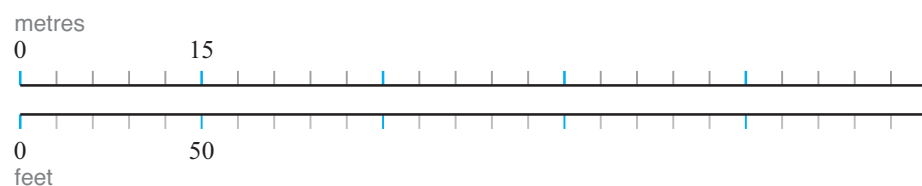
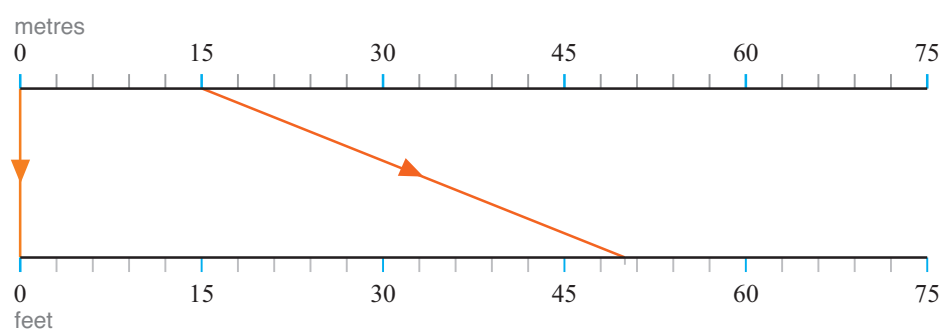
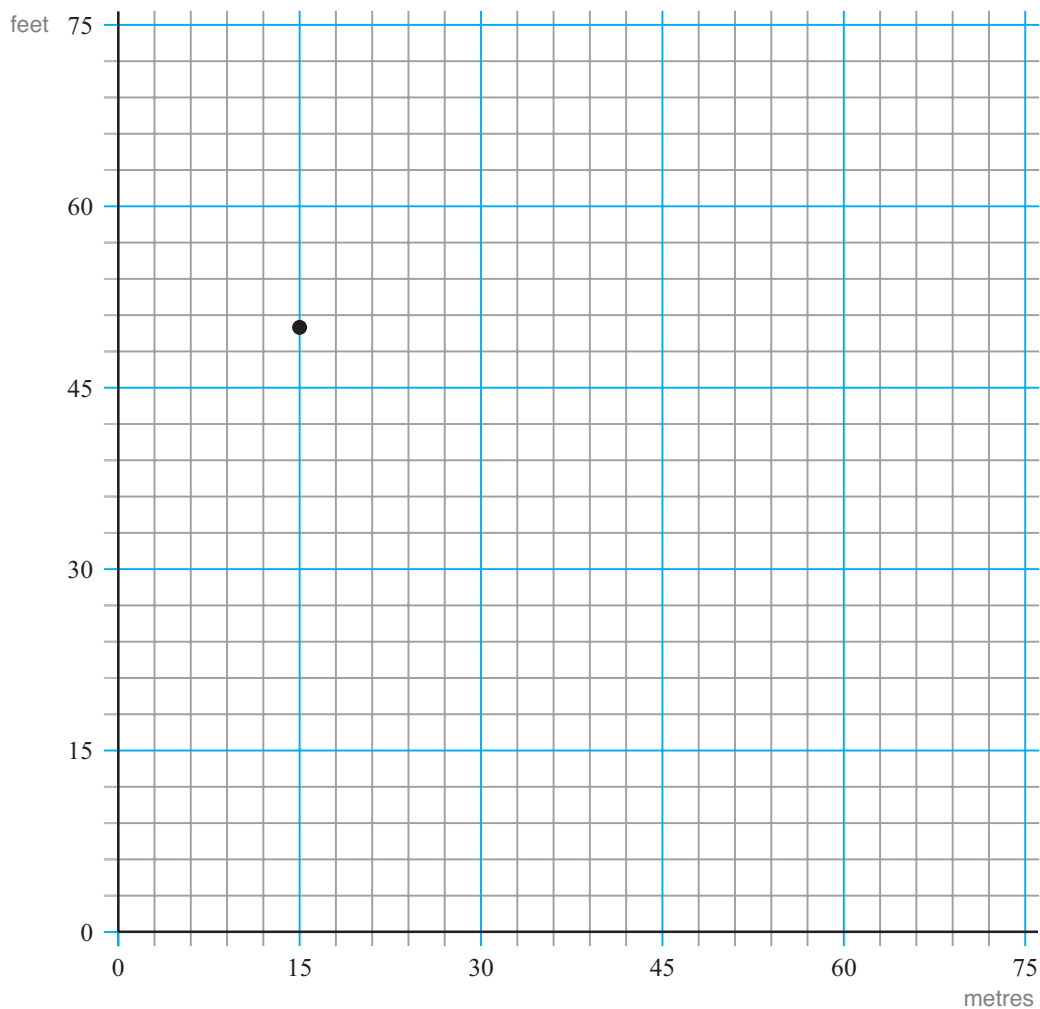
If we rotate one of the number lines through 90°, we can change the mapping diagram into a Cartesian graph, as in the two steps shown below.



[We can of course go straight to Cartesian graphs if we want a device for reading-off conversions. Here we can draw the linear scales on the axes first, using any scale that suits the paper and ruler we are using (conceptually, there is a lot to be said for keeping the scales on the two axes the same). Then we usually only need one pair of values (eg £52 buys \$70) which we plot as a point and then join to the origin (usually!) with a straight line (usually!).]

Multiplicative Reasoning 3B

Lesson

6B**6B-MR-3B Worksheet**Double
number
lineMapping
diagramCartesian
graph

Multiplicative Reasoning 4AB

Mini-assessment

7AB**Double number line rules**

Look at the double number line. →

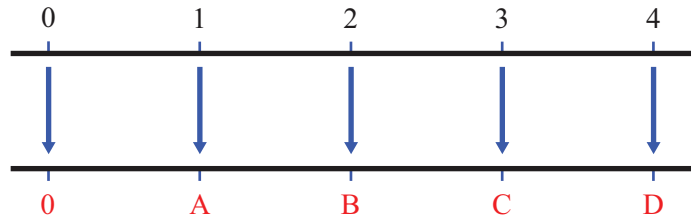
The top scale starts at 0 and is numbered evenly.

The bottom scale also starts at 0 and is also to be numbered evenly.

Find a rule that maps the numbers on the top line onto the numbers on the bottom line, when $D = 10$.

Choose another value for D .

Find the new rule.

**Commentary**

Here we look at a family of double number lines (DNLs) with the zeros lined-up and with linear scales.

The resulting mappings are purely multiplicative (ie of the form $x \rightarrow ax$).

How readily do students see the multiplicative relationship for different values of D ?

Do students first try to find some of the intermediate values A , B and C ? If so, how?

Does finding the intermediate values help? If so, how?

You might want to select one or more slides from the file DNL-times7.pdf for homework or class discussion.

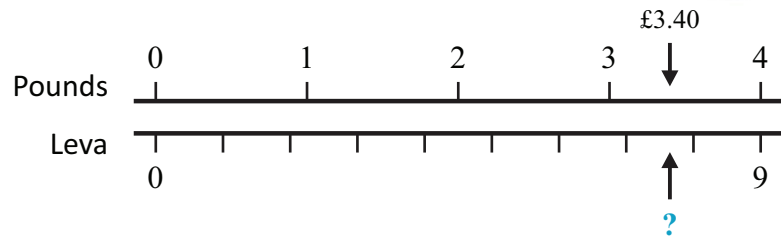
Multiplicative Reasoning 4A

Lesson

7A**Converting Pounds to Leva**

This DNL can be used to convert British Pounds to Bulgarian Leva.

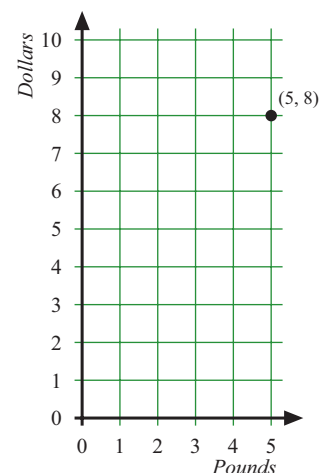
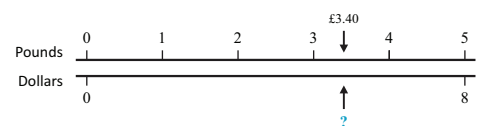
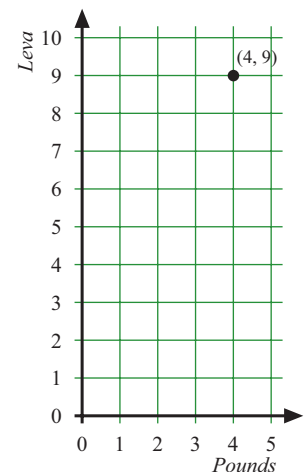
Volen wants to know what £3.40 is in Leva. What methods could he use?

**Summary**

This lesson involves conversion, like *Westgate Close*, but this time of currency: Pounds to Bulgarian Leva and Pounds to US Dollars. We use a given DNL to help with the conversions and compare it to the Cartesian graph (and, as a possible extension activity, to the mapping diagram).

Outline of the lesson

- Use the Pounds-Leva DNL to convert £3.40 to Leva.
 - Show Slide 1 (above).
Ask for a quick visual estimate for £3.40 in Leva.
 - Look for more precise ways of using the DNL to convert £3.40 to Leva. Discuss students' methods.
 - "How could we use a calculator to find the answer?"
- Record some Pounds-Leva values on a graph.
 - Use the DNL to list several precise Pounds-Leva conversions.
 - Record the values as points on a Cartesian graph (Slide 2).
 - Discuss the pattern made by the points.
 - "How could we use the graph to convert £3.40 to Leva?"
- Consolidation: convert £3.40 to US Dollars.
 - Show the Pounds-Dollars DNL (Slide 3).
Ask for a quick visual estimate for £3.40 in Dollars.
 - Give students a copy of Slide 3.
Ask them carefully to number the Dollars scale.
 - Discuss how students numbered the Dollars scale.
Make a more precise estimate of £3.40 in Dollars.



- Consolidation: use a Pounds-Dollars graph.
 - Show (or distribute) the Pounds-Dollars graph (Slide 4).
"How can we use the graph to convert £3.40 to Dollars?"

Multiplicative Reasoning 4A

Lesson

7A

Overview

Mathematical ideas

In this lesson we consider ratio in a conversion context, like *Westgate Close*, but this time involving currency: Pounds to Leva and to Dollars. We visit the same two problems several times (the value of £3.40 in Leva and in Dollars). Thus our interest is not so much in the answer to the problems than in the various ways we can model and solve them.

We start with a problem modelled by a double number line (DNL) and we explore how to use the model. We then look at a related model - the Cartesian graph.

The multipliers here ($\times 2.25$ and $\times 1.6$) are more complex than in the 7AB Mini-assessment ($\times 1.5$) but students might find using a DNL easier than constructing one.

Students' mathematical experiences

Students may discover some of the following

- the relationship between currencies is multiplicative, not additive
- we can model currency conversions with a DNL, where the scales on each line are uniform
- we can model currency conversions with a straight line Cartesian graph that goes through the origin.

Key questions

If £4 are worth 9 Leva,

- what roughly are £5 worth?
- are £8 worth twice as much, or 4 Leva more?

How could we work out a conversion using a calculator?

On the Cartesian graph, which is steeper, the Pound-Leva line or the Pound-Dollar line? Why?

Assessment and feedback

Try to assess how students use the DNL here. Is it in a similar way to *Westgate Close*?

- Which students can make good visual estimates?
- What sorts of analytic methods do students use?
- Which students resort to the (inappropriate) addition strategy and thus get the answer 8.40 Leva?
[$4 + 5 = 9$, so $3.40 + 5 = 8.40$;
or $4 - 0.60 = 3.40$, so $9 - 0.60 = 8.40$.]

Can students find the answer using a calculator? If so, can they explain their steps in terms of any of the models?

[You might want to ask students to *evaluate* such explanations.]

Adapting the lesson

It can be helpful to use ratio tables to record data and analyse methods. You might want to use simpler conversion rates, eg Pounds - Swiss Francs ($\times 1.5$ approx), or conversions where the multiplier is much larger, eg Pounds - Malaysian Ringgits ($\times 4.8$ approx) or Pounds - Japanese Yen ($\times 130$ approx). You might also want to use the reverse conversions, ie Leva to Pounds and Dollars to Pounds, where the multiplier is less than 1.

In later lessons you might want students to experience other conversion contexts, such as inches - centimeters ($\times 2.54$ approx), gallons - litres ($\times 4.6$ approx), pound weight - kilogram ($\times 2.2$ approx), imperial pint - US pint ($\times 1.2$ approx). And you might want to use genuine scaling situations, where we consider a set of objects at different scales but measured in the same units - eg the lengths of lines in different enlargements of a photograph or on maps of a region drawn to different scales; or the changing price of groceries subject to a specific rate of inflation.

Multiplicative Reasoning 4A

Lesson

7A**Outline of the lesson (annotated)**

1. Use the Pounds-Leva DNL to convert £3.40 to Leva.

- Show Slide 1 (above).
Ask for a quick visual estimate for £3.40 in Leva.
- Look for more precise ways of using the scales to convert £3.40 to Leva. Discuss students' methods.

As well as referring to the DNL, you might want to use ratio tables to support the discussion.

Students might decide to find the value of £1. This can be done by using the DNL ($4 \rightarrow 9$, so $2 \rightarrow 4.5$, so $1 \rightarrow 2.25$), or by simply dividing 9 by 4.

They might then scale up to £3.40 ($3.40 \times 2.25 = 7.65$ Leva) or build up (eg $\text{£}3.40 = \text{£}1 + \text{£}2 + 50\text{p} - 10\text{p} \rightarrow 2.25 + 4.50 + 1.125 - 0.225 = 7.65$ Leva).

Or they might find the value of 10p (0.225 Leva) and scale up by $\times 34$; or find the value of 50p and 10p and subtract from 9 Leva.

Can we find a multiplier that maps £3.40 onto Leva? How? What would it be?

$[9 \div 4 = 2.25$, so $4 \times 2.25 = 9$; similarly $3.40 \times 2.25 = 7.65]$

- "How could we use a calculator to find the answer?"

2. Record some Pounds-Leva values on a graph.

- Use the DNL to list several precise Pounds-Leva conversions.
- Record the values as points on a Cartesian graph (Slide 2).
- Discuss the pattern made by the points.
- "How could we use the graph to convert £3.40 to Leva?"

— eg, (4, 9), (2, 4.5), (1, 1.25), (0, 0).

— Do they lie on a straight line?

What is its slope? Why?

Does it go through the origin? Why?

3. Consolidation: convert £3.40 to US Dollars.

- Show the Pounds-Dollars DNL (Slide 3).
Ask for a quick visual estimate for £3.40 in Dollars.
- Give students a copy of Slide 3.
Ask them carefully to number the Dollars scale.
- Discuss how students numbered the Dollars scale.
Make a more precise estimate of £3.40 in Dollars.

— We have produced two versions of Slides 1, 3 and 5, with scales of 1 cm : £1 and 2 cm : £1. The former is numerically more straightforward but can be rather small.

— "How could we find the answer using a calculator?"

4. Consolidation: use a Pounds-Dollars graph.

- Show (or distribute) the Pounds-Dollars graph (Slide 4).
"How can we use the graph to convert £3.40 to Dollars?"

— A nice thing about the graph, in contrast to the DNL, is that we don't have to construct the numbering for the Dollars scale. Any linear scale will do.

Multiplicative Reasoning 4A

Lesson

7A

Background

The use of models

In this lesson we use two related models: the double number line and the Cartesian graph. We also offer a Revisit where we use the mapping diagram shown below.

Each model can be viewed as serving two purposes: as a model *of* mathematical relationships, in this case involving ratio, and as a model *for* solving problems involving those relationships.

The models may serve these purposes to varying degrees. The DNL, for example, can be difficult to construct because of the different scales on the two lines, in contrast to the Cartesian graph where each axis can be drawn to any scale, as long as it is linear. On the other hand, for the Conversion context (and in contrast to, say, *Shadows* in Lesson 8B) the visual aspect of the Cartesian graph is more remotely related to the context than is the visual aspect of the DNL.

The mapping diagram and geometric enlargement

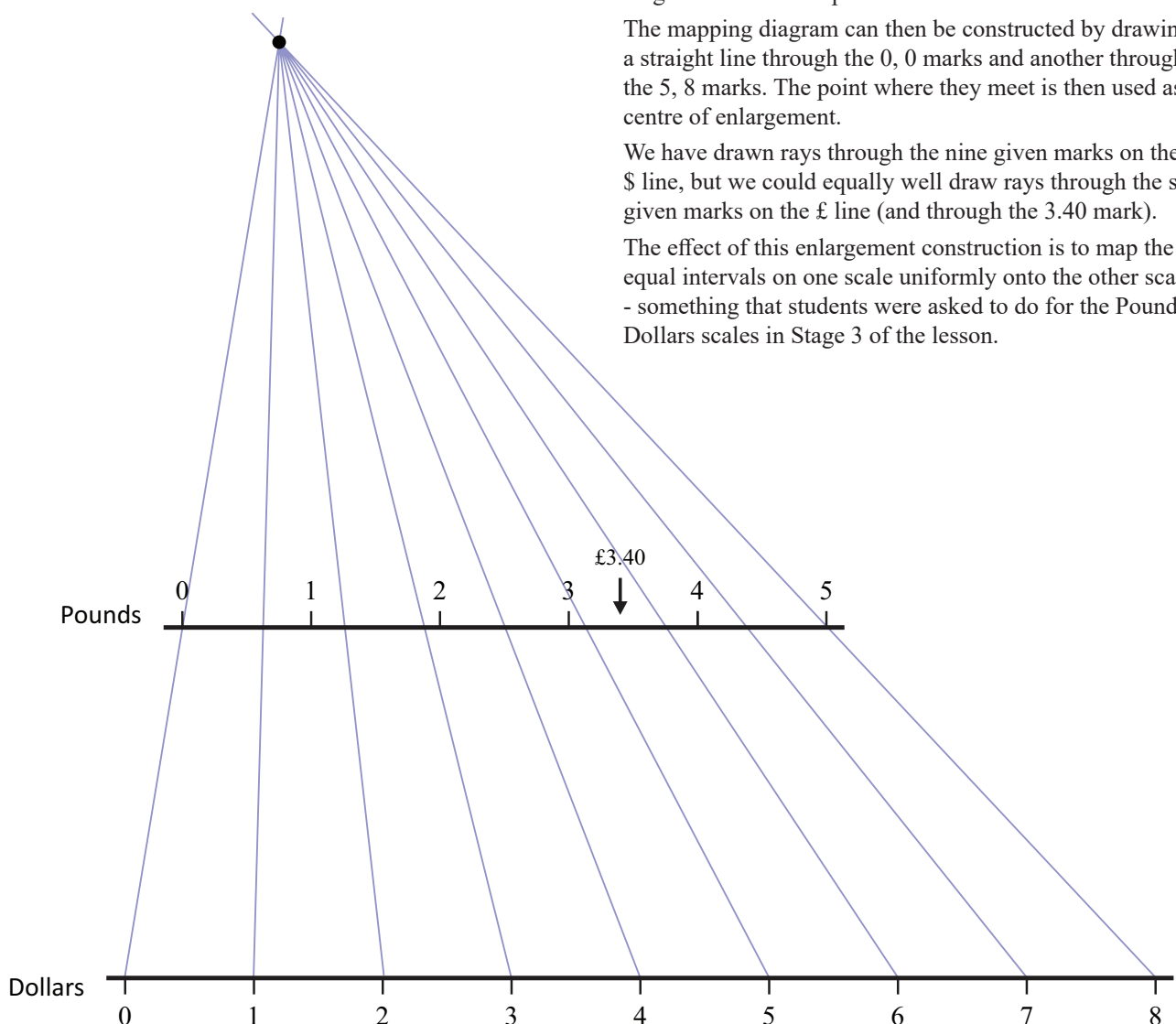
We commonly draw mapping diagrams ‘after the event’, ie when we know the ordered pairs that we want to map. However, in the Revisit to this lesson, the mapping diagram (below) for Pounds-Dollars is based on just two ordered pairs, namely (0, 0) and (5, 8), that can be read-off directly from the given DNL. The other ordered pairs are found by means of a geometric construction, using the fact that the conversion relationship can be represented by a geometric enlargement. This is a complex and subtle idea that might be of interest to teachers as much as students.

In the mapping diagram below, the Pounds and Dollars lines are drawn to the same scale (2 cm : £1 : 1 Dollar). This is not essential, but it *is* important that the two lines are parallel. Beyond that, the lines can be ‘anywhere’ (as long as they are not on the same line). It can be a nice exercise for students to construct their own versions of the mapping diagram - and to compare them.

The mapping diagram can then be constructed by drawing a straight line through the 0, 0 marks and another through the 5, 8 marks. The point where they meet is then used as a centre of enlargement.

We have drawn rays through the nine given marks on the \$ line, but we could equally well draw rays through the six given marks on the £ line (and through the 3.40 mark).

The effect of this enlargement construction is to map the equal intervals on one scale uniformly onto the other scale - something that students were asked to do for the Pounds-Dollars scales in Stage 3 of the lesson.

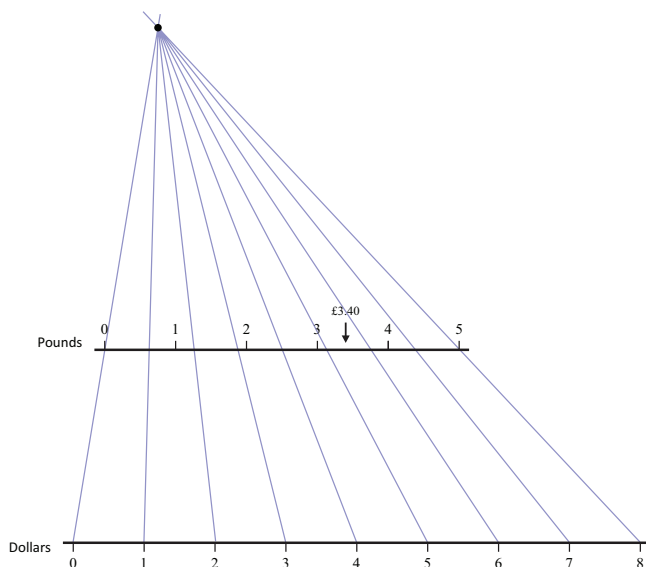
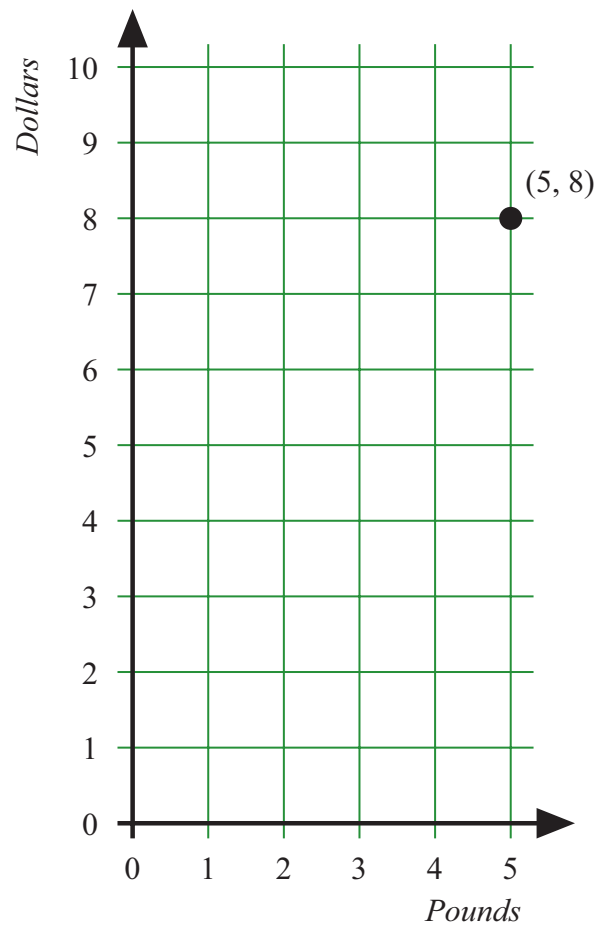
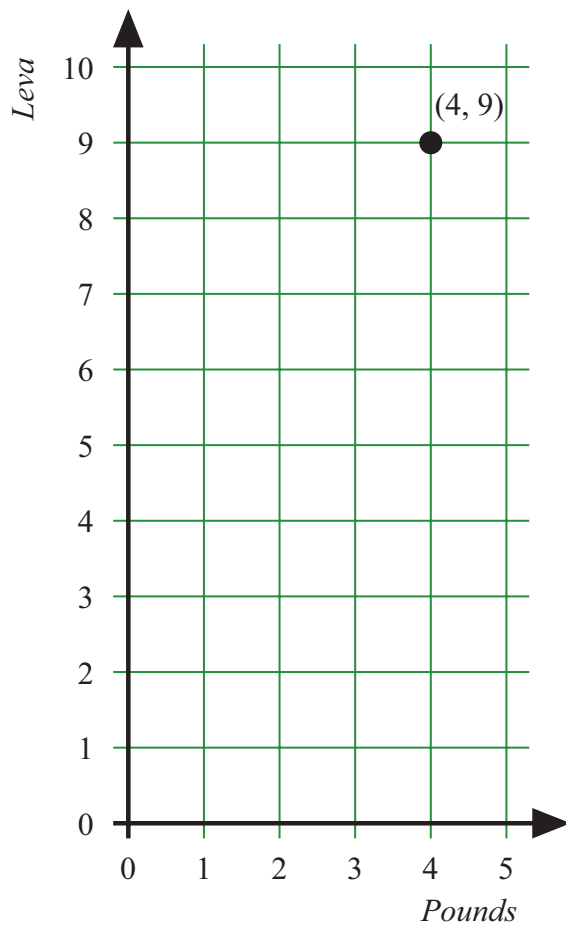
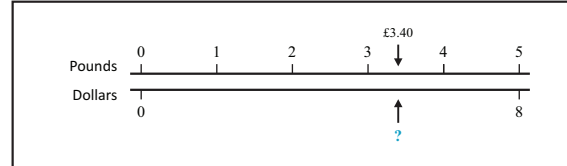
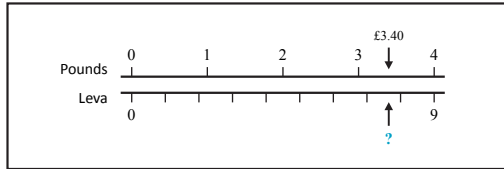


Multiplicative Reasoning 4A

Lesson

7A

Slides 1, 2, 3, 4, 5 shown at a scale of 1 cm : £1 (slides 1, 3, 5 are also available at a scale of 2 cm : £1)



Lesson

7A

Notes

Multiplicative Reasoning 4B

Lesson

7B**Potato pancakes**

This recipe is for 4 people.

How many medium potatoes are needed for 10 people?

POTATO PANCAKES

Serves 4

6 medium potatoes

1 medium onion

2 eggs

30 g flour

$\frac{1}{2}$ teaspoon salt

cooking oil

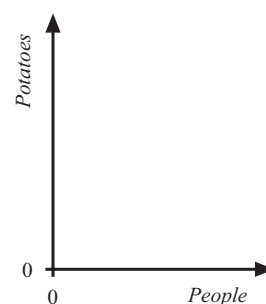
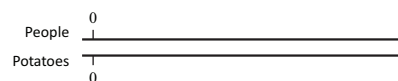
**Summary**

This lesson involves the double number line and ratio table (and graph), but in contrast to Lesson 7A, this time the emphasis is more on how such models can be used as models *of* a problem situation, rather than as models *for* solving it. Also, in contrast to our usual approach (but somewhat in line with Mini-assessment 7AB), we home in on two specific methods, namely the use of the multipliers $\times 1.5$ and $\times 2.5$. These are relatively simple and, in contrast to a later lesson like *Post Shadows* (Lesson 8B), this lesson involves one of the most accessible ratio contexts, namely that of a recipe.

Outline of the lesson

- Students find ways to solve the above problem.
 - Ask students to solve the problem in several ways.
 - Discuss their methods. Try to draw out the methods that involve $\times 1.5$ and $\times 2.5$ (ie $10 \times 1.5 = 15$ and $6 \times 2.5 = 15$).
- Apply the $\times 1.5$ and $\times 2.5$ methods.
 - Ask students to make up problems involving the pancake recipe that can be solved by simply
 - multiplying by 1.5
 - multiplying by 2.5.
 Note some of their problems on the board.
- Represent the problem and the two methods in a ratio table.
 - Ask students to represent the original problem in this ratio table.
 - Add arrows to show $\times 1.5$ and $\times 2.5$.
- Represent the problem on a double number line (DNL).
 - Ask students to represent the original problem on a *sketch* of this DNL.
 - Discuss how we could use or modify the drawing to represent some of the students' problems from Stage 2 of the lesson.
- Represent the problem on a Cartesian graph.
 - Ask students to represent the original problem on a *sketch* of this Cartesian plane.
 - Discuss how we could use or modify the graph to represent some of the students' problems from Stage 2 of the lesson.

People	
Potatoes	



Multiplicative Reasoning 4B

Lesson

7B

Overview

Mathematical ideas

In previous lessons we have examined models of ratio situations from two points of view: to illuminate the concept of ratio, and to find or analyse methods of solving ratio problems. These viewpoints are still present in this lesson, but our prime focus is on *constructing* models of a given ratio problem: we want the models to become thinking tools which students can draw on spontaneously, as and when needed.

It is likely that students will find this quite challenging and so we have chosen a relatively accessible ratio context (recipes) and relatively simple numerical relationships ($\times 1.5$ and $\times 2.5$). We have also imposed some structure on the models that students are asked to complete by labelling various rows, lines or axes (the possible effects of this are discussed on page 111). You can of course vary these aspects of the lesson as appropriate for your students.

Students' mathematical experiences

Students may discover some of the following

- it is harder to construct a model than simply 'read' a given model
- using a model can make the details of a problem clearer
- using a model can make the relationships between numbers more visible
- using a model can make it easier to explain methods.

Key questions

How did you solve it? Can you think of another method?

Did you use any models to help you solve the problem?

What do you like about the models? In what way are they hard?

Assessment and feedback

Try to assess what spontaneous methods students use to solve the initial problem. In this recipe context, it is unlikely that many will think purely additively (which would lead to the answer '12 potatoes'), but they might use rated addition ($4+4+\text{half-of-}4 = 10$, $6+6+\text{half-of-}6 = 15$). The Mini-assessment might have revealed students who quite readily use a multiplicative approach. Do they do so here?

Do any students spontaneously refer to models? If so, which ones?

In the light of these assessments, you might want to change some of the numbers in the problem, especially if the problem seems too hard (it doesn't matter so much if it seems too easy - the next stages of the lesson may still be quite challenging for the students). You could also turn this into a self-assessment activity for the students: "How could I change the numbers in this problem to make it easier/harder?"

Adapting the lesson

You might, in this lesson or in a subsequent lesson, want to change the context used in the problem, change the numbers (and hence the numerical relationships) or remove the structuring (ie the labels) that we have put on the models that students are asked to construct.

You could involve the students in assessing the effects of these changes - the easiest to consider is probably changing the numbers. If you want students to think about context, you could present them with a numbered DNL or Cartesian graph, but without labels, and ask them to invent a story to go with the numbers. Or you could explore what a non-ratio situation would look like on a DNL or graph, as in this Lunch Queue problem (see Lesson 17B-MR-11B):

Meg is in a very long queue waiting for her lunch.

Carl joins the queue.

When Carl has been in the queue for 8 seconds,

Meg has been in the queue for 21 seconds.

When Carl has been in the queue for 32 seconds,
how long has Meg been in the queue?

Multiplicative Reasoning 4B

Lesson

7B**Outline of the lesson (annotated)**

1. Students find ways to solve the above problem.

- Ask students to solve the problem in several ways.
- Discuss their methods.
Try to draw out the methods that involve $\times 1.5$ and $\times 2.5$ (ie $10 \times 1.5 = 15$ and $6 \times 2.5 = 15$).

$\times 1.5$ is a functional multiplier - it maps number of people onto number of potatoes;

$\times 2.5$ is a scalar multiplier - it maps number of people onto number of people, and number of potatoes onto number of potatoes.

2. Apply the $\times 1.5$ and $\times 2.5$ methods.

- Ask students to make up problems involving the pancake recipe that can be solved by simply
i. multiplying by 1.5 ii. multiplying by 2.5.
Note some of their problems on the board.

We can use $\times 1.5$ to calculate the number of potatoes needed for any number of people - eg the number of potatoes needed for 8 people is 8×1.5 . [Of course, we can also use a new scalar multiplier $\times 2$ here, giving $6 \times 2 = 12$.]

We can use $\times 2.5$ to calculate the amounts of any of the ingredients needed for 10 people - eg the amount of flour needed for 10 people is 30×2.5 g.

3. Represent the problem and the two methods in a ratio table.

- Ask students to represent the original problem in this ratio table.
- Add arrows to show $\times 1.5$ and $\times 2.5$.

We have labelled the rows so that the numbering will correspond to the numbering on the similarly labelled DNL in Stage 4. However, you might want to leave this open.

The given ratio table will eventually look like this:

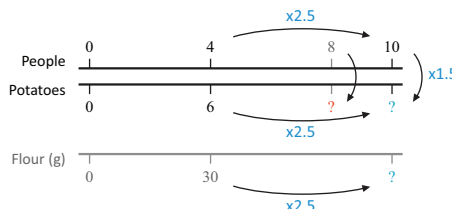
		$\times 2.5$
People	4	10
Potatoes	6	?
		$\times 1.5$

4. Represent the problem on a double number line.

- Ask students to represent the original problem on a *sketch* of this DNL.
- Discuss how we could use or modify the drawing to represent some of the students' problems from Stage 2 of the lesson.

The basic DNL will look like this, above (but see page 111 for a viable but less coherent alternative).

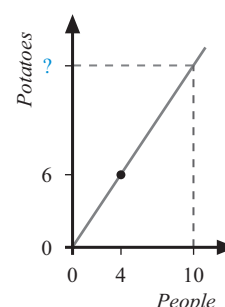
We can easily add marks to show the number of potatoes needed for 8 people, say. However, we need another line to show amounts of flour, say:



5. Represent the problem on a Cartesian graph.

- Ask students to represent the original problem on a *sketch* of this Cartesian plane.
- Discuss how we could use or modify the graph to represent some of the students' problems from Stage 2 of the lesson.

Here is the basic graph. We can easily use it to read off the number of potatoes needed for 8 people, say. To use it to find the amount of flour, say, for 10 people, we would need to draw another line, through (0, 0) and (4, 30); or we could add another scale to the y axis, with 30 coinciding with the existing 6.



Multiplicative Reasoning 4B

Lesson

7B

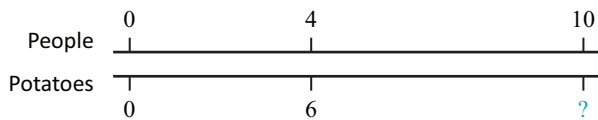
Background

Constructing models of problems

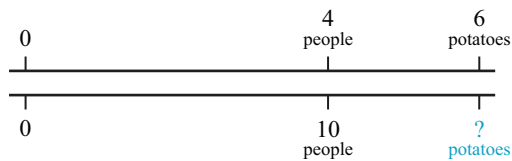
The DNL

The focus of this lesson is more on constructing than on using models. To keep things relatively simple, we have structured the modelling activity by attaching the labels *People* and *Potatoes* (or, strictly speaking, Number of people and Number of potatoes) to the rows of the blank ratio table, to the DNL number lines, and to the Cartesian axes. However, you might want to do without this structuring to allow students to become aware of alternative versions of the models.

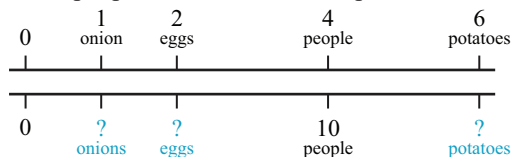
This is the 'standard' DNL for our original problem:



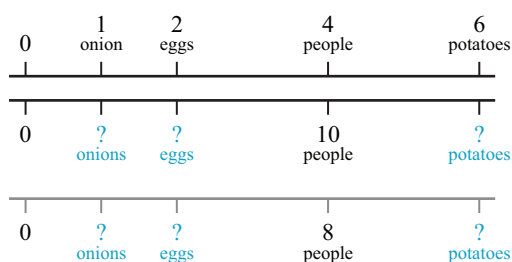
If the lines are not labelled from the outset, students might construct this alternative version, below. Here the top line represents various quantities for 4 people and the bottom line various quantities for 10 people.



Clearly, the same numerical relations apply as before and we can thus still use the DNL to represent or find methods of solution. However, the lines themselves are rather hard to interpret, since, at present, each line represents numbers of people *and* numbers of potatoes. We could also use them to represent all the other ingredients of the pancake, for the 4 people and 10 people versions of the recipe:



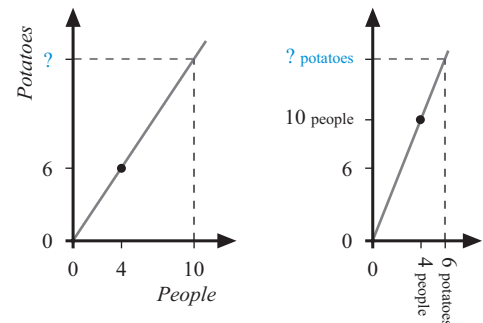
If we now wanted to find the quantities of ingredients for, say, 8 people, we would have to add a new number line:



The Cartesian graph

We can get an alternative version of the Cartesian graph in a similar way.

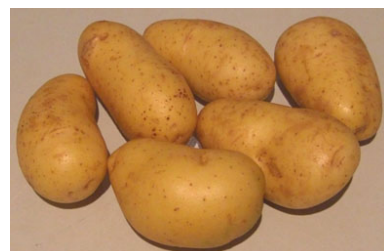
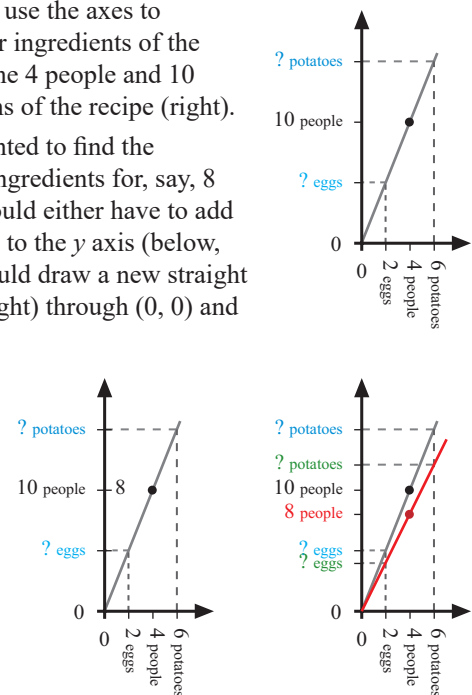
Below, left is the 'standard' graph of the original problem. If the axes are not labelled from the outset, students might construct this alternative version (below, right):



Clearly, the same numerical relations apply as before and we can thus still use the graph to represent or find methods of solution. However, as with the alternative DNL, the axes are rather hard to interpret, since each one represents numbers of people *and* numbers of potatoes.

We could also use the axes to represent other ingredients of the pancake, for the 4 people and 10 people versions of the recipe (right).

If we now wanted to find the quantities of ingredients for, say, 8 people, we would either have to add a second scale to the y axis (below, left), or we could draw a new straight line (below, right) through (0, 0) and (4, 8).



Lesson

7B

Notes

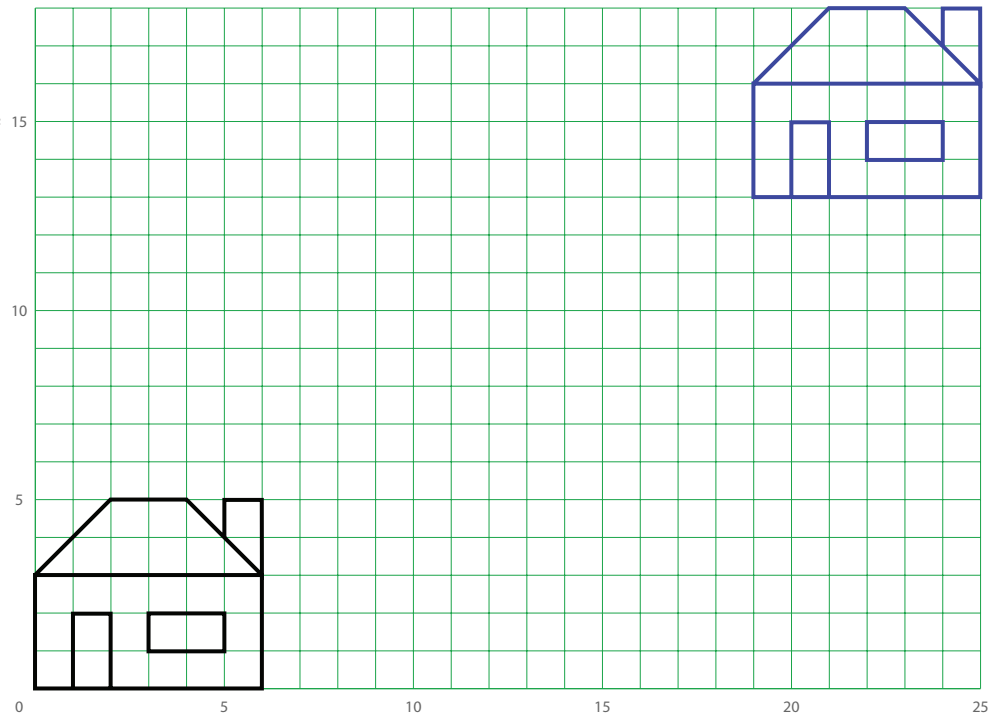
Multiplicative Reasoning 5AB

Mini-assessment

8AB**Stretched house**

We are going to stretch the black house *across* the page and then *up* the page.

Is the resulting black house the same shape as the blue house?

**Commentary**

Use the file **HOUSE-1cm-grid.docx**, which shows a locked blue drawing of a house and free-moving black copy.

‘Expand’ the black house in this way (but *don’t* try to achieve an exact, mathematical enlargement):

- click on the top-right corner of the black house
- drag across the page by a convenient amount
- now drag up the page by a convenient amount.

Ask the class to vote on whether, for its new width, the expanded house is too short, too tall or about right.

Ask one or two students to justify their choice, but leave the answer unresolved (don’t, for example, calculate height/width - that can come later).

Repeat two or three times.

How well do students judge the expanded versions?

What kinds of argument and what mathematical language do they use to support their decisions?

Note: it is worth practising with the software before using it with the class. Make sure it is possible to stretch the house in one direction at a time, ie horizontally then vertically. [If this is not possible, ensure that the ‘lock aspect ratio’ option is unchecked under the ‘format picture’ menu. Alternatively, stretch the house using the middle handle on the right-hand edge, then the middle handle on the top edge.]

Multiplicative Reasoning 5A

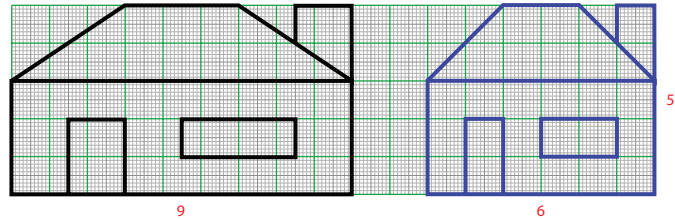
Lesson

8A**Expanded house**

The black house (left) is a stretched version of the blue house (right).

The black house is now to be stretched vertically, until it is exactly the same shape as the blue house.

How tall will the black house be?

**Summary**

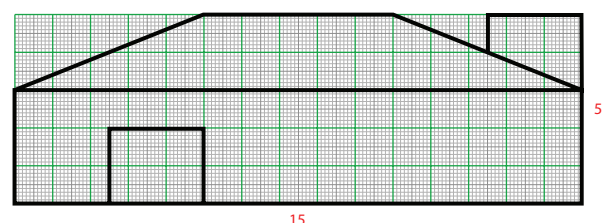
Here we introduce the notion of stretching which is taken further in Lesson 11A (*Stretched ruler*) and the 11AB Mini-assessment (*Elastic strip*), and in the work on enlarging in Lessons 24A (*Tangram*) and 24B (*Stars*).

We use the same drawing of a house, and a similar file (**HOUSE-graphpaper.docx**) as in the current Mini-assessment (*Stretched house*). We examine the effects of a horizontal stretch and use this to help us create houses which have 'exactly the same shape' as (ie are *similar* to) the 6 units by 5 units house.

Outline of the lesson

- Find the desired height of the stretched house.
 - Present the above task. Let students discuss it in small groups.
 - Record suggested heights on the board.
 - Use the file **HOUSE-graphpaper.docx** to display two or three of the proposed houses in turn. Ask the class to vote on the houses. Ask individual students to justify their solutions.
 - Examine one or two of the proposed houses in detail. Find features that seem to support or refute the claim that the house is 'exactly the same shape' as the blue house.
- Examine the original stretch. Record the results additively.
 - Compare horizontal distances on the two given houses. Record results as additive mappings, like this. → Draw out the fact that *all* horizontal distances of a given length are stretched the same amount.
 - Express the stretch *multiplicatively*.
"Can I find the stretched distances by multiplying?"
 - Imagine applying the same stretch vertically.
"What happens to the height of the house?"
"What can you say about the new shape of the house?"
- Perform another stretch.
 - Use the file **HOUSE-windowless.docx**. Present students with another stretch (eg where the width of the house becomes 15 units long). Ask students to draw the missing window.
 - Imagine applying the same stretch vertically.
"What happens to the height of the house?"
- Find other pairs of stretches that preserve shape.
 - Pose problems of this sort (or ask students to do so). →

Width of house:	6	→ ⁺³	9
Width of door:	1	→ ^{+0.5}	1.5
Width of window:	2	→ ⁺¹	3
Length of roof ridge:	2	→ ⁺¹	3
Width of chimney:	1	→ ^{+0.5}	1.5
Gap between wall and door:	1	→ ^{+0.5}	1.5
Gap between door and window:	1	→ ^{+0.5}	1.5



The black house is stretched until it is 20 units wide. What is its height if it is the same shape as the blue house?

Multiplicative Reasoning 5A

Lesson

8A

Overview

Mathematical ideas

Here we explore what it would take for different drawings of a house to be ‘exactly the same shape’ (ie ‘similar’) by considering geometric properties of the shape and by focussing on the ‘uniform’ effects of a one-way-stretch.

Rather than using a simple shape such as an ‘empty’ rectangle, we have chosen the drawing of a house because it is more figurative and contains ‘internal’ features (the door and window) which make the properties of a stretch (and an enlargement) more visible. We start with a stretch involving a scale factor of $\times 1\frac{1}{2}$ and then a scale factor of $\times 2\frac{1}{2}$.

We look at a horizontal stretch and start by analysing it *additively*. This gives students the chance to observe that *all* horizontal distances of a given length are increased by the same amount but that (in contradiction to the addition strategy) this amount depends on (indeed, is proportional to) the value of the length. Further, students might discover that this property can be expressed *multiplicatively*: all horizontal lengths are multiplied by the same scale factor.

We then consider the effect of applying identical horizontal and vertical stretches, and use this idea to transform the drawing of a house while preserving its shape (ie to perform an *enlargement*).

Students’ mathematical experiences

Students may discover some of the following

- a stretch stretches distances uniformly
- a stretch multiplies all distances in the direction of the stretch by the same factor
- there are lots of geometric cues that can help us determine whether two shapes are the same (ie similar).

Assessment and feedback

Think about the quality of students’ arguments during the Mini-assessment: how sensitive were they to key geometric properties of the shape and how rich was their mathematical language? Be prepared to spend a substantial amount of time on Stage 1 of the lesson, and look out for students who can help you bring out geometric features and introduce richer mathematical language. Even if students seem able to cope successfully with the initial task in Stage 1 (which involves a scale factor of $\times 1\frac{1}{2}$), it can still be useful to spend time analysing and describing the effects of such a scale factor.

Key questions

What happens to horizontal distances?

What rule best describes how the distances change?

Describe properties of the house that might help us compare the shape of different versions.

Adapting the lesson

You should make use of the files **HOUSE-graphpaper.docx** and **HOUSE-windowless.docx** if at all possible.

However, you also need to find the right moments to do so: seeing a stretch as it happens, and seeing the result, both provide powerful feedback, but it is also important for students to be given the opportunity to *predict* what such a stretch might look like. Students would also benefit from using the software files themselves, eg as a subsequent homework activity. If that is not practicable, we present some tasks in the Revisits section that could be done at home (or in class).

In Stage 3 of the lesson, you might want to provide students with a worksheet showing the outline of the stretched house, to save them from having to draw this themselves; the work can be made more challenging by omitting the image of the door.

You might want to repeat some stages of the lesson using a different scale factor. You could also, in this or a future lesson, represent the dimensions of various expanded houses on a Cartesian graph - houses that are similar will have points that lie on a straight line through the origin.

Multiplicative Reasoning 5A

Lesson

8A**Outline of the lesson (annotated)**

1. Find the desired height of the stretched house.

- Present the above task. Let students discuss it in groups.
- Record suggested heights on the board.
- Use the file **HOUSE-graphpaper.docx** to display two or three of the proposed houses in turn. Ask the class to vote on the houses. Ask individual students to justify their solutions.
- Examine one or two of the proposed houses in detail. Find features that seem to support or refute the claim that the house is 'exactly the same shape' as the blue house.

If students' explanations during the Starter activity are rather halting or vague, it is worth spending lots of time on this part of the lesson. One approach would be to cycle between small-group and whole-class discussion, and to draw on those students who identify salient mathematical features and use rich mathematical language.

It is worth discussing lots of features, even if students have found the height of the house successfully.

The features might include:

- the door and window should be congruent
- the front wall should be composed of two squares
- the roof should slope at 45 degrees
- the gaps between door and wall, door and window, window and wall, top of window and roof, should all be the same
- the heights of the roof and door should be the same.

2. Examine the original stretch. Record the results additively.

- Compare horizontal distances on the two given houses. Record results as additive mappings, like this. Draw out the fact that *all* horizontal distances of a given length are stretched the same amount.
- Express the stretch *multiplicatively*.
"Can I find the stretched distances by multiplying?"
- Imagine applying the same stretch vertically.
"What happens to the height of the house?"
"What can you say about the new shape of the house?"

Each stretched horizontal length is 'half as much again' as the original length, or $1\frac{1}{2} \times$ the original length. Thus the 'stretch factor' is $\times 1\frac{1}{2}$.

Refer back to the gap between the red and blue lines on the *Elastic strip* (Starter 3AB) and to 'MADE IN ENGLAND' on the *Stretched ruler* (Lesson 3A). "What happened when they were stretched?"

The new height will be $5 \times 1\frac{1}{2}$ units or $7\frac{1}{2}$ units. The shape will be the same as (ie similar to) the original house.

3. Perform another stretch.

- Use the file **HOUSE-windowless.docx**. Present students with another stretch (eg where the width of the house becomes 15 units long). Ask students to draw the missing window.
- Imagine applying the same stretch vertically.
"What happens to the height of the house?"

You might want to give out copies of the stretched, window-less house, for students to draw on.

To get a (rough and ready) sense of the stretch, students could be given a cut-open elastic band, with a 6 cm section marked to show the top of the door and window, like this:

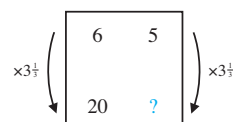
You can make the task more demanding by simply *stating* the new width of the house, rather than actually *showing* the stretched outline of the house and door.

Instead of stretching horizontally and then vertically, we could achieve the same result with a *single* stretch. "What would the direction be of this stretch?"

4. Find other pairs of stretches that preserve shape.

- Pose problems of this sort (or ask students to do so).
The black house is stretched until it is 20 units wide.
What is its height if it is the same shape as the blue house?

You might want to use a ratio table to represent and analyse the problems.



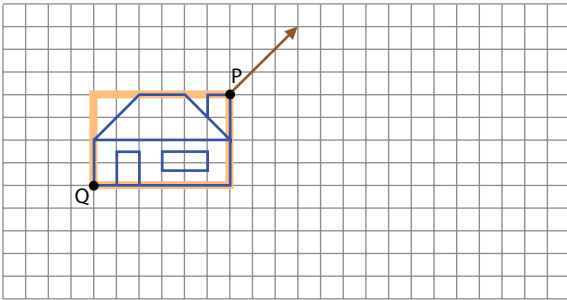
Multiplicative Reasoning 5A

Lesson

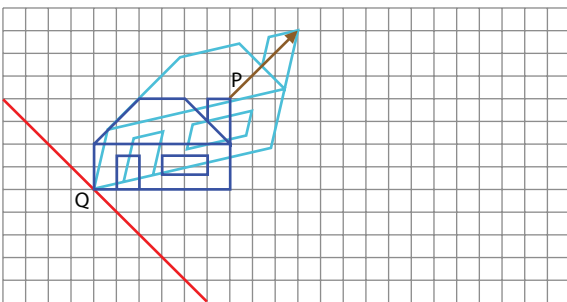
8A

Background Dragging and transforming

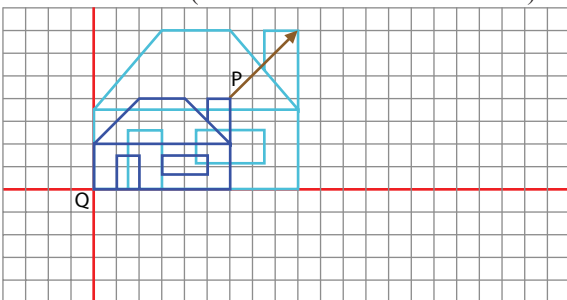
In most computer drawing packages (and more general programs like MS Word), an object such as a drawing of a house is bounded by a horizontally/vertically oriented rectangle as in the diagram below. When one corner (eg, point P) of the rectangle is dragged, the opposite corner (point Q) stays fixed.



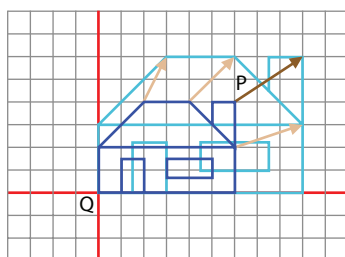
One might expect the resulting transformation to be a one-way-stretch, with an invariant line through the fixed point and perpendicular to the direction of the drag:



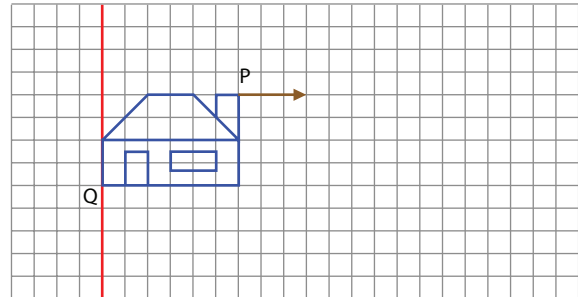
However, this is not normally the case. Instead, drawing packages are so designed that the drag results in two stretches, one horizontal (with a vertical invariant line), the other vertical (with a horizontal invariant line):



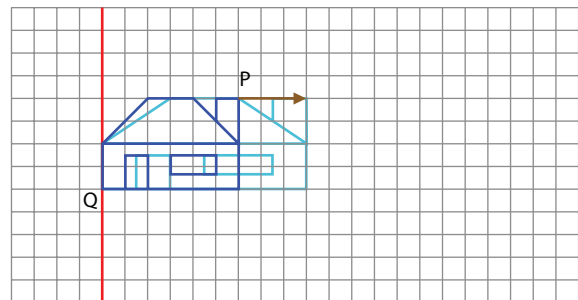
When the drag is in the direction of the bounding rectangle's diagonal (ie in the direction QP), the result is an enlargement with centre Q. Note here that effectively *all* points on the object are being dragged along lines that radiate from the fixed point.



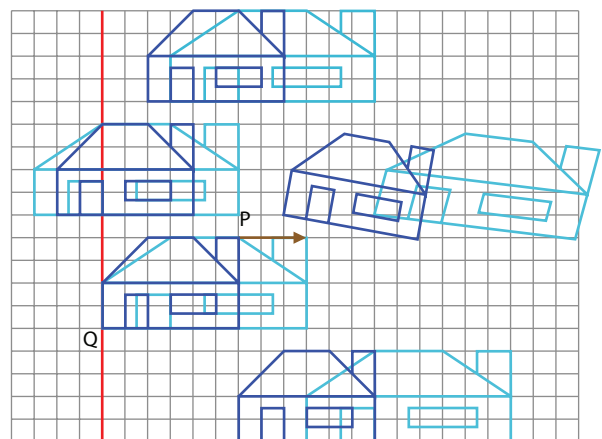
When a corner of the bounding rectangle is dragged horizontally (or vertically), this *does* result in a single one-way-stretch. [In essence, the second stretch, a vertical one in this case, has a scale factor of $\times 1$, ie it has no effect.]



Note, however, that the stretch only transforms the object within the bounding rectangle:



Mathematical transformations, including the one-way-stretch, transform the *whole plane*, ie all the points and objects on it. In the diagram below, all the objects (ie the dark blue houses) are identical. This means that all their stretched images (the light blue houses) will be identical too (with the exception of the slanting house), even though corresponding points on the objects (eg the points corresponding to P) may have moved by different amounts.



Note:

The left-hand column shows

1. a stretch with scale factor $\times 1.6$
2. a $\times 1.5$ horizontal stretch and a $\times 1.75$ vertical stretch
3. a $\times 1.5$ horizontal stretch and a $\times 1.5$ vertical stretch, ie an enlargement with scale factor $\times 1.5$.

The right-hand column shows a $\times 1.5$ horizontal stretch.

Multiplicative Reasoning 5B

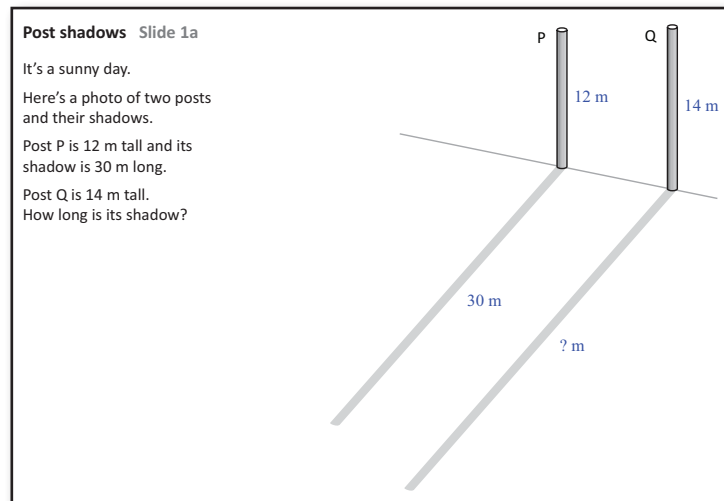
Lesson

8B**Post shadows**

Solve the problem on Slide 1a.

Depending on students' responses, move on to Slides 1b and 1c, or move on to Slide 2.

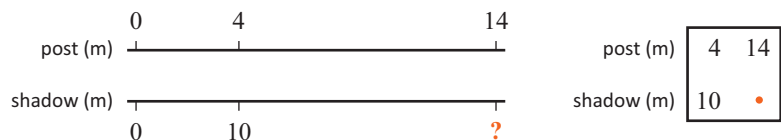
Investigate Slides 3a and 3b.

**Summary**

This lesson considers ratio in a fresh context, namely the length of shadows of vertical posts, cast by parallel rays from the sun. There is an emphasis on assessment: activities are selected from a range of tasks, in order to elicit additive and multiplicative strategies (and mixtures of the two) and to provoke discussion. Use is made of the DNL and ratio table (and perhaps graph) to explore strategies and models further.

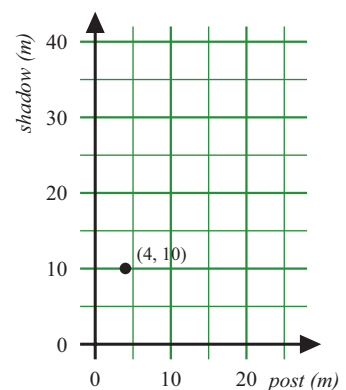
Outline of the lesson

- Show Slide 1a. Find the shadow of post Q.
 - Ask for quick responses, then for considered responses after discussion in pairs. Record answers.
 - Discuss students' methods.
 - If several students give the additive response 32 m, examine this further using Slides 1b and 1c.
 - If most students give the multiplicative response 35 m, challenge them with the problem on Slide 2.
- Show Slide 3a. Examine different methods.
 - Show Slide 3a. Ask for different ways to solve the problem. Try to elicit these methods:
 - rated addition* (see Slide 3b) and
 - scaling* (direct, and by using an intermediary post of length 2 m or 1 m).
- Show Slide 3a again. Examine methods using a DNL and ratio table.
 - Represent the given information using a DNL and ratio table. Use mapping arrows to show the various methods (see page 121).



- Extension: Show Slide 3a again. Represent information on a graph.

- Represent the given information (the 4 m post U has a 10 m shadow) as the point (4, 10) in the Cartesian plane.
- Plot the point for post V.
- Consider points for other posts. Describe the set of points.
- Compare how information is conveyed by the graph, DNL and ratio table.



Multiplicative Reasoning 5B

Lesson

8B

Overview

Mathematical ideas

In this lesson we consider ratio in another geometric context, this time involving shadows. By presenting students with appropriate variants of the basic task, students are likely on occasion to adopt (or at least consider) an additive or a mixed additive/multiplicative approach. The tasks include situations where such approaches come into conflict with normal expectations (eg by producing a shadow that is much too long) or with other salient relations (eg trebling), and students are encouraged to discuss their methods and solutions, to allow such conflicts to emerge.

A double number line and ratio table (and perhaps graph) are used to represent and examine methods more closely. The Mini-assessment focussed on one such method using the DNL: the multiplier that maps any number on the top line onto the number directly below. For the DNL used in the present context this is a *functional* multiplier which maps post-height onto shadow-length.

Students' mathematical experiences

Students may discover some of the following

- mathematical discussion can bring out the conflict between different ideas and can sometimes help one choose between them
- the relationship between a post's height and the length of its shadow is multiplicative, not additive
- we can use a variety of (related) methods to find the length of a post's shadow
- we can use a DNL and ratio table to represent and analyse the relationship between height and shadow length.

Assessment and feedback

Select slides (see page 6 and the file 8B-MR-5B-SLIDES.pdf) that provide an appropriate degree of challenge for your students. Information gleaned from the multipliers that students coped with in the Mini-assessment activity should help you in this, though the use of a new context (shadows) will add a degree of uncertainty.

Try to determine which students tend to see the relationship between height and shadow length as additive and which as multiplicative.

Identify students who can explain why the relationship is not additive. Is the DNL helpful here?

You might want to ask students to invent a problem like that on Slide 1a, say, but designed to be easier or harder.

Key questions

At what time of day is your shadow very short / very long?

How short / long can it get?

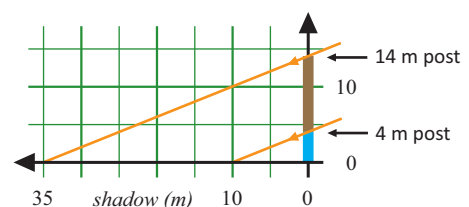
Imagine partitioning a post into equal vertical sections - what can you say about the shadow of each section?

Someone shorter than you is walking on (and in!) your shadow. At what point does their shadow become visible?

Adapting the lesson

As well as selecting specific slides that are appropriate for your class, you might, for a given slide, want to add posts with heights chosen (by you or the students) to make the work easier or more demanding. There is particular scope for this in the graphing activity in Stage 4. You might also want to repeat the lesson using a different context, to increase the range of contexts that students 'see' as multiplicative (see page 121 for some examples).

To analyse the mathematical relations in more detail, you might want to draw a *side elevation* for one of the slides, as here for Slide 3a. This can bring out the fact that the sun-rays are parallel and that any post and its shadow will be of different lengths (unless the rays are at an angle of 45°). You might also want to link this representation to the Cartesian graph (see page 121).



Multiplicative Reasoning 5B

Lesson

8B**Outline of the lesson (annotated)**

1. Show Slide 1a. Find the shadow of post Q.

- Ask for quick responses, then for considered responses after discussion in pairs. Record answers.
- Discuss students' methods.
- If several students give the additive response 32 m, examine this further using slides 1b and 1c.
- If most students give the multiplicative response 35 m, challenge them with the problem on Slide 2.

The answer 32 m comes from the *addition strategy*; it can arise in these two ways:

- Q is 2 m taller than P so its shadow is 2 m longer
- P's shadow is 18 m longer than its height, so Q's shadow is 18 m longer than its height.

If students can solve this task, which involves difficult numbers, it is still worth going on to analyse the simpler task on slide 3a.

2. Show Slide 3a. Examine different methods.

- Show Slide 3a. Ask for different ways to solve the problem. Try to elicit these methods:
 - *rated addition* (see Slide 3b) and
 - *scaling* (direct, and by using an intermediary post of length 2 m or 1 m).

Rated addition involves using the correspondence 4–10 to go to 8–20 then to 12–30 then, via 2–5, to 14–?

We can use the scalar multiplier $\times 3.5$ to map U's height directly onto V's height and U's shadow length onto V's shadow length; or we can use the functional multiplier $\times 2.5$ to map post height onto shadow length.

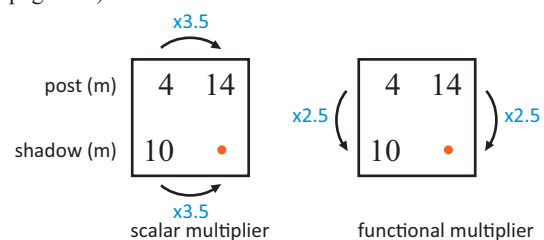
If we first find the shadow length of a 2 m or 1 m post we can use a whole number multiplier ($\times 7$ or $\times 14$) to find the shadow length of V.

To counter the tendency to think additively, compare posts U and P, whose heights (and shadow lengths) are in the easily discernable ratio 1 : 3.

3. Show Slide 3a again. Examine methods using a DNL and ratio table.

- Represent the given information using a DNL and ratio table. Use mapping arrows to show the various methods (see page 121).

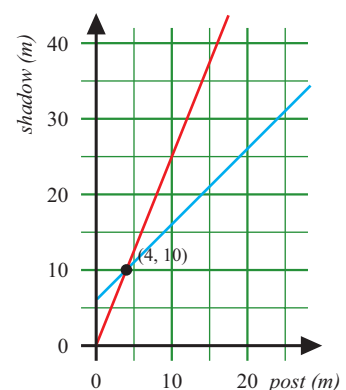
Here are examples of mapping arrows (more are shown on page 121). Similar arrows can be drawn for the DNL.



4. Extension: Show Slide 3a again. Represent information on a graph.

- Represent the given information (the 4 m post U has a 10 m shadow) as the point (4, 10) in the Cartesian plane.
- Plot the point for post V.
- Consider points for other posts. Describe the set of points.
- Compare how information is conveyed by the graph, DNL and ratio table.

- The points lie on a straight line (the red line, right).
 - What can we say about its slope?
 - Why must it go through the origin?
- You might like to consider points that would arise from using the addition strategy (the blue line, right).
 - Do they lie on a straight line? Why?
 - What is the line's slope? Why?
 - Does it go through the origin? Why not?



Multiplicative Reasoning 5B

Lesson

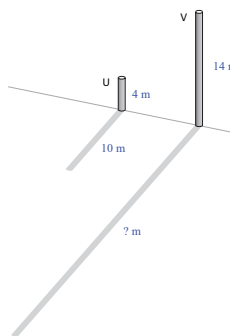
8B

Background

Using ratio tables

It is worth using ratio tables to make the various methods and numerical relations more visible and thus easier to talk about. The examples here are for posts U and V on Slide 3a.

Arrows can be marked on the DNL in a similar way.

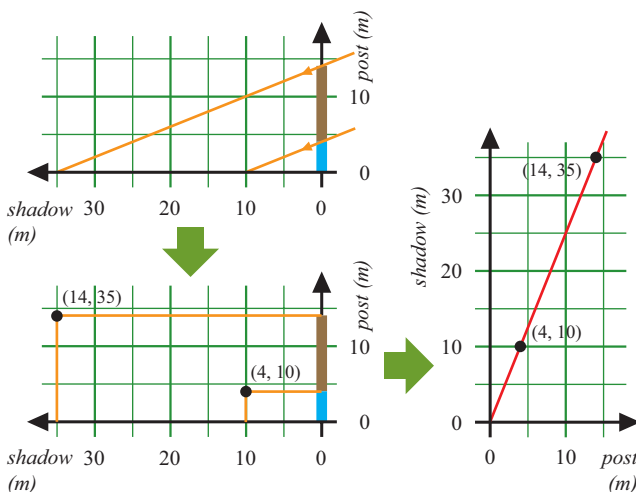


Addition strategy	
$\begin{array}{r} 4 \quad 14 \\ 10 \quad \bullet \end{array}$ $\begin{array}{l} +10 \\ +6 \\ +10 \end{array}$	$\begin{array}{r} 4 \quad 14 \\ 10 \quad \bullet \end{array}$ $\begin{array}{l} +6 \\ +6 \end{array}$
Rated addition	
$\begin{array}{r} 4 \quad 8 \quad 12 \quad 4 \div 2 \quad 14 \\ 10 \quad 20 \quad 30 \quad 10 \div 2 \quad \bullet \end{array}$	
Use of a rate (5 for every 2)	Unitary method (2.5 for every 1)
$\begin{array}{r} 4 \quad 2 \quad 14 \\ 10 \quad 5 \quad \bullet \end{array}$ $\begin{array}{l} \div 2 \quad \times 7 \\ \div 2 \quad \times 7 \end{array}$	$\begin{array}{r} 4 \quad 1 \quad 14 \\ 10 \quad 2.5 \quad \bullet \end{array}$ $\begin{array}{l} \div 4 \quad \times 14 \\ \div 4 \quad \times 14 \end{array}$
Scalar multiplier	Functional multiplier
$\begin{array}{r} 4 \quad 14 \\ 10 \quad \bullet \end{array}$ $\begin{array}{l} \times 3.5 \\ \times 3.5 \end{array}$	$\begin{array}{r} 4 \quad 14 \\ 10 \quad \bullet \end{array}$ $\begin{array}{l} \times 2.5 \\ \times 2.5 \end{array}$

Using graphs

You might want to relate the Cartesian graph of post heights and shadow lengths (below, right) to the *side elevation* of posts and their shadows (below, upper left).

This brings home the fact that though the graph is clearly and directly related to the physical situation it represents, it is an abstraction and not a direct 'photographic' portrayal.

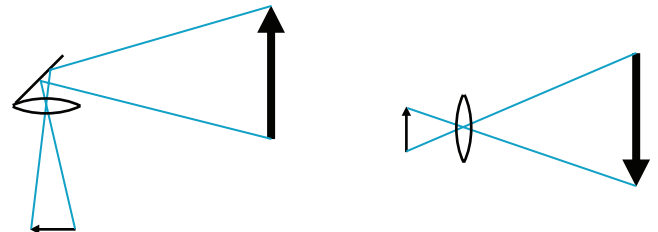


Using different contexts

Students might recognise that mathematical relationships are multiplicative in a task set in one context, but not in another. Thus, they might recognise a recipe task as being multiplicative, but resort to additive strategies in a task involving geometric enlargement.

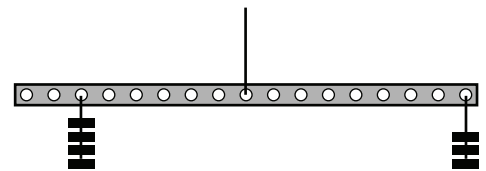
It is thus worth helping students to extend the range of contexts that they 'see' as multiplicative.

There are contexts that bear a relationship to 'shadows', for example the size of images produced by an OHP or by a slide projector. Here the independent variable could be *size of object* or *distance of projector lens* from the screen. It is worth bearing in mind that these contexts differ from 'shadows' in that the rays of light are not parallel.

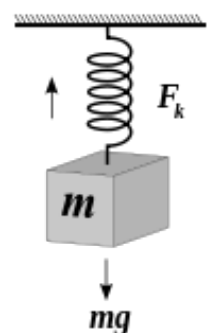


A context that might seem more remote from 'shadows'

is the see-saw balance, where the independent variable could be the left-hand mass or its distance from the fulcrum, and the dependent variable the right-hand mass or its distance from the fulcrum.



Hooke's law of elasticity provides another seemingly remote but interesting context. This states that for certain materials (eg a spring or an elastic strip) from which a mass is suspended, the *extension* of the material is proportional to the mass, within certain limits. Note here that if we consider the *total* length of the material, then we don't have a simple ratio relationship - put another way, the graph of total length against mass gives a straight line but not through the origin.



* From Wikimedia Commons [licensed under the Creative Commons Attribution Share Alike licence]

A physicist's view of Hooke's Law*

Lesson

8B**Background, continued****Slides 1a, 1b, 1c, 2, 3a, 3b (see page 123)****Slides 1a, 1b and 1c**

The addition strategy

An aim of the task shown in Slide 1a is to see whether students realise that the context involves ratio.

Students find it relatively difficult to recognise ratio relationships in a geometric setting, all the more so when these relationships can not be expressed as whole number multipliers.

The numbers in Slide 1a have been deliberately chosen so that the relationship between a post and its shadow involves a fractional multiplier, $\times 2\frac{1}{2}$. Similarly the relationship between the two posts (or their two shadows) involves a fractional multiplier, $\times 1\frac{1}{6}$.

(The first relationship is called a functional relationship, the second a scalar relationship.)

Students who do not see that the task involves ratio may well use the *addition strategy*, which leads to the conclusion that post Q's shadow is not 35 m but 32 m long (derived from $14\text{ m} + 18\text{ m}$, or $30\text{ m} + 2\text{ m}$).

Thus it is of interest to see what proportion of students give the answer 32 m. If this is high, you might want to go on to Slide 1b (or Slide 1c, where the 2 m section is brought out more explicitly). Here the use of the addition strategy leads to an answer of 20 m for post R (and the 2 m section of Q). This is far larger than the actual shadow length of 5 m, and may therefore prompt students to rethink their strategy.

Slide 2

Testing students' understanding of ratio

If students come up with the correct answer of 35 m on Slide 1a (and thus don't use the addition strategy), you might want to put their understanding of ratio to the test by going on to Slide 2. In this slide the sun has moved to a slightly lower elevation, causing the shadow of the 12 m post to lengthen by 1 m.

The (functional) relationship between post length and shadow length is now numerically even more complex ($\times 2\frac{1}{12}$ rather than $\times 2\frac{1}{2}$), which may lead some students to resort to the addition strategy and hence give an answer of 36 m ($35\text{ m} + 1\text{ m}$) or 33 m ($31\text{ m} + 2\text{ m}$, or $14\text{ m} + 19\text{ m}$) for the new shadow of post Q, rather than $36\frac{1}{6}\text{ m}$.

Slides 3a and 3b

Using rated addition and scaling

In Slide 3a, the relationship between the length of a post and its shadow is again given by the fractional multiplier $\times 2\frac{1}{2}$. However, the numbers here lend themselves quite strongly to a 'rated addition' approach, which can be summarised as follows:

every 4 m length of post has a 10 m shadow;
the 14 m post can be partitioned into these lengths:

$4\text{ m} + 4\text{ m} + 4\text{ m} + \text{half of } 4\text{ m};$

it therefore has a shadow of length

$10\text{ m} + 10\text{ m} + 10\text{ m} + \text{half of } 10\text{ m}$, which is 35 m.

Some students might use a hybrid strategy that includes an inappropriate additive element. For example, instead of adding 'half of 10 m' at the end of the above step, they might simply add 2 m. Put another way, some students might argue that the post's height, 14 m, is equal to $3 \times 4\text{ m} + 2\text{ m}$, so its shadow is $3 \times 10\text{ m} + 2\text{ m}$.

Rated addition is not as powerful or universal a method as scaling (eg by using the single multiplier $\times 2\frac{1}{2}$ in the case of Slide 3a); however, it is a more grounded method and may thus be more accessible to students with a propensity for the addition strategy. Thus if lots of students come up with 32 m on Slide 1a you might want to use Slide 3a with them at some stage (perhaps followed by Slide 3b, which emphasises the rated addition approach more strongly).

You might also ask students to compare post U on Slide 3a with post P on Slide 1a. Here the multiplicative relation between the post lengths (and between the lengths of their shadows) is relatively easy to spot, since it involves scaling by a whole number ($\times 3$).

Slides 4a and 4b

Using functional and scalar relations

Slide 4a and Slide 4b are cut-down versions of Slide 1a and Slide 3a, and you might want to show these before (or after) their full-grown versions. Slide 4a focusses attention on the relation between post height and shadow length, which is a *functional relation*, while Slide 4b focusses attention on the relation between two post heights, and hence later between their two shadow lengths, both of which are *scalar relations*.

It is important that students gain experience of both kinds of relation (and in a variety of contexts). Research suggests that students tend to prefer using scalar relations, although this might not be the case here, where there is a clear causal link between an object and its shadow.

The items below are on the *Mini Ratio Test*. We gave the items to comparable samples of secondary school students (mostly Year 8, $N = 77$ and $N = 74$). Item A turned out to be much easier than item B (with facilities of 91% and 51% respectively). This supports the conjecture that students prefer scalar relations, since this relation is much simpler for item A ($\times 3$) than for item B ($\times 2\frac{1}{11}$).

A

Ant is making a spicy soup for 11 people. He uses 25 ml of tabasco sauce. Bea is making the same soup for 33 people. How much tabasco sauce should she use?
--

B

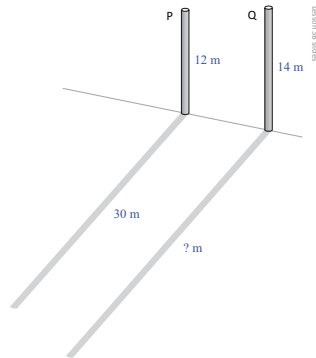
Ant is making a spicy soup for 11 people. He uses 33 ml of tabasco sauce. Bea is making the same soup for 25 people. How much tabasco sauce should she use?
--

Multiplicative Reasoning: Lesson 5B Post shadows (continued)

Slides 1a, b, 1c, 2, 3a, 3b, 4a, 4b (see file 8B-MR-5B-SLIDES.pdf for A5 versions of each slide)

Post shadows Slide 1a

It's a sunny day.
Here's a photo of two posts and their shadows.
Post P is 12 m tall and its shadow is 30 m long.
Post Q is 14 m tall.
How long is its shadow?

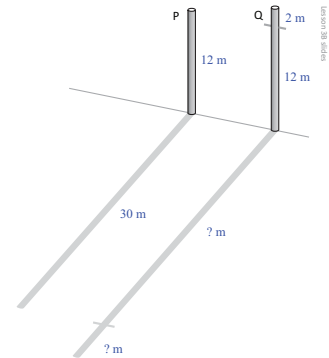


page 1

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Post shadows Slide 1b

A rod is fixed through post Q. It separates the post into a 12 m part and a 2 m part.
How long is the shadow of the 12 m part of post Q?
How long is the shadow of the 2 m part of post Q?



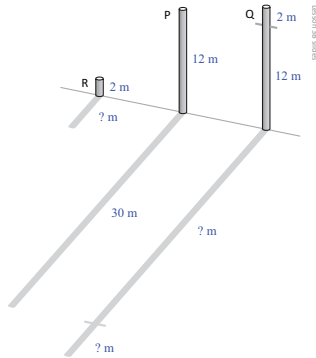
page 2

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Post shadows Slide 1c

Post R is 2 m tall.
How long is its shadow?

Think again about post Q.
How long is its shadow?



page 3

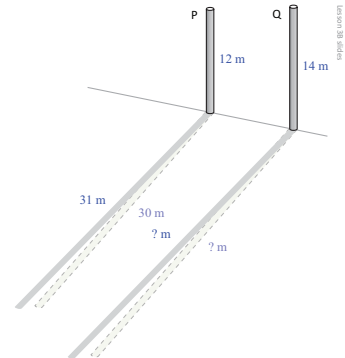
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Post shadows Slide 2

The sun has moved to a slightly lower angle in the sky.

The shadow of post P has lengthened by 1 m.

How long is the new shadow of post Q?

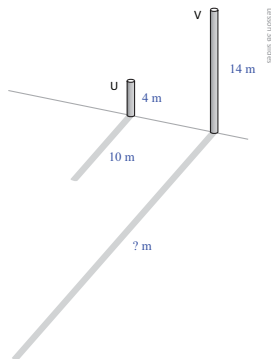


page 4

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Post shadows Slide 3a

It's a sunny day.
Here's a photo of two posts and their shadows.
Post U is 4 m tall and its shadow is 10 m long.
Post V is 14 m tall.
How long is its shadow?



page 5

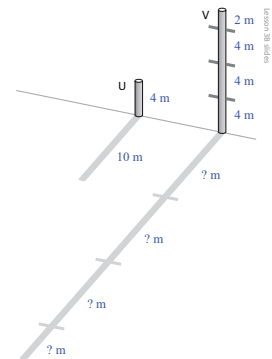
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Post shadows Slide 3b

Some rods are fixed through post V. They divide the post into four parts.

Find the length of the shadow of each part.

Find the total length of the shadow of post V.

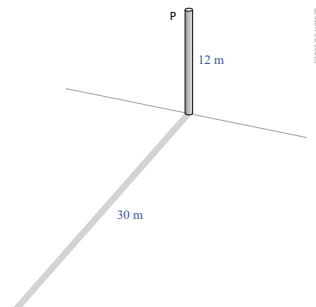


page 6

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Post shadows Slide 4a

It's a sunny day.
Here's a photo of Post P and its shadows.
The post is 12 m tall and its shadow is 30 m long.
How can we use this information to work out the shadow of a post Q, if we know the post's height?

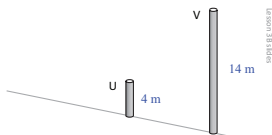


page 7

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Post shadows Slide 4b

It's a sunny day.
Here's a photo of two posts.
Post U is 4 m tall and Post V is 14 m tall.
How can we use this information to work out the shadow of Post V if we know the length of the shadow of Post U?



page 8

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Lesson

8B

Notes

Multiplicative Reasoning 6AB

Mini-assessment

9AB**Two fractions**

Look at the fractions.

Which is larger?

Justify your answer
in different ways.

$$\frac{1}{2}$$

$$\frac{3}{7}$$

Commentary

Students will have met fractions like these many times in the past, but it can be very illuminating to see how they deal with them now.

Do any students choose the wrong fraction ($\frac{3}{7}$)?

What different methods and representations do the students use?

How fluently can the students explain their methods?

Multiplicative Reasoning 6A

Lesson

9A**Comparing fractions****Look at the fractions.****Which is larger?**

$$\frac{1}{2} \qquad \frac{3}{7}$$

Summary

In this lesson students compare fractions. Fractions can be construed in many different ways (see page 129), but here we focus on a measurement and part-whole interpretation. Rather than working procedurally, students are encouraged to use number lines, rectangles and circles to represent and compare the fractions.

If the students have explored the initial $\frac{1}{2}$ v $\frac{3}{7}$ problem in depth during the Mini-assessment, then use a more challenging version (eg, $\frac{3}{4}$ v $\frac{8}{10}$).

Outline of the lesson

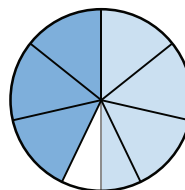
1. Which is bigger?

- Remind students of the Mini-assessment. Give students time (in pairs) to remember and to revisit some of the solutions.
- Discuss their solutions.
“Can you draw and describe your method on the board?”
“Can you say how the approaches are similar or different?”

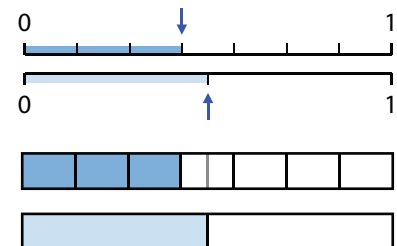
— You might need to remind students how to construct their own diagrams, emphasising equal partitioning.

2. Find the difference between $\frac{1}{2}$ and $\frac{3}{7}$.

- Use the Worksheet (page 130) or ask students to draw their own diagrams. Circulate to decide which students to ask to contribute first. Include some incorrect or partially correct solutions.
- Record students' methods on the board. Give them ‘think time’ and ask, “Do you agree with all the approaches?”



3.5 is a half of 7,
so a half is the
same as $\frac{3.5}{7}$.



How much is a half of a seventh?

3. Invent similar problems.

- Ask students to generate and then solve similar problems. Encourage them to invent easier and harder problems:
“Were all your ‘easier’ problems really easier?”
“Can you solve it in more than one way?”

4. Peer-assessment.

- Students swap and then attempt to solve each other's problems. (Think carefully about the groupings at this stage.)
- Ask students to give each other feedback on their solutions. Encourage them to state one thing that has been done well and to suggest one way to improve.

Multiplicative Reasoning 6A

Lesson

9A

Overview

Mathematical ideas

In this lesson, students use a variety of representations in order to develop a “feel” for fractions. The focus is principally on the measurement aspect of fractions by, for example, partitioning number lines to measure a distance from 0. The problem of “how big is the difference” relates this to part-whole comparisons that underlie equivalence.

We broaden our view in Lesson 9B by also considering decimal notation and percentages, while in Lessons 10A and 10B we focus on sharing.

Students' mathematical experiences

Students should

- use number lines and area drawings (circles and rectangles) to represent fractions
- construct their own diagrams as well as using pre-prepared diagrams (this allows them to make mistakes and perhaps overcome them)
- compare and discuss the different approaches
- invent and solve their own similar problems.

Assessment and feedback

Observe:

- how well the students use the various representations
- whether students who use formal methods can explain why they work
- the range and difficulty of the problems that students generate
- whether any students use fractions greater than 1.

In the final part of the lesson, students generate their own problems. This should help them understand the learning objective. Finally, they use a peer-assessment in which they give each other feedback:

- what makes a problem easier or harder?
- are there any problems that you could not solve?

Key questions

Can you explain how Ben (see page 131) drew the arrow in that position?

What's good about his answer?

How could he improve it?

Adapting the lesson

Use the Mini-assessment to adapt the lesson for your students. You might want to use a different pair of fractions as the starting point or to challenge students to come up with harder comparisons – see page 128 (bottom, right) on ways of making the numbers more or less challenging.

When introducing the peer assessment task you might want to discuss how to give feedback - you could use an example of students' work from page 131 for this purpose.

You could extend the lesson by asking students to find fractions *between* $\frac{1}{2}$ and $\frac{3}{4}$.

Multiplicative Reasoning 6A

Lesson

9A**Outline of the lesson (annotated)****1. Which is bigger?**

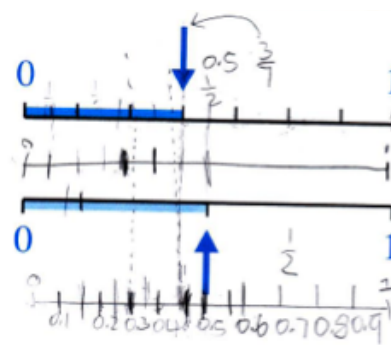
- Remind students of the Mini-assessment. Give students time (in pairs) to remember and to revisit some of the solutions.
- Discuss their solutions.
“Can you draw and describe your method on the board?”
“Can you say how the approaches are similar or different?”

If students suggest a formal computational procedure (eg ‘common denominator’), ask them to justify it, and try to ensure that they relate it to the various fraction representations that arise.

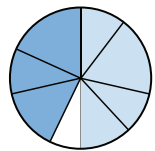
It can be a fruitful challenge to ask students to represent the fractions, especially $\frac{3}{7}$, on a number line marked in intervals of 0.1.

Some students’ work might become quite messy as they express their evolving ideas. You will need to judge whether the messiness is appropriate or not. The aim is for students to be able to use these diagrams when they encounter fractions in the future.

A messy but helpful diagram



A neat but less helpful diagram

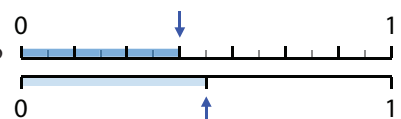
**2. Find the difference between $\frac{1}{2}$ and $\frac{3}{7}$.**

- Use the Worksheet (page 130) or ask students to draw their own diagrams. Circulate to decide which students to ask to contribute first. Include some incorrect or partially correct solutions.
- Record students’ methods on the board. Give them individual or paired ‘think time’ and ask, “Do you agree with all the approaches?”

For example, “ $\frac{3}{7}$ is like three $\frac{1}{7}$ pieces. If we split each of the pieces in half, this is the same as six $\frac{1}{14}$ pieces ...”

Some students might refer to decimals. Use a double number line to estimate the decimal equivalent of $\frac{3}{7}$.

If students use a story, ask them to draw a diagram to illustrate the story.

**3. Invent similar problems.**

- Ask students to generate and then solve similar problems. Encourage them to invent easier and harder problems:
“Were all your ‘easier’ problems really easier?”
“Can you solve it in more than one way?”

There are several factors that influence whether a comparison is more or less challenging, including:

- Very familiar fractions such as $\frac{1}{2}$ and $\frac{1}{4}$ make the task more straightforward because they can be used as reference points for comparison.
- Some fractions might be amenable to particular strategies: e.g. $\frac{5}{8} < \frac{7}{8}$ because there are fewer of the same-size parts, or $\frac{3}{5} > \frac{3}{7}$ because each of the parts is bigger.
- Smaller differences are less amenable to estimation strategies.
- Some fractions might be unfamiliar to the students, such as $\frac{3}{77}$ or fractions greater than 1.
- Students will find it easier to show some fractions on diagrams, eg halves, quarters, eighths, etc.

4. Peer-assessment.

- Students swap and then attempt to solve each other’s problems. (Think carefully about the groupings at this stage.)
- Ask students to give each other feedback on their solutions. Encourage them to say one thing that has been done well and to suggest one way to improve.

Multiplicative Reasoning 6A

Lesson

9A

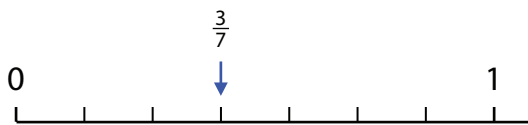
Background

The meaning of fractions

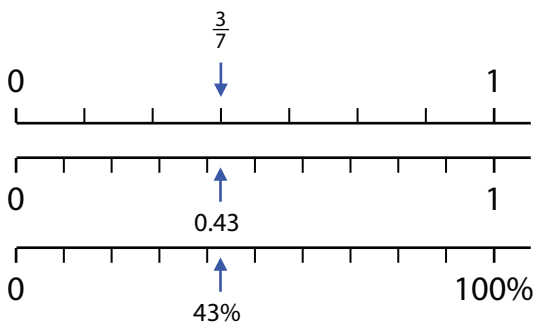
There are many different meanings and representations of a fraction such as $\frac{3}{7}$ which students need to grasp as they develop a feel for fractions. Some involve measurement, while others involve division (eg, sharing), part-whole comparisons, ratio or scaling.

Measurement

$\frac{3}{7}$ can be represented as a number on a number line:

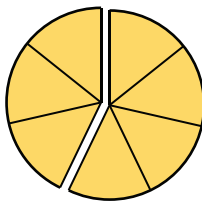


The number might also be represented as a decimal or percentage (see Revisit section):



Part-whole

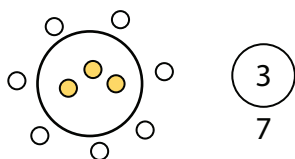
A pizza, say, is split into 7 equal parts. Three of the parts are $\frac{3}{7}$ of the pizza:



Division (sharing)

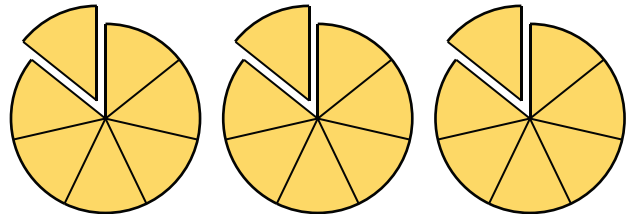
Three pizzas are shared equally between 7 people, as shown by these 'table' diagrams (see Lessons 6A and 6B):

Each person gets $\frac{3}{7}$ of a pizza, or $\frac{3}{21}$ or $\frac{1}{7}$ of the pizzas.



Division (computation)

The fraction $\frac{3}{7}$ can be thought of as $3 \div 7$, which can be evaluated as the recurring decimal 0.428571 .



Ratio

The fraction $\frac{3}{7}$ can represent a speed (eg 3 miles in 7 minutes) or some other rate (3 ice creams for £7).

Scale factor of an enlargement

The picture (below, left) has been enlarged by a scale factor of $\frac{3}{7}$ (below, right).



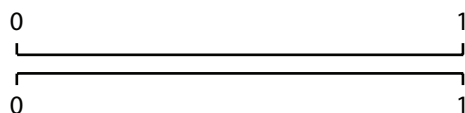
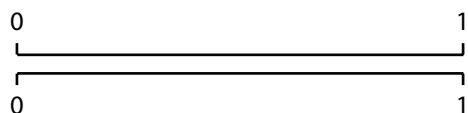
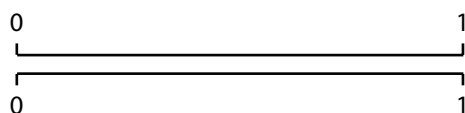
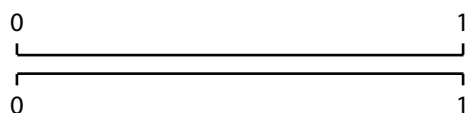
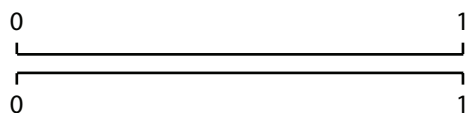
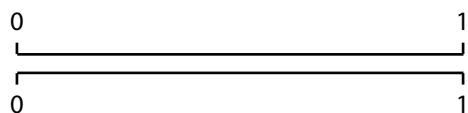
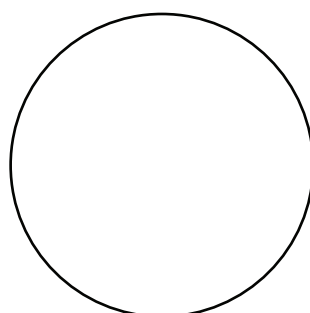
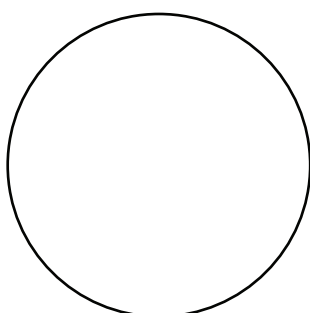
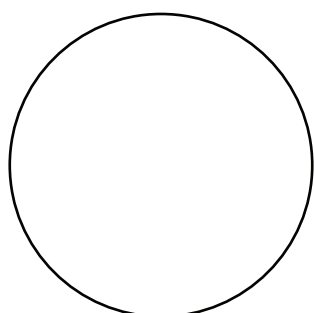
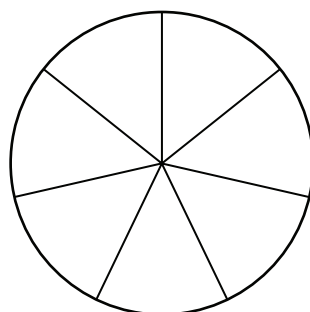
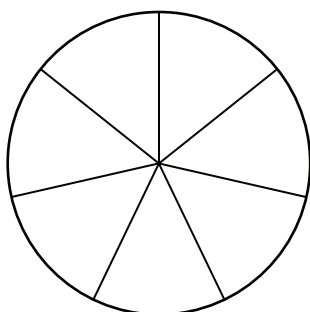
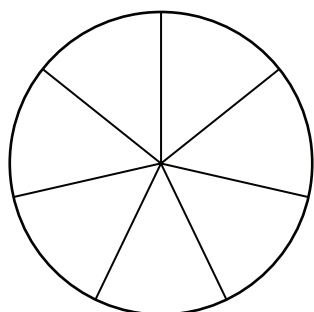
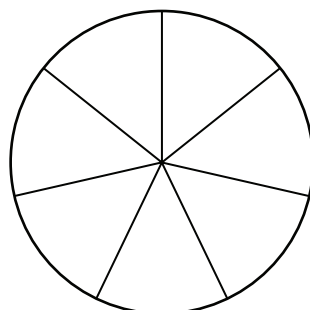
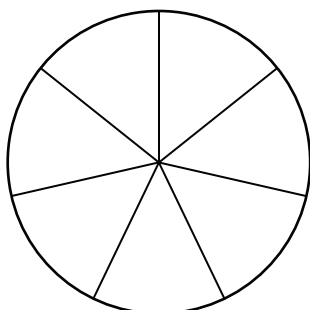
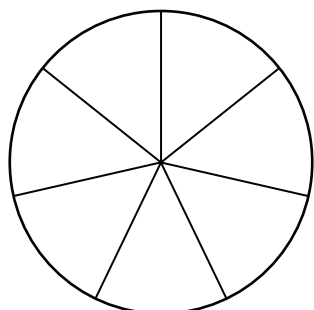
Some follow-up tasks can be found in the Revisits section.

Multiplicative Reasoning 6A

Lesson

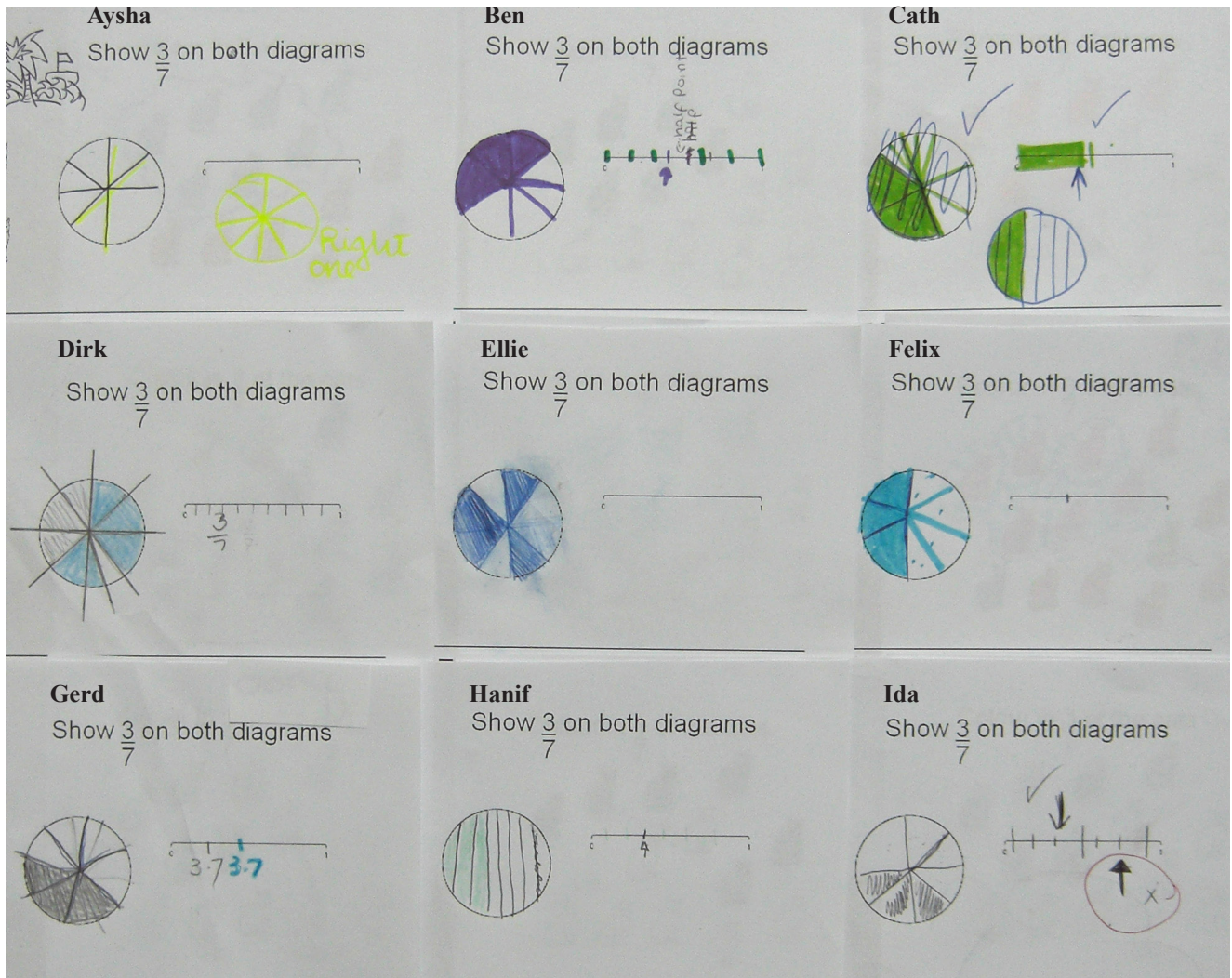
9A

Worksheet



Multiplicative Reasoning 6A

Lesson

9A**Examples for peer assessment**

Multiplicative Reasoning 6B

Lesson

9B**Ordering numbers**

Put these in order,
smallest first.

0.29 0.3 $\frac{1}{3}$ 3%

Summary

In this lesson students are asked to order a mixed set of numbers expressed as common fractions, decimals and percentages. We consider a variety of methods and representations that could be used to order the numbers.

You will need to decide on the numbers to use. For example, if the above numbers turn out to be very challenging for your students, you could present them with simpler and/or fewer numbers.

Outline of the lesson

1. Present a mixed set of numbers. Put them in order.

- Present a mixed set of numbers of your choosing.
“We want to order these, from smallest to largest.”
- Ask for some initial attempts to order them.
- Let students discuss the problem.
Note their considered solutions - is there consensus?

— For example, these
are easier numbers
(perhaps):

0.3 $\frac{1}{2.9}$ 0.29 $\frac{1}{0.3}$ 2.9%

And here is a more challenging set of numbers:

2. Harvest students' methods.

- Ask students to briefly explain their solutions.
Try to elicit a range of ideas and representations
and make a quick note of them on the board.

3. Discuss specific methods.

- It is likely that many of the ideas that
students use will overlap.
Try to identify key features and spend
time discussing *some* of these in turn.
Try to probe students' thinking.

— You could classify students' ideas in terms of
these three dimensions (see page 135):

- the nature of rational numbers
- ways of representing rational numbers
- ways of transforming rational numbers
(calculation procedures).

4. Peer-assessment.

- Ask students to invent easier/harder comparison
tasks involving just two numbers.
Ask them to solve each other's problems.
- Ask students to give each other feedback on their
solutions. Encourage them to say one thing that has
been done well and to suggest one way to improve.

— Write some of the resulting tasks on the board.
Ask the class to rank them in terms of difficulty.

Multiplicative Reasoning 6B

Lesson

9B

Overview

Mathematical ideas

In this lesson students compare a mixed set of rational numbers presented in the form of common fractions, decimals and percentages. Students are encouraged to consider a variety of interpretations and representations and to try to apply any chosen perspective that seems to fit one form to the other two forms.

(We compare sets of numbers again in Lesson 16B, but in a context this time - the 'sweetness' of breakfast cereals.)

Students' mathematical experiences

Students should be reminded that

- we can interpret rational numbers in a variety of ways
- we can represent rational numbers in a variety of ways
- we can interpret and represent a rational number in a particular way, regardless of whether it is in the form of a fraction, decimal or percentage.

Key questions

Is this number less than or more than a half?

Which number is closest to zero / one?

Can you think of a fraction / decimal / percentage that is bigger than 1?

Can you think of a fraction / decimal / percentage that lies between these two numbers?

How could we represent this number?

Assessment and feedback

Observe:

- How readily do students find ways to interpret and represent rational numbers?
- Do they focus on the 'complexity' of a number or can they pick out crucial features?
[“I can't possibly know what $\frac{1}{2.00001}$ is as a decimal” versus “ $\frac{1}{2.00001}$ is a tiny bit less than 0.5”.]
- Which interpretations and representations do they tend to use?
- Do they see the different representations as essentially showing the same thing?
- Can students justify their numerical procedures?
What explanations do they use?

Adapting the lesson

Choose numbers to compare that you think are suitable for your class, and be ready to modify them and to use more/ fewer numbers in the light of students' responses. You can make a task easier by including numbers that are very different in size (eg $\frac{1}{9}$ and 0.9, in contrast to, say, $\frac{1}{9}$ and 9%), or by including some familiar numbers like 0.25 that can serve as 'anchor points', or by using numbers that are near such 'anchor points' (eg $\frac{24}{50}$ is slightly less than a half).

It is also interesting to use unorthodox fractions, decimals or percentages, such as $0.3\frac{1}{2}$ or $\frac{1}{2.9}$ or $3\frac{1}{2}\%$, which can jolt us into thinking about the structure of these forms.

You might also want to probe some common misconceptions, such as these:

$\frac{1}{9} > \frac{1}{8}$ because $9 > 8$; $0.29 > 0.3$ because $29 > 3$; $0.31 < 0.3$ because 'the longer the decimal, the smaller the number'.

If students struggle to interpret and represent the numbers, then you might want to focus on a particular interpretation and representation. However, it can be illuminating to use and to be able to compare several interpretations and representations.

Multiplicative Reasoning 6B

Lesson

9B**Outline of the lesson (annotated)**

1. Present a mixed set of numbers. Put them in order.

- Present a mixed set of numbers of your choosing.
“We want to order these, from smallest to largest.”
- Ask for some initial attempts to order them.
- Let students discuss the problem.
Note their considered solutions - is there consensus?

— It can be helpful to have at least one ‘straightforward’ number, eg a ‘simple’ number close to 0, $\frac{1}{2}$ or 1 (eg $\frac{1}{20}$, $\frac{11}{20}$ and $\frac{19}{20}$ respectively); and perhaps include an ‘extreme’ number like $\frac{1}{2.00001}$, which might prompt students to estimate and to focus on key features.

— Encourage students to estimate - we don’t always need to be precise when ordering numbers.

2. Harvest students’ methods.

- Ask students to briefly explain their solutions.
Try to elicit a range of ideas and representations and make a quick note of them on the board.

— To help clarify the discussion, you might want to focus just on pairs of numbers.

3. Discuss specific methods.

- It is likely that many of the ideas that students use will overlap.

Try to identify key features and spend time discussing *some* of these in turn.
Try to probe students’ thinking.

— Try to draw explanations from the students rather than the explanations coming from you - though you might well want to reflect them back to the class in a clearer form, as well as probing further.

— You could classify students’ ideas in terms of these three dimensions (see page 135):

Rational number ‘subconstructs’:

- part-whole relationship
- ratio
- division (sharing)
- operator
- measure (number)

Representations or models, eg:

- area model (eg using a circle, rectangle, pizza)
- sharing model (eg ‘table symbols’)
- number line

Calculation procedures, eg:

- changing fractions into equivalent fractions (by multiplying the numerator and denominator by the same number)
- changing fractions into decimals (by dividing numerator by denominator)

4. Peer-assessment.

- Ask students to invent easier/harder comparison tasks involving just two numbers.
Ask them to solve each other’s problems.
- Ask students to give each other feedback on their solutions. Encourage them to say one thing that has been done well and to suggest one way to improve.

Multiplicative Reasoning 6B**Multiplicative Reasoning 6B**

Lesson

9B**Background****Coordinating perspectives on rational numbers**

Students might well think about common (or vulgar) fractions, decimals and percentages differently, ie they might have favourite perspectives for each. For example they might tend to think of fractions in terms of part-whole relationships (eg parts of a pizza), of decimals as numbers (perhaps marked-off on a number line), and of percentages as operators, ie ‘percentage of’ something.

Thus you might want to challenge students to apply a particular perspective to all three forms of rational number. “Sketch 0.3 of a pizza?”, “What is 0.3 of £20?”, “Where does $\frac{3}{7}$ go on the number line?”, “Mark 20% on the number line”.

It is likely that some interpretations (or subconstructs) of rational number, and some models and calculation procedures, will be drawn on more frequently by students than others. Focus on the more popular ones in this lesson. It is not necessary (or feasible) to cover them all; the important thing is to remind students that a variety of perspectives exist and that it can be useful to draw on some of these, as and when it seems appropriate, to support their thinking and to help them explain what is going on.

Rational number subconstructs**Part-whole relationship**

This is the common way in which fractions are introduced to students. Thus $\frac{3}{7}$ can be thought of as 3 of 7 equal parts of a pizza, say. Here it is important to be clear about what the unit is (one whole pizza of a certain size) and that the parts into which it is cut are equal. There is evidence to suggest that initially students find a sharing perspective (see below) more accessible.

Ratio

The fraction $\frac{3}{7}$ can be thought of as the ratio 3:7. In the context of pizzas, this might refer to 3 eaten pieces of 7 not-yet-eaten pieces or 3 eaten pieces out of 7 pieces to start with. Here the pieces have to be equal again, but the units do not. If $\frac{3}{7}$ of a small pizza has been eaten and $\frac{3}{7}$ of a large pizza has been eaten, then the *proportion* that has been eaten is the same, though the *amount* is not. If we are thinking in terms of ratio, we can not simply add the numbers.

Division (sharing)

We can think of $\frac{3}{7}$ as the instruction $3 \div 7$, which, if we carry it out, eg with a calculator, neatly turns the fraction into a decimal. Less formally, we can think of $\frac{3}{7}$ in terms of sharing, for example sharing 3 identical pizzas equally amongst 7 people. As with the part-whole relationship, the unit here is one whole pizza of a certain size (each person gets $\frac{3}{7}$ of a pizza). However, students might take the unit to be ‘all the things being shared’, with each person getting $\frac{1}{7}$.

Operator

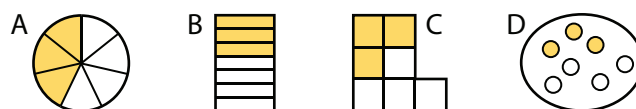
We tend to view percentages from the perspective of operators. This aspect also comes to the fore when we think of scaling, as in geometric enlargement.

Measure (number)

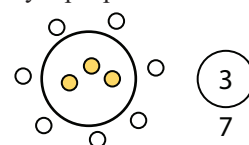
We can think of a rational number as measuring a quantity, where the unit might be 1 cm or 1 kg, say, or as a pure number, where the unit is simply 1. Nowadays we tend to express the number as a decimal.

Rational number representations/models**Part-whole model**

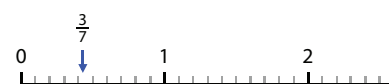
The part-whole view of rational number is commonly represented in terms of area, as in cases A, B and C for $\frac{3}{7}$, below, where the whole shape is regarded as the unit. However, we can also use discrete elements, as in D, where all 7 elements make one unit.

**A sharing model**

We can think of rational numbers in terms of division or ‘sharing’. We can model this by a diagram like the one below (left) for $\frac{3}{7}$, which can be thought of as, say, 3 identical cakes being shared equally by 7 people seated around a table. In turn, we can represent this schematically by what Streefland calls a ‘table symbol’ (far right). This model is explored in lessons MR-6AB.

**The number line**

We can represent rational numbers on a number line. This is a versatile model that one would expect students to make progressively more use of. Here care has been taken to use 1 as the unit, not the length of any given line segment.



It can be a fruitful

challenge to represent $\frac{3}{7}$ on a number line marked off in tenths rather than sevenths.

Calculation procedures

Many students will know that they can form equivalent fractions by multiplying the numerator and denominator by the same number. Similar rules apply elsewhere in mathematics (eg ‘do the same to both sides’ for transforming equations) and it is easy to overgeneralise the rule, as here: $\frac{3}{7} \equiv \frac{5}{9}$ (add 2); $\frac{3}{7} \equiv \frac{9}{49}$ (square). It is important to ask students to explain why they think the rule works so that mathematics is not reduced to unsubstantiated (mindless) procedures.

Changing fractions into decimals by dividing the numerator by the denominator provides a very efficient way of comparing fractions and decimals. The sharing model of rational number might help students conceptualise this. It is also worth going the other way, ie representing decimals as common fractions.

Lesson
9B

Notes

Multiplicative Reasoning 7AB

Mini-assessment

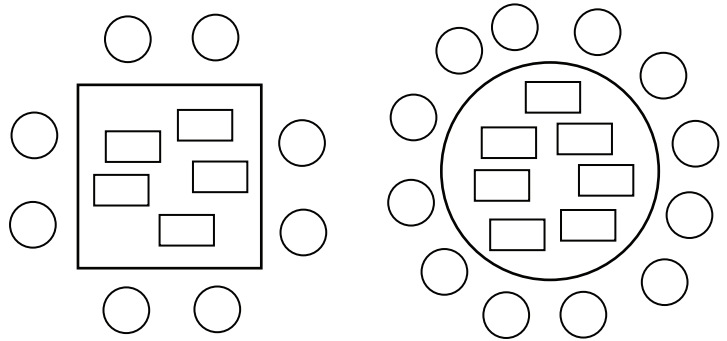
10AB**Sharing chocolate**

The square table has 5 bars of chocolate which are being shared by 8 people.

The round table has 7 bars of chocolate which are being shared by 12 people.

Which table produces the bigger share?

Find different ways to justify your answer.

**Commentary**

This Mini-assessment involves thinking of fractions in terms of sharing.

What methods do students come up with for comparing the shares? Try to elicit lots of ways.

Do students make use of the context, for example in these kinds of ways:

splitting the chocolate bars (eg, “On both tables, each person gets half a bar, plus a piece of the bar left over which will be a larger piece on the square table”)

creating ‘equivalent’ tables (eg, “5 bars shared by 8 people is the same as $2\frac{1}{2}$ bars shared by 4 people, or $7\frac{1}{2}$ bars shared by 12 people”)

adding chocolate and people (eg, “If 4 people with only half a bar each join the square table, we get the round table”)

Do students use fractions to represent the situation?

If so, do they use formal methods to compare them, eg by using the common denominator 24 ?

And if so, what explanations can they give for why this works?

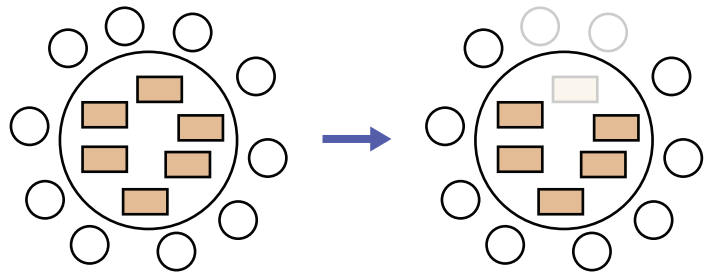
Two related activities are shown in the Lesson 10A Revisits section.

Multiplicative Reasoning 7A

Lesson

10A**Sharing chocolate bars**

Eight girls and two boys are sitting at a table.
 They want to share six bars of chocolate.
 The two boys run off with one of the bars.
 So there are now 8 girls with 5 bars.
 Does this leave the girls with
 more, the same, or less chocolate each?

**Summary**

The aim of this lesson is to increase students' familiarity with a *sharing* interpretation of fractions, using the context of sharing bars of chocolate, and using the model of people seated around a table bearing the chocolate. Students are asked to find different ways of sharing the chocolate equally.

The emphasis is on finding methods based on the table arrangements model, rather than on the formal manipulation of fractions, though the lesson provides plenty of scope for linking the two.

Outline of the lesson

- Discuss the given task.
 - Students might be quick to see that each girl's share increases (albeit not by very much).
Ask for a *range* of explanations; focus on the context.
- Sharing the chocolate.
 - "How could the 8 girls share the 5 bars equally? Find different methods."
Ask students to work on this in groups, then discuss with the class.
- Sharing by halving.
 - Discuss this scenario (right):
 - What would happen if instead of halving, they always 'quartered' instead?
 - What would happen if they always cut things into 3 equal pieces?
- Sharing pieces of chocolate.
 - Present this scenario (right):
What fraction of this **one** bar does each person get? (one eighth)
 - If all 5 bars are shared in this way, how much does each girl get?

The 8 girls decide to halve each of the 5 bars.
 They take a half-bar each.

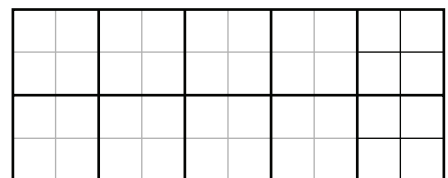
They then halve the remaining pieces and keep halving until there are enough pieces to go round.

How much chocolate does each girl end up with?

The girls take one bar and break it into 10 pieces.

They take a piece each.

They then halve the remaining pieces and keep halving until there are enough pieces to go round.



Multiplicative Reasoning 7A

Lesson

10A

Overview

Mathematical ideas

Here we look at the notion of fraction as *distribution* or *sharing*. In particular we use the model of people sharing items while seated around a table.

Stages 1 and 2 of the lesson allow students to explore the sharing model. This shouldn't be rushed. Stages 3 and 4 focus on specific methods of sharing, while Stage 5 allows students to consolidate ideas. The Mini-assessment activity should help you anticipate how much encouragement and time students might need to explore the model (Stages 1 and 2), and how readily they might pick-up on the specific methods (Stages 3 and 4).

The emphasis is on *using* the model to find different (but equivalent) ways of sharing (for example by using the process of repeated halving). Thus the approach is very grounded. However, most students will already have met formal notation for fractions and formal procedures for operating with them, and so the lesson gives students the opportunity to make links between their formal knowledge and the model. In this way the lesson counters the tendency of some students to abandon models as they are introduced to more formal mathematics.

Students' mathematical experiences

Students might discover some of the following

- it is possible to compare shares qualitatively (ie without finding their precise value)
- there are lots of practical ways of finding the value of an equal share
- the result is the same whether we share things out one at a time or all at once
- we can use the sharing model to explain (or at least illuminate) formal procedures.

Assessment and feedback

The Mini-assessment activity should indicate how readily students pick up on and are willing to use the sharing-chocolate-around-a-table model. In this lesson, you might want to remind the class of fruitful responses to the activity made by particular students.

Do students understand the model?

Do/can students come up with a variety of sharing methods?

Do students use formal methods? If so, can they relate them to the sharing model?

Key questions

Why does the girls' share get bigger?

Why is $\frac{1}{10}$ of a bar the same as $\frac{4}{40}$ of a bar?

Can we share the chocolate equally in a different way?

Adapting the lesson

The difference in the size of share when the boys are present and absent is very small. You might thus want to alter the numbers to make the change more obvious. However, it is useful to end up with 8 people as this lends itself to the repeated halving strategy. Thus you could start with one boy instead of two, who then runs off with the one bar all to himself. Or start with 7 bars rather than 6 and have the 2 boys run off with a bar each.

It is worth spending a few minutes yourself trying out different combinations - you may find that it is not that easy to avoid situations such as '8 people with 4 bars', where the sharing process becomes trivial!

You might want to use the *Sharing water* activity in the Revisits section as an extension activity or for a further lesson.

Multiplicative Reasoning 7A

Lesson

10A**Outline of the lesson (annotated)**

1. Discuss the given task.

- Students might be quick to see that each girl's share increases (albeit not by very much).
Ask for a *range* of explanations; focus on the context.

2. Sharing the chocolate.

- "How could the 8 girls share the 5 bars equally? Find different methods."
Ask students to work on this in groups, then discuss with the class.

3. Sharing by halving.

- Discuss this scenario (halving):

- What would happen if instead of halving, they always 'quartered' instead?
- What would happen if they always cut things into 3 equal pieces?

4. Sharing pieces of chocolate.

- Present this scenario (tenths):
What fraction of this **one** bar does each person get? (one eighth)
- If all 5 bars are shared in this way, how much does each girl get?

5. The boys return with their bar of chocolate.

- Imagine that the boys return with their chocolate. So we have 10 people and 6 bars. Find ways of sharing the chocolate equally.
- Compare the resulting share with the girls-only share. Explain why it is smaller. How much smaller is it?

For example:

— "The boys go away with a half-bar each, which is a bit less than their share would have been, which leaves a bit more for the girls."

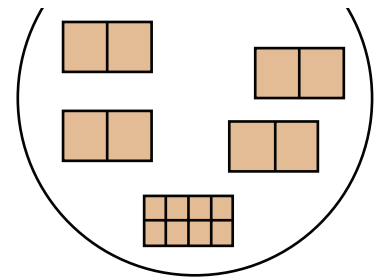
"If everyone was given half a bar, there'd be 1 bar left to share. The boys leave their share of this to the girls."

"Everyone would have got half a bar plus a tenth; now the girls get half a bar plus an eighth."

"6 for 10 is the same as 12 for 20; 5 for 8 is the same as 12½ for 20."

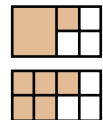
— Students might come up with methods like those in Stages 3 and 4 of the lesson.

Some might suggest simply splitting each bar into 8 equal pieces, making $\frac{5}{8}$ of a bar each. However, some might interpret this as 5 pieces out of 40 pieces, making *one* eighth each. Care needs to be taken here: what is the unit?



— We could represent the halving process like this. →

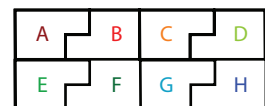
Each person gets one half and one eighth of a bar,
or 4 eighths and 1 eighth, ie 5 eighths,
ie each person gets $\frac{1}{2} + \frac{1}{8} = \frac{4}{8} + \frac{1}{8} = \frac{5}{8}$.



— Each person gets $\frac{1}{4} + \frac{1}{4} + \frac{1}{8} = \frac{2}{8} + \frac{2}{8} + \frac{1}{8} = \frac{5}{8}$.

— Here we enter the realm of 'recurring fractions'!
Each person gets $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \dots$

— Each person gets one eighth of the one bar. We can verify this with a drawing like this:



Formally, each person gets

$$\frac{1}{10} + \frac{1}{40} = \frac{4}{40} + \frac{1}{40} = \frac{5}{40} = \frac{1}{8}.$$

5 times one eighth is 5 eighths.

Or, if all the bars are first cut into ten pieces, this is $\frac{6}{10}$ each, with $\frac{2}{10}$ or $\frac{8}{40}$ left over, which is another $\frac{1}{40}$ each. Again, formally, $\frac{6}{10} + \frac{1}{40} = \frac{24}{40} + \frac{1}{40} = \frac{25}{40} = \frac{5}{8}$.

— The difference is actually very small: $\frac{1}{40}$ of a bar!

Multiplicative Reasoning 7A

Lesson

10A

Background

Fractions as sharing

The sharing context used in this lesson is likely to be very familiar, or at least meaningful, to students. It thus has the potential of providing students with a well grounded model of fractions.

An interesting feature of the model is that it readily evokes the fact that we get the same overall portion whether we share equally all the items in one go, or in batches, or one at a time. Thus we can imagine being in a pizza restaurant, say, where all the pizzas are served together, or they come in relays, as and when they are ready. Anyone who has eaten Flammkuchen (Tarte flambée) with a group of people in Alsace will have experienced the latter. Streefland* refers to this situation as 'French division'.



This phenomenon highlights the very different behaviours of the numerator and denominator: as each batch of Flammkuchen arrives, their number is added to the numerator, while the denominator stays the same. It thus embodies the fact that division is distributive over addition.

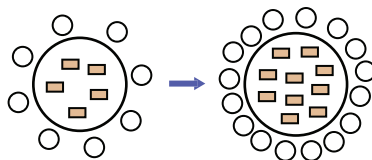
Note: the distributive law only works in one direction here: $(a + b) \div c = a \div c + b \div c$ but $c \div (a + b) \neq c \div a + c \div b$. so, for example,

$$\frac{5+10}{30} \text{ equals } \frac{5}{30} + \frac{10}{30} \text{ but } \frac{30}{5+10} \text{ does not equal } \frac{30}{5} + \frac{30}{10}.$$

Equivalent fractions

The sharing context also readily leads to the notion of equivalent arrangements, and hence to equivalent fractions.

8 girls sharing 5 chocolate bars each get the same amount as 16 girls sharing 10 bars.

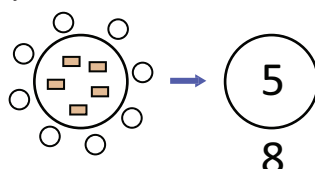


Similarly, if 24 people arrive at a restaurant wanting to share 9 pizzas, they get the same shares if they split into groups of 8 people at a table, each sharing 3 pizzas.

We look more closely at equivalent arrangements in Lesson 10B.

Symbolising fractions

It can be helpful to use 'table symbols' like these, alongside the schematic drawings of tables and people. It is a small step from this to our standard fraction notation.



Extension activity

We provide the outline of another activity/lesson, *Sharing water*, in the Revisits section, should you wish students to pursue the sharing model in more depth. It involves a similar sharing activity, but asks students to compare a larger number of arrangements, including several arrangements that are equivalent (something we look at in Lesson 10B). It also includes a graphing activity (something we also look at in Lesson 10B).

The Revisits section also contains some shorter and/or more straightforward sharing tasks (concerning apples).

* Leen Streefland, 1991. *Fractions in Realistic Mathematics Education*. Kluwer: Dordrecht.

Multiplicative Reasoning 7B

Lesson

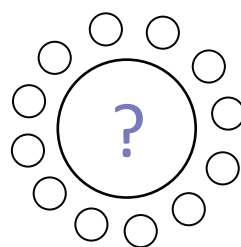
10B

Sharing pancakes

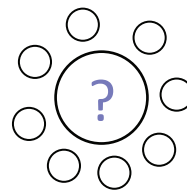
A group of 20 people has ordered 45 pancakes at Molly's Café.

The people don't all fit around Molly's big table, so 8 people sit at another table.

How many pancakes should Molly put on each table so everyone can have the same size share?



12 people



8 people

Summary

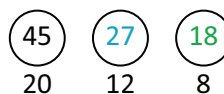
In this lesson we again consider a *sharing* interpretation of fractions using the model of table arrangements. The focus is on *equivalent* table arrangements and how they can be used to compare shares (of pancakes). We also consider how table arrangements, and the resulting shares, can be represented by a Cartesian graph and a double number line (DNL). The emphasis is again on methods based on the model, rather than on the formal manipulation of fractions, although there are again opportunities to link the two.

Outline of the lesson

1. Discuss the given task.

- Let students discuss the problem. Note some of their answers. Try to field a range of explanations, correct and incorrect. Don't push for a resolution if one isn't forthcoming at this stage.

— You might want to use 'table symbols' like these:



2. Other table arrangements for 20 people and 45 pancakes.

- Imagine other seating possibilities for the 20 people. For each possibility, find the table arrangements (of people and pancakes) that are fair.

— Include a variety of possibilities, like these:

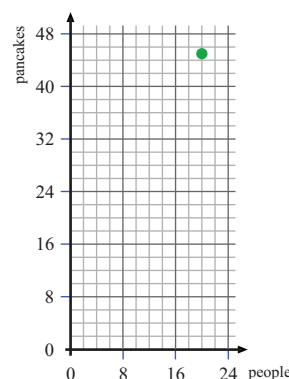
- two tables of 10 people (easy?)
- a table for 18 people and for 2 (challenging?)
- tables for 10, 5 and 5
- tables for 4, 4, 4, 4 and 4.

3. Comparing shares.

- In another room at Molly's, 9 people are sharing 4 pancakes and 12 people are sharing 5 pancakes. Each group shares them equally. Which group gets the larger shares?
- Think of some 'equivalent table arrangements' that would make it easy to compare the shares. Find lots of pairs.

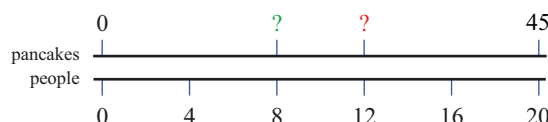
4. Represent arrangements on a Cartesian graph.

- Represent (number of people, number of pancakes) by points on a graph.
- Discuss how the graph can be used to compare shares for different table arrangements (eg those in Stage 2 or Stage 3 of the lesson).



5. Represent arrangements on a double number line.

- This DNL is for the original 45 pancakes and 20 people. Find the numbers represented by the question marks.
- Mark other numbers on the top line.
- Describe the relation between the numbers on the two lines.



Multiplicative Reasoning 7B

Lesson

10B

Overview

Mathematical ideas

In this lesson we use table arrangements to model a sharing interpretation of fractions, in the context of sharing pancakes. We create *equivalent* table arrangements and use these to compare the shares resulting from different arrangements. We also consider how the Cartesian graph and the DNL (and table symbols) can be used to represent the arrangements.

Students' mathematical experiences

Students might discover some of the following

- different table arrangements can produce the same size share
- we can compare shares by comparing carefully chosen *equivalent* table arrangements.
- we can represent arrangements in various ways, eg using table symbols, a Cartesian graph, a DNL.

Key questions

If some people move to another table, how many pancakes should they take with them?

Can you think of a smaller/larger table arrangement that would result in the same share as before?

Assessment and feedback

Along with the Mini-assessment and Lesson 10A, the first Stage of the lesson provides a good opportunity to assess students' understanding of the table-arrangements model. Orchestrate the discussion, and perhaps encourage the use of efficient notation such as table symbols, but allow students to present a range of ideas (to you and the class) with minimal guidance from you.

If there is widespread use of additive methods, then use Stage 2 of the lesson to 'confront' the class with arrangements involving simple multiplicative relations like halving, or splitting into, say, 5 equal groups.

Adapting the lesson

You can make the problems easier or harder by changing the numbers in the problems so as to modify the (functional) relationship between numbers of people and pancakes or the (scalar) relationship between numbers of people at different tables.

You might want to adjust the amount of time given to the four main Stages of the lesson: producing equivalent arrangements; using equivalent arrangements to compare shares; representing arrangements on a Cartesian graph; representing arrangements on a DNL.

Multiplicative Reasoning 7B

Lesson

10B**Outline of the lesson (annotated)**

1. Discuss the given task.

- Let students discuss the problem. Note some of their answers. Try to field a range of explanations, correct and incorrect. Don't push for a resolution if one isn't forthcoming at this stage.

— Give students 'space', ie use this Stage of the lesson to assess their ideas and understanding - then adjust your use of Stage 2 accordingly.

2. Other table arrangements for 20 people and 45 pancakes.

- Imagine other seating possibilities for the 20 people. For each possibility, find the table arrangements (of people and pancakes) that are fair.

— Students are likely to treat 'two tables of 10 people' multiplicatively: each table gets half of the 45 pancakes, rather than $45 \div 10$.

However, they might see 'a table for 18 people ...' as additive: 18 is $20 - 2$ so they get $45 - 2$ pancakes.

"Which view is right?"

3. Comparing shares.

- In another room at Molly's, 9 people are sharing 4 pancakes and 12 people are sharing 5 pancakes. Each group shares them equally. Which group gets the larger shares?
- Think of some 'equivalent table arrangements' that would make it easy to compare the shares. Find lots of pairs.

— Allow a free rein for this part of Stage 3:

eg, the use of division:

$4 \div 9 = 0.4444...$, $5 \div 12 = 0.41666...$;

or a part-whole view of fractions:

each person gets either $\frac{4}{9}$ or $\frac{5}{12}$ of a pancake, and $\frac{4}{9}$ is bigger than $\frac{5}{12}$ because it is nearer $\frac{1}{2}$...

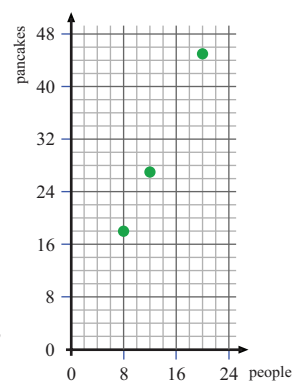
— eg, 36 people, 16 pancakes and 36, 15; or 3, $1\frac{1}{3}$ and 3, $1\frac{1}{4}$; or 18, 8 and 18, $7\frac{1}{2}$; or 1, $\frac{4}{9}$ and 1, $\frac{5}{12}$; etc.

Or 45 people, 20 pancakes and 48, 20; or $22\frac{1}{2}$, 10 and 24, 10; or $2\frac{1}{4}$, 1 and $2\frac{2}{5}$, 1; etc.

4. Represent arrangements on a Cartesian graph.

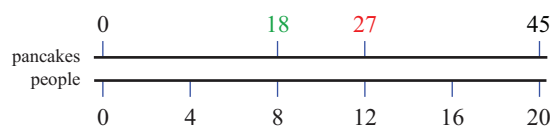
- Represent (number of people, number of pancakes) by points on a graph.
- Discuss how the graph can be used to compare shares for different table arrangements (eg those in Stage 2 or Stage 3 of the lesson).

— Points for equivalent arrangements lie on a straight line.
Why? Where does the line cut the axes?
What does its slope tell us?
How can we compare non-equivalent arrangements?



5. Represent arrangements on a double number line.

- This DNL is for the original 45 pancakes and 20 people. Find the numbers represented by the question marks.
- Mark other numbers on the top line.
- Describe the relation between the numbers on the two lines.



— The relation is *multiplicative*. We can express it in various ways; eg, if the number of people is x and the corresponding number of pancakes y , then

$$y = 2\frac{1}{4}x; x = \frac{4}{9}y; y \div x = 2\frac{1}{4}; x \div y = \frac{4}{9}.$$

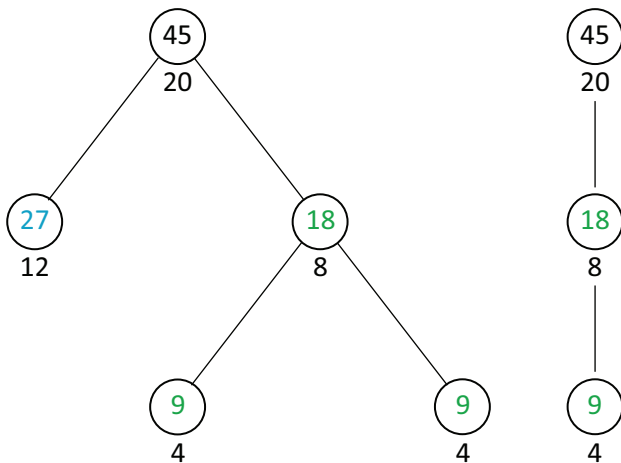
Multiplicative Reasoning 7B

Lesson

10B**Background****Table arrangements and tree diagrams**

Streefland* describes the use of tree diagrams to represent equivalent tables. For our original task this would produce a diagram something like this (below, left):

Streefland goes on to suggest (ibid, p78) that at some stage such a tree might be 'pruned back to no more than trunks, thereby producing a class of equivalent tables similar to ratio tables' (eg below, right).



* Leen Streefland, 1991. *Fractions in Realistic Mathematics Education*. Kluwer: Dordrecht.

'Adding' table arrangements

It is tempting to summarise the relationship between our original table arrangements in this sort of way:

$$\begin{pmatrix} 45 \\ 20 \end{pmatrix} = \begin{pmatrix} 27 \\ 12 \end{pmatrix} \oplus \begin{pmatrix} 18 \\ 8 \end{pmatrix}$$

And we might want to go a step further and use standard fraction notation (below). [We have used the symbol $\oplus$ instead of $+$ to show that this is not our normal form of addition.]

$$\frac{45}{20} = \frac{27}{12} \oplus \frac{18}{8}$$

This looks rather like the standard addition of fractions, but, of course, the statement $\frac{45}{20} = \frac{27}{12} + \frac{18}{8}$ is not true (though some students might think so, as we know from the occurrence of the classic error $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$).

This raises the question: can we invent a sharing story where, by observing certain constraints, the addition of table arrangements *does* model the addition of fractions?

Consider this scenario:

Weekend pizza (A)

At the weekend I ate pizza twice.

On Saturday 8 of us shared 5 large pizzas equally.

On Sunday 6 of us shared 2 large pizzas equally.

How much pizza did I have altogether?

We could symbolise the arrangements as fractions, then 'add' the arrangements:

$$\frac{5}{8} \oplus \frac{2}{6} = \frac{7}{14}$$

But what does this mean? Well, it can be interpreted as the average of all 14 shares. But it is not necessarily quite the average of 'my' two portions (and it is definitely not their sum, even though we are 'adding').

We could also transform the story like this without changing the size of my shares:

Weekend pizza (B)

At the weekend I ate pizza twice.

On Saturday 24 of us shared 15 large pizzas equally.

On Sunday 24 of us shared 8 large pizzas equally.

How much pizza did I have altogether?

So now we can argue along these lines:

Altogether the 24 of us shared 23 large pizzas equally, so we each had $\frac{23}{24}$ large pizzas over the weekend.

We can symbolise the arrangements in this version of the story as shown below. Note that here the $+$ sign *does* carry its usual meaning.

$$\frac{5}{8} + \frac{2}{6} = \frac{15}{24} + \frac{8}{24} = \frac{23}{24}$$

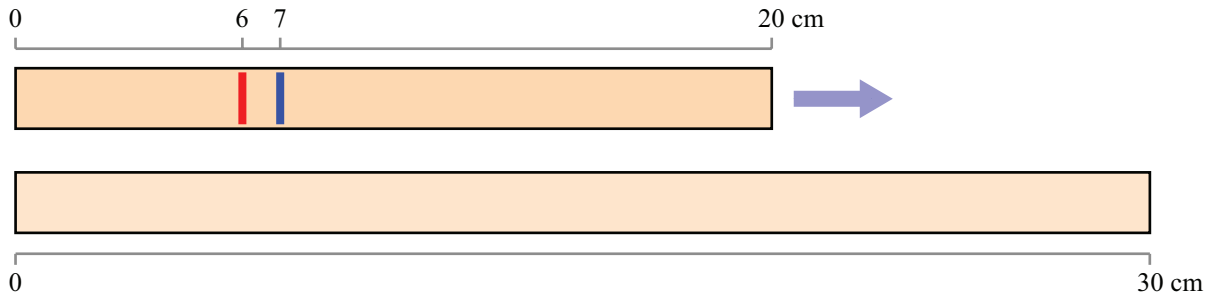
Lesson

10B

Notes

Multiplicative Reasoning 8AB

Mini-assessment

11AB**Elastic strip**

A strip of elastic is 20 cm long.

The red line is 6 cm from the left hand end.

The blue line is 7 cm from the left hand end.

The elastic is stretched until it is 30 cm long.

1. How far from the left end is the red line now?
2. How big is the gap between the red and blue lines now?
3. What happens to the middle of the elastic strip?

Commentary

Here is another context where scaling applies, namely a uniform one-way-stretch of scale factor $\times 1.5$.

It is worth demonstrating the task with an actual elastic strip.

White hemline elastic is good for this purpose (available from any good haberdasher's).

Use a 2 m length, say, rather than just 20 cm.

Alternatively, use the GeoGebra file MR-8AB-mini-GEOGEBRA.ggb

or, failing that, use the set of slides MR-8AB-mini-SLIDES.pdf

What different (successful and unsuccessful) strategies do the students use?

Do any students use an addition strategy (eg, "The red line moves 10cm")?

Some students might, quite reasonably, assume that the elastic stretches more in the middle than near the ends. You might at some stage want to say, "Assume that this is an ideal (mathematical!) strip that stretches evenly".

Multiplicative Reasoning 8A

Lesson

11A

Stretched ruler

Some of the markings have worn off this old ruler.

The blue part is 6 cm long.

The ruler is stretched until the blue part is 15 cm long.

Where is the 4 cm mark on the stretched ruler?



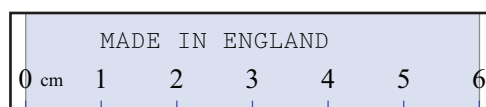
Summary

This lesson provides further experience of the double number line and helps students appreciate that geometric enlargement (or, in this case, a one-way-stretch) involves a multiplicative rather than an additive relationship.

Outline of the lesson

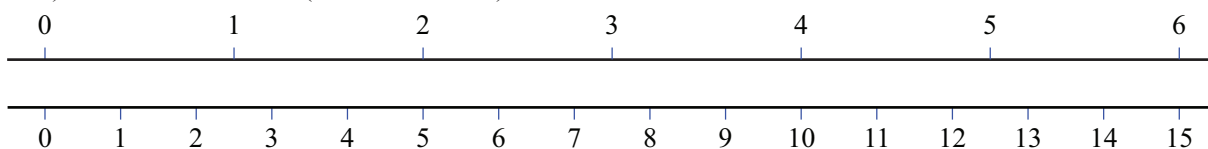
- Consider the position of the 4 cm mark on the stretched ruler.
 - Show the stretched ruler (above).
Visually estimate the position of the 4 cm mark.
 - “How could we work out the exact position of the 4 cm mark?”

- Estimate the position of the marks on a stretched *new* ruler.
 - Show this new version of the ruler. “Imagine it is stretched in the same way. What happens to ‘MADE IN ENGLAND’?”
 - Use Worksheet 1. “*Sketch* the positions of all the cm marks.”
 - Discuss the students’ sketches.
“How could we work out the exact positions of the marks?”



- Determine the position of the marks on a stretched new ruler.
 - Use Worksheet 2. “Draw the cm marks as accurately as possible.”

- Examine the resulting double number line.
 - The two scales on Worksheet 2 form a double number line. Scrutinise the DNL. “Describe all the things you notice.”
 - Find a rule to map the numbers on the top scale (the stretched ruler) onto the bottom scale (the real-life scale).



- Check the rule by applying it to ‘known’ values (eg $6 \rightarrow 15$).
 - Apply the rule to more ‘obscure’ values like 3.5.
“Does our rule still seem to work with these?”
- Extension: explore other stretches.
 - Depending on the class, you might want to consider an easier or harder stretch (eg imagine the ruler had been 5 cm or 7 cm long).

Multiplicative Reasoning 8A

Lesson

11A

Overview

Mathematical ideas

In this lesson we examine a one-way-stretch and explore how it can be modelled by a double number line (DNL). We use the context of a ruler, but are really thinking of an idealised (ie mathematical) stretch, where all horizontal distances are stretched *uniformly*, which means they are all changed by the same scale factor, which in this case is $\times 2.5$. This is modelled by the DNL, which consists of two linear scales with the zeros lined up and with 6 on the top scale in line with 15 on the bottom scale. Any number on the bottom scale is $\times 2.5$ the number directly above.

In the *Elastic strip* Mini-assessment we offer a potential counter to additive thinking by asking about the midpoint. Here we provide a similar counter by considering a strip (in this case a ruler) with evenly spaced marks. However, we start with a ruler with most of the marks missing, to allow any additive thinking that occurred with the Mini-assessment to emerge again.

Students' mathematical experiences

Students may discover some of the following

- the stretch is *not* additive; $6 \rightarrow 15$, but this does *not* mean that all horizontal distances on the stretched ruler have been increased by 9 cm
- the stretch is multiplicative; all horizontal distances have been increase by the same scale factor, $\times 2.5$
- we can use a DNL to model the stretch
- the scales on the DNL are linear and the zeros line-up.

Assessment and feedback

In the light of responses to the Mini-assessment activity, which students tend to see the stretch as additive and which as multiplicative?

Can some students explain why the stretch is *not* additive? What sorts of argument do these students use? For example,

visual, such as “I can see that the 1 cm mark will be a bit further away from the end, but not as much as 9 cm further away”;

or logical, such as “If 8 becomes 18, and 7 becomes 17, and 6 becomes 16, etc, then these marks would still be the same distance apart so this part of the ruler would not be stretched”.

Do students expect there to be a consistent rule that applies to all (horizontal) distances on the ruler (including the length of ‘**MADE IN ENGLAND**’)?

Key questions

What happens to the 1 cm gaps between the marks on the original ruler? Will they still all be equal to each other?

What happens to the midpoint of the ruler?

How are aligned numbers on the DNL related?

Adapting the lesson

Stage 5 of the lesson suggests ways of changing the difficulty of the task (see page 151 as well). In the light of responses to the Mini-assessment activity, you might want to do this from the outset if you think that the given task (ie a $\times 2.5$ stretch) is not appropriate for your class.

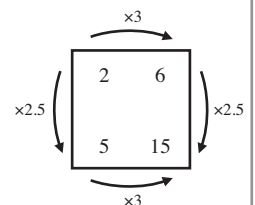
A useful extension is to represent the stretch on a Cartesian graph:

Do the points lie on a straight line? Does the line go through the origin? Why?

What is the slope of the line? What does this tell us?

It is also worth putting values from the DNL in a ratio table, either as an aid to finding a missing value or to highlight relationships. For example, in this ratio table (right) we can observe a ‘horizontal’ relation of $\times 3$ and a ‘vertical’ relation of $\times 2.5$.

Thus $2 \times 3 = 6$ and $5 \times 3 = 15$; similarly $2 \times 2.5 = 5$ and $6 \times 2.5 = 15$.



Multiplicative Reasoning 8A

Lesson

11A**Outline of the lesson (annotated)**

1. Consider the position of the 4 cm mark on the stretched ruler.

- Show the stretched ruler (above).
Visually estimate the position of the 4 cm mark.
- “How could we work out the exact position of the 4 cm mark?”

— Visually, students might well sense that the 4 cm mark is about two-thirds of the way along the stretched ruler. However, arithmetically students might opt for the ‘addition strategy’ and argue that as the length of the ruler has increased by 9 cm, so the 4 cm mark will be $4+9$ cm along the stretched ruler.

After students have *thought about* the stretch, they could use a cut-open elastic band, marked off in six 1 cm intervals, to test their ideas and to get an approximate feel for what happens.

2. Estimate the position of the marks on a stretched *new* ruler.

- Show this new version of the ruler. “Imagine it is stretched in the same way. What happens to ‘**MADE IN ENGLAND**’?”

— ‘**MADE IN ENGLAND**’ is 3 cm long on the original ruler - an easy magnitude since it is half that of the ruler. It, and any such horizontal 3 cm length, will be 7.5 cm long after the stretch.

Each *individual letter* sits in a 2 mm wide field, which becomes 5 mm after the stretch.

You could ask students to sketch the result:

MADE IN ENGLAND

- Use Worksheet 1. “*Sketch* the positions of all the cm marks.”
- Discuss the students’ sketches.
“How could we work out the exact positions of the marks?”

— The evenly spaced markings on the new ruler should help students see that the stretch will be *uniform* and therefore *multiplicative* - the markings will still be evenly spaced after the stretch and thus wider apart.

Note: we are dealing with a *mathematical* stretch in this lesson, not a physical stretch. It is quite conceivable that an actual ruler made of rubber, say, would get thinner in the middle when it is stretched and hence stretch more in the middle than at the ends.

3. Determine the position of the marks on a stretched new ruler.

- Use Worksheet 2. “Draw the cm marks as accurately as possible.”

4. Examine the resulting double number line.

- The two scales on Worksheet 2 form a double number line. Scrutinise the DNL. “Describe all the things you notice.”
- Find a rule to map the numbers on the top scale (the stretched ruler) onto the bottom scale (the real-life scale).
- Check the rule by applying it to ‘known’ values (eg $6 \rightarrow 15$).
- Apply the rule to more ‘obscure’ values like 3.5.
“Does our rule still seem to work with these?”

— For example,

- the zeros are aligned
- the scales are uniform (linear)
- all the 1 cm gaps between the marks are now 2.5 cm gaps
- the numbers on the bottom scale are all 2.5 times the numbers immediately above.

5. Extension: explore other stretches.

- Depending on the class, you might want to consider an easier or harder stretch (eg imagine the ruler had been 5 cm or 7 cm long).

— See page 151.

Multiplicative Reasoning 8A

Lesson

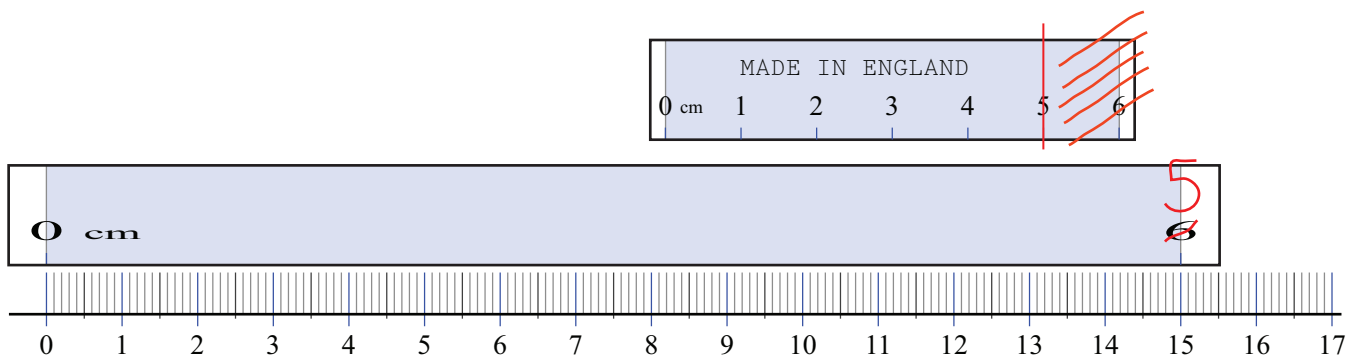
11A

Background

Modifying the Worksheets

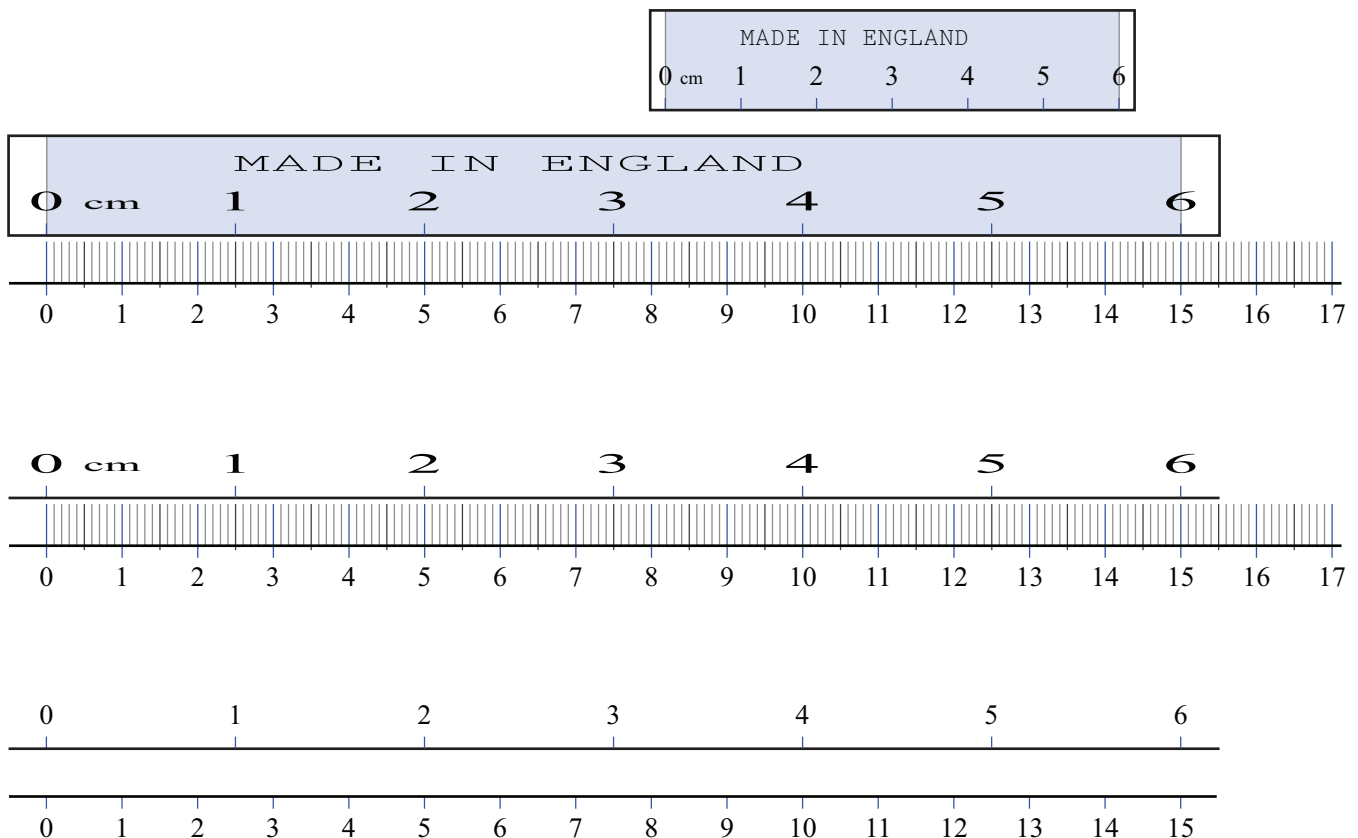
For each of the student worksheets (shown on the next page), we have provided two copies of the ruler that students are asked to mark. This allows students to have a second attempt at each task. It also makes it possible for you or the students to modify the tasks as in the diagram below. Here a 5 cm ruler, rather than the original 6 cm ruler, is stretched until it is 15 cm long.

This particular modification makes the task much easier. The opposite would be true if we had extended the ruler by 1 cm and changed the 6 to a 7.



The stretched ruler and the double number line

In Stage 3 of the lesson students are asked to use Worksheet 2 to draw the marks on the stretched new ruler. The first diagram below shows what the stretch looks like. The outcome is a double number line, which is shown in progressively more abstract ways in the second and third diagrams. You might want to show one or several of these diagrams in Stage 4.

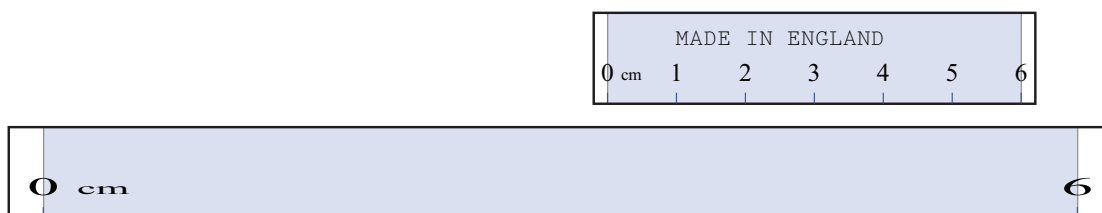
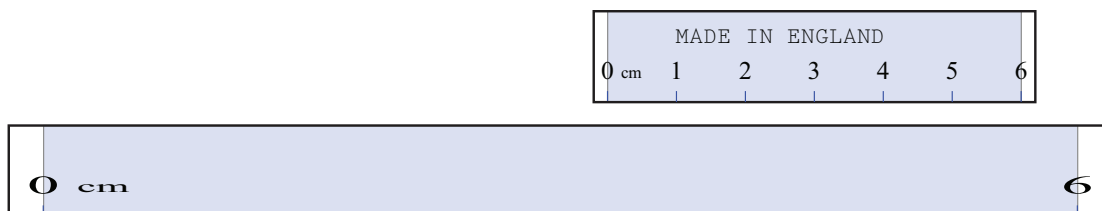


Multiplicative Reasoning 8A

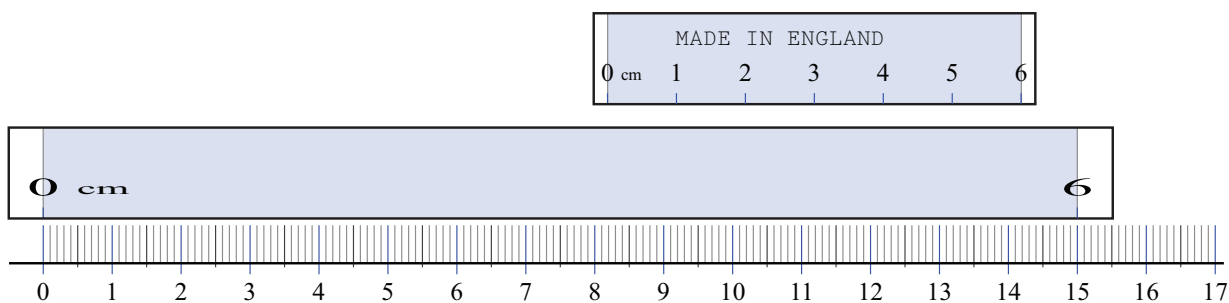
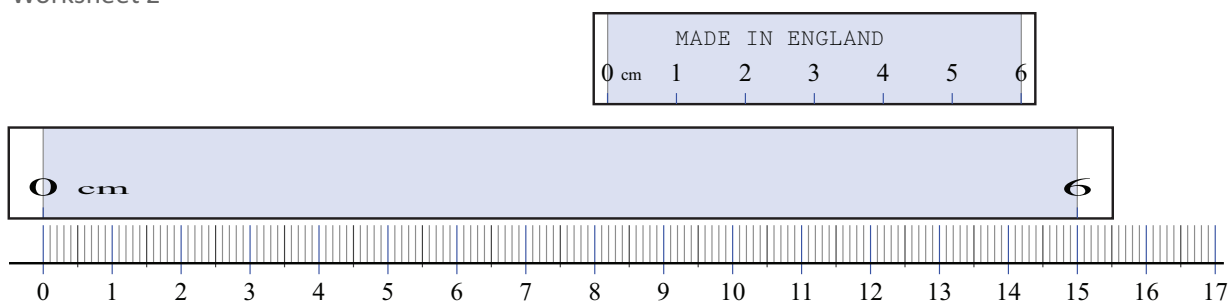
Lesson

11A

Worksheet 1



Worksheet 2



Lesson

11A

Notes

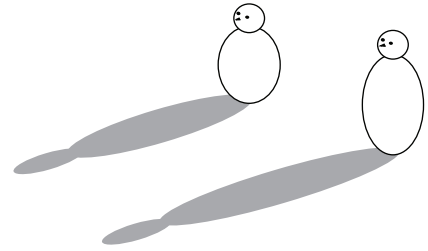
Multiplicative Reasoning 8B

Lesson

11B**Snowmen**

Mike's snowman is 4 ft tall
and its shadow is 10 ft long.

Lisa's snowman is 5 ft tall.
How long is its shadow?

**Summary**

As with Lesson 8A (*Expanded house*) we consider what is in effect a one-way-stretch, but this time we restrict ourselves to 1 dimension rather than applying a stretch in 2 dimensions (ie an enlargement). The lesson involves shadows, a context that we first met in Lesson 8B (*Post shadows*).

We make use of a range of slides (selected from the file **SNOWMEN-slides.pdf** - see page 158) to probe the addition strategy once more and to guide students towards the unitary method. We then adopt a more geometric approach (based on similar triangles), by considering side-elevations, superimposed on a grid.

We provide a peer-assessment homework task in the Revisits section.

Outline of the lesson

- Find the shadow-length of Lisa's snowman.
 - Present the above task (Slide A).
Ask for some quick responses, and then more considered responses after students have discussed the task.
[If the task proves to be too easy, change the 10 ft shadow-length to 11 ft, say.]
- Investigate the addition strategy and the unitary method.
 - Discuss the addition strategy using Slide B (and some or all of Slides C, D, E and F).
 - Consider the unitary method, if it emerges from the discussion.
- Vary the task and find a general rule.
 - Show Slide G (and then H). Let students devise easier and/or harder versions of the task.
 - Try to find a general rule for solving the task.
- Consider a side elevation view of height and shadow-length.
 - Discuss how we can use Slides I and J (or K) to solve the Slide A task.

Snowmen

Mike's snowman is 4 ft tall and its shadow is 10 ft long.

Lisa's snowman is 5 ft tall. How long is its shadow?

Snowmen

Lisa says
My snowman is 1 ft taller than yours.
So its shadow will be 1 ft longer than yours,
and so it will be 11 ft long.

Is she right? Find some reasons to justify your answer.

Snowmen

Mike's sister has built a snowman that is ft tall.
How long is its shadow?

Choose a number for the height - make it easy or hard.
Solve the problem.

Snowmen

This shows a sun-ray for Mike's 4 ft snowman.
Imagine a similar ray for Lisa's 5 ft snowman.
How long is its shadow?

Slide I

Multiplicative Reasoning 8B

Lesson

11B

Overview

Mathematical ideas

Here we revisit the context of shadows, as in Lesson 8B, but this time involving snowmen rather than posts. We again scrutinise the addition strategy (Slides B, C, D, E and F) by making deliberate use of the segmented nature of the snowmen: if we increase a snowman's height by adding an extra head, what do we add to the length of the snowman's shadow - the height of the head (addition strategy) or the length of the head's shadow (rated addition)?

We also use the segmentation to reinforce the idea that a shadow can be thought of as a uniform stretch (eg Slide C), and, by using several identical segments (towers made of 1 ft snowballs, Slide E), we give students an opportunity to consider the unitary method.

We then shift to a more overtly geometric approach (in essence, the geometry of similar triangles) by using side elevations to represent heights and shadow-lengths (Slides I, J and K). As a follow-up, we provide a homework activity in the Revisits section involving peer assessment.

Students' mathematical experiences

Students may discover some of the following

- when we increase the height by a specific amount, the shadow-length is also increased by a specific amount, but these amounts are not usually the same - rather, they are in proportion
- the shadow of an object is like a one-way-stretch of the object
- a side elevation provides a powerful way of representing and analysing the relation between height and shadow-length.

Assessment and feedback

For what kinds of number-pairs do students see the relation between height and shadow-length as multiplicative? When do they resort to the addition strategy?

How readily do students discern the unitary method? Do they use it?

Key questions

If we increase the height by 1 ft, what happens to the shadow-length?

If we know the height, how can we find the shadow-length?

Adapting the lesson

You can easily alter the numbers in a task if students find it too easy (or too difficult). For the 4 ft snowman on Slide A, changing the 10 ft shadow-length to, say, 11 ft changes the multiplier that maps height onto shadow-length from $\times 2.5$ to $\times 2.75$. Many students would find this far less easy to discern and more demanding to apply.

You might want to change the emphasis of some stages of the lesson, eg by devoting more time to 'snow towers' like those on Slide E as a way of emphasising the unitary method. Or you could give further emphasis to the one-way-stretch by imagining snowmen that are stretched versions of other snowmen.

You could extend the work on elevations by using a Cartesian graph to represent height and shadow-length.

You could also bring some 'earth geometry' into the proceedings:

The town of Tjålme is in the far north of Sweden, at 68.2°N , 20.2°E .

Here a 4 ft snowman would have a 10 ft shadow at about noon on the first day of spring.

Why?!

Multiplicative Reasoning 8B

Lesson

11B**Outline of the lesson (annotated)**

1. Find the shadow-length of Lisa's snowman.

- Present the above task (Slide A).
Ask for some quick responses, and then more considered responses after students have discussed the task.
[If the task proves to be too easy, change the 10 ft shadow-length to 11 ft, say.]

— You might want to show the information in a ratio table:

	Mike	Lisa
Height of snowman	4	5
Length of shadow	10	?

2. Investigate the addition strategy and the unitary method.

- Discuss the addition strategy using Slide B (and some or all of Slides C, D, E and F).
- Consider the unitary method, if it emerges from the discussion.

— Slide C: It is possible to find a rule to map height onto shadow-length which gives the same result when it is applied to the head and body separately as when it is applied to the snowman as a whole (the distributive law in action!). We can think of the mapping as a one-way-stretch - the effect is uniform along the snowman's height.

Slide D: Students who use the addition strategy are saying, in effect, that the head's shadow is the same as its height - yet the overall shadow is greater than the overall height.

Slide E: We can find the total shadow by finding the shadow of each head - the unitary method.

Slide F: this provides a simple multiplicative situation (doubling) which most students will agree is valid, including those attracted to the addition strategy.

3. Vary the task and find a general rule.

- Show Slide G (and then H). Let students devise easier and/or harder versions of the task.
- Try to find a general rule for solving the task.

— Try to elicit a range of (equivalent) rules:

- unitary method: each 1 ft height has a 2.5 ft shadow
- a functional relationship: the shadow is 2.5 times the height
- a scaling relationship: Lisa's snowman is 1.25 as tall as Mike's so its shadow will be 1.25 times as long.

4. Consider a side elevation view of height and shadow-length.

- Discuss how we can use Slides I and J (or K) to solve the Slide A task.

— The sun's rays are parallel. The triangles formed by the height, shadow and sun ray for different snowmen are similar.

Changing the snowman's height by a given amount changes the shadow-length by a *different* amount (unless the sun's rays are at 45° to the horizontal).

The shadow length for Mike's snowman is 13 ft rather than 10 ft in Slide K.

Multiplicative Reasoning 8B

Lesson

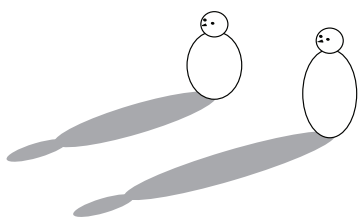
11B**Shadows and the one-way stretch**

We can think of the shadow of an object as an elongated version of the object, ie as a one-way-stretch (at least, in the case of a vertical object whose shadow is cast onto a horizontal plane by the parallel rays of the sun).

As we saw in Lesson 8A (Expanded house), the one-way stretch is *uniform*. For this reason we have used an object (a snowman) formed of several sections rather than just one (as in the case of a pole, say), to bring out the idea that each section is stretched in the same way and that the stretch must be multiplicative.

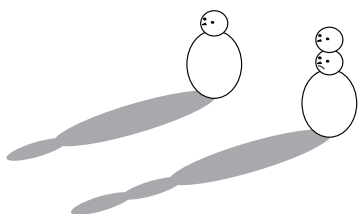
We can, of course, also see the relation between the heights of our different snowmen as a one-way-stretch (and likewise the relation between the lengths of their shadows). However, the snowmen that we have used are not themselves stretched versions of each other. Rather they have been formed, at least in part, by adding pieces rather than stretching them. For example, in the case of Slide A, Lisa's snowman can be thought of as having a stretched version of Mike's snowman's body, but with the same size head.

Mike's snowman is 4 ft tall and its shadow is 10 ft long. Lisa's snowman is 5 ft tall. How long is its shadow?



And in Slide D, Mike has created a snowman with the same height as Lisa's by simply adding another identical head to his snowman.

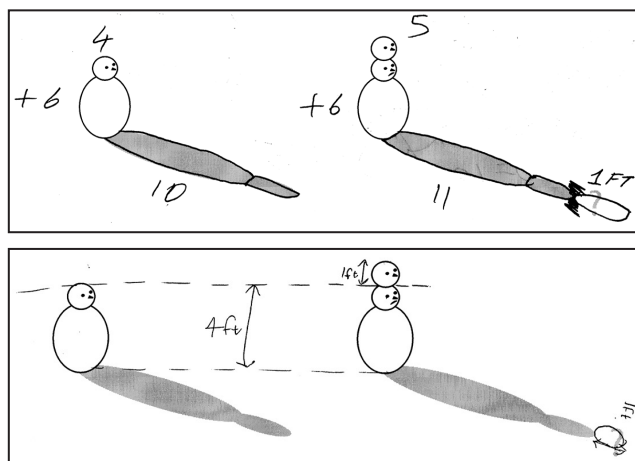
Mike's snowman is 4 ft tall and its shadow is 10 ft long. He increases the height by 1 ft. How long is its shadow now?



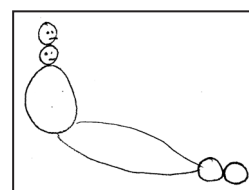
This additive element is deliberate, in that it allows us to use a simple additive approach (rated addition) to scrutinise the addition strategy. For example, in adding an extra 1 ft head to the snowman in Slide D, we have not simply added 1 ft to the shadow; rather we have added an extra *shadow* of a 1 ft head, and in this case the shadow happens to be *longer* than 1 ft.

Of course, it will not be obvious to all students that the head's shadow is longer than 1 ft, or, if students are aware of it, they will not all keep track of this fact.

The drawings below are of an earlier version of Slide D, where students were asked to draw the extra shadow. The first student has drawn a nicely elongated shadow, to match the given shadow of the first head, yet the shadow has been labelled as 1 ft long, and the total shadow as 11 (ft) rather than 12.5 ft. The second student has done something similar.



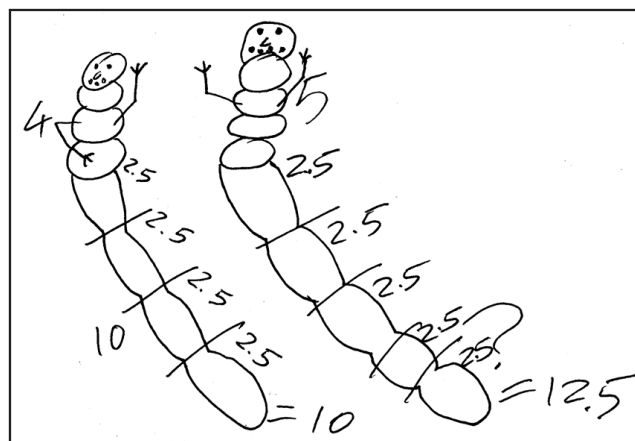
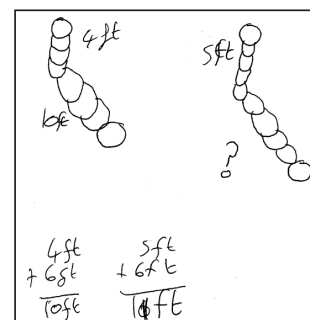
Here (right), the student has drawn an elongated shadow for the body but not for the heads. The student has also written "If the snowman's only 1 ft taller the shadow should be 1 ft taller so it will be 11 ft".



The drawing below, right, is for an earlier version of Slide E. Here the shadows are perhaps slightly elongated, but the student has also used the addition strategy to arrive at 11 ft for the length of the shadow.

"If the snowman's only 1 ft taller the shadow should be 1 ft taller so it will be 11 ft."

On the other hand, Slide E has worked for the student below, who has determined that all the heads have shadows of length 2.5 (ft), with correct total lengths of 10 and 12.5.



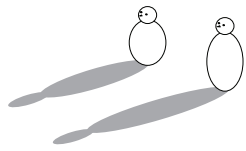
Multiplicative Reasoning 8B**Multiplicative Reasoning 8B**

Lesson

11B**Slides in the file SNOWMEN-slides.pdf**

Snowmen

Mike's snowman is 4 ft tall and its shadow is 10 ft long. Lisa's snowman is 5 ft tall. How long is its shadow?



Slide A

The given numbers are quite likely to provoke the addition strategy, since the relation $4+1=5$ is far more obvious than $4 \times 1.25=5$.

Snowmen

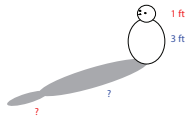
Lisa says
My snowman is 1 ft taller than yours.
So its shadow will be 1 ft longer than yours,
and so it will be 11 ft long.
Is she right? Find some reasons to justify your answer.

Slide B

Here the addition strategy is stated explicitly. It is useful for students to try to explain *why* this doesn't work even if they already *know* it doesn't.

Snowmen

Mike's snowman is 4 ft tall and its shadow is 10 ft long.



The snowman's body is 3 ft tall and its head is 1 ft tall.
How long are the shadows of the body and the head?

Slide C

We need a rule (eg $\times 2.5$) that works for the whole snowman *and* for the separate parts: $4 \times 2.5 = 10$ *and* $1 \times 2.5 + 3 \times 2.5 = 10$.

Students might propose inconsistent rules like $3 \times 3 + 1 = 10$.

Snowmen

Mike's snowman is 4 ft tall and its shadow is 10 ft long. He increases the height by 1 ft. How long is its shadow now?

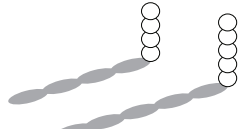


Slide D

It is tempting to say that "If the height increases by 1 ft, so does the shadow". However, it can be seen that the shadow of the extra 1 ft head is longer than 1 ft.

Snowmen

Mike and Lisa have also built some towers using 1 ft high snowballs.
How long are the shadows?



Slide E

This slide is likely to prompt the unitary method: the total shadow-length is equal to the shadow-length of one head multiplied by the number of heads.

Snowmen

Mike's snowman is 4 ft tall and its shadow is 10 ft long. Arnie's snowman is 8 ft tall. How long is its shadow?



Slide F

This offers a powerful counter example to the addition strategy: "Arnie's snowman is twice as tall, so its shadow is twice as long" is likely to be more compelling than "Arnie's snowman is 4 ft taller so its shadow is 4 ft longer".

Snowmen

Mike's sister has built a snowman that is ft tall.
How long is its shadow?

Choose a number for the height - make it easy or hard.
Solve the problem.

Slide G

Students are asked to devise similar problems, but easier or harder, and to express a general rule for finding the shadow-length.

Snowmen

Mike's sister has built a snowman that is ft tall.
How long is its shadow?

Choose a number for the height - make it easy or hard.
Solve the problem.

Find a general rule for working out the length of the shadow.

Slide H

Snowmen

This shows a sun-ray for Mike's 4 ft snowman.
Imagine a similar ray for Lisa's 5 ft snowman.
How long is its shadow?

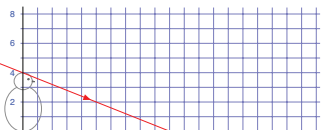


Slide I

A sketch of Lisa's ray might provide students with a general sense that a 1 ft increase in height produces more than a 1 ft increase in shadow-length.

Snowmen

This shows a sun-ray for Mike's 4 ft snowman.
Imagine a similar ray for Lisa's 5 ft snowman.
How long is its shadow?

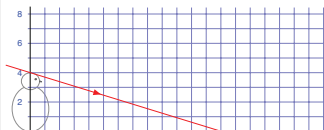


Slide J

The grid might enable students to take a more precise, analytic approach to determining Lisa's ray and hence the shadow-length.

Snowmen

It is late afternoon and the sun is lower in the sky.
This again shows a sun-ray for Mike's 4 ft snowman.
How long is the shadow for Lisa's 5 ft snowman now?



Slide K

The numbers are more challenging here but should provide a good test of students' general rules.

Lesson

11B

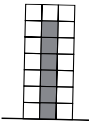
Notes

Algebra

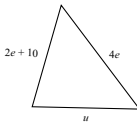
Mini-test

Mini algebra test

miniAlGtest Name Class Date
 NO CALCULATORS Please show all your working/jottings

1.  ← This is a 'tower'.
 It is made of a column of grey tiles, surrounded by a layer of white tiles.
 How many white tiles are there in a tower which has
 a) 7 grey tiles b) 50 grey tiles? c) m grey tiles?

2. Look at the triangle:
 Here is an expression for its perimeter: $2e + 10 + 4e + u$.
 a) Write the expression more neatly.
 b) For what value of e are the two slanting sides equal?



3. Olga and Matt both type this number into their calculators: **54997**.
 Olga says, "I'm going to add 4 to the number, then multiply the result by 27".
 Matt says, "I'm going to multiply the number by 27, then add 4 to the result".
 Whose answer is bigger, or are they the same?
 Explain:

4. William cleans windows. He charges £3 for each window.
 This is his expression for what he charges: $3n$ (the n stands for the number of windows).
 Kate also charges £3 for each window. But she also charges £4 to come to your house.
 Write an expression for how much Kate charges altogether for cleaning n windows.
 Harry charges twice as much as Kate charges altogether.
 Write an expression for how much Harry charges altogether for cleaning n windows.

Commentary

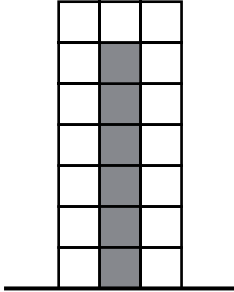
This is a short test which should be given to students sometime before Lesson 3A (*Boat hire*) and again at the beginning of Year 8. **Allow about 15 minutes.** The test touches on some key ideas in algebra and should give you an indication of the variety of student responses in your class.

The items vary in difficulty and you should alert students to this: they shouldn't worry if some items appear strange or difficult, and they should have a go nonetheless. So for example, in the *Towers* question students might determine by drawing that the tower with 7 grey tiles has 17 white tiles, but resort to an erroneous numerical approach (eg, 'there are 10 more white tiles than grey tiles') for the tower with 50 grey tiles.

miniALGtest Name Class Date

NO CALCULATORS Please show all your working/jottings

1.



← This is a 'tower'.

It is made of a column of grey tiles, surrounded by a layer of white tiles.

How many white tiles are there in a tower which has

a) 7 grey tiles

b) 50 grey tiles?

c) m grey tiles?

.....

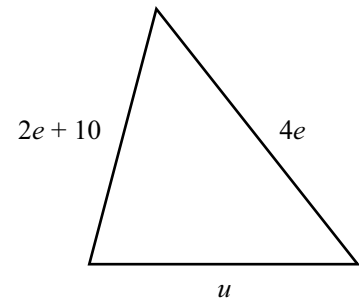
.....

.....

2. Look at the triangle:

Here is an expression for its perimeter: $2e + 10 + 4e + u$.

a) Write the expression more neatly.

b) For what value of e are the two slanting sides equal?3. Olga and Matt both type this number into their calculators: **54997**.

Olga says, "I'm going to add 4 to the number, then multiply the result by 27".

Matt says, "I'm going to multiply the number by 27, then add 4 to the result".

Whose answer is bigger, or are they the same?

Please explain:

4. William cleans windows. He charges £3 for each window.

This is his expression for what he charges: $3n$ (the n stands for the number of windows).

Kate also charges £3 for each window. But she also charges £4 to come to your house.

Write an expression for how much Kate charges altogether for cleaning n windows.

Harry charges twice as much as Kate charges altogether.

Write an expression for how much Harry charges altogether for cleaning n windows.

Mini-test

Mini algebra test

Notes

Algebra 4AB

Mini-assessment

12AB**Same or different?**

Look at this expression:

$$4 \times 319$$

What happens if we add 1 to the first number and subtract 1 from the second?

Is the value the same or different?

$$5 \times 318$$

Commentary

The aims of this Mini-assessment are

- to produce some quick information on how readily students sense how an expression ‘operates’
- to help students realise that we can look at structure without immediately performing a calculation.

If the task turns out to be too easy, you could try numbers that are closer together or the same (eg 100×100 and 101×99).

Are students willing to compare the expressions *without* evaluating them?

Do students think they are the same or different?

What methods do students use?

Algebra 4A

Lesson

12A**Order matters**

It costs £8 to visit Adventure Land and £3 for each ride.

Jo's mum says, "Here's £20. That's enough for 4 rides".

Jo says, "Thanks mum, but 4 rides will cost £44".

Who is right?

entrance	cost	number
fee	per ride	of rides

$$8 + 3 \times 4$$

Summary

In this lesson, we look at numerical expressions involving two operations. The aim is to increase students' awareness that the order of operations matters. We focus on multiplication and addition and explore how they combine (which in effect means looking at the distributive property of multiplication over addition).

We start with contextualised problems and use the contexts to make sense of how the operations interact. We deliberately resist introducing brackets until the end of the lesson, to avoid triggering procedures that may not be well understood - the lesson should help provide a basis for the procedures to emerge in a meaningful way.

Outline of the lesson

- Present the above problem to the class and focus attention on the numerical expression.
 - Ask the class, "How might the expression $8+3\times 4$ lead to the answers 20 and 44? Which is right?"
 - Discuss the difference, 24, between the answers. "What does the 24 mean in the story?"
 - "Could you change the story so that Jo is right?" [Jo exits Adventure Land after each ride!]
- Repeat Step 1 using the Pizza problem.
 - Some students might suggest the use of brackets. Acknowledge this but defer discussion: "Brackets would help avoid the problem. Could we talk about it later?"
- Repeat Step 1 using the Candle problem.
- Repeat Step 1 using the Temperature problem.
- Consider other numerical expressions involving addition (or subtraction) and multiplication (or division).
 - Make up an expression such as $3\times 4+8$. Consider what happens when the operations are applied in a different order [ie $\times$ first or $+$ first].
 - Explain how the answers differ.
 - Discuss the use of brackets.

A basic pizza costs £6 plus £1.20 for each topping.
Kim and Lou order a pizza with 3 toppings.
Kim says it will cost £21.60. Lou says it will cost £9.60.
Who is right?

basic cost		cost of topping		number of toppings
6	+	1.20	$\times$	3

A candle for a birthday cake is 50 mm tall.
It burns down 3 mm every minute.
Meg lights the candle.
She says, "It will be only 20 mm tall in 10 minutes".
Ned says, "No, it will be 470 mm tall".
Who is right?

height in mm		mm per minute		number of minutes
50	—	3	$\times$	10

It is a very hot day. The temperature reaches 90°F.
Olga and Per convert this to °C by using this rule:
Subtract 30, divide by 2.
Olga says, "90°F is about 30°C".
Per says, "No, it's about 75°C".
Who is right?

90	—	30	$\div$	2
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Algebra 4A

Lesson

12A**Overview****Mathematical ideas**

In this lesson, the students explore the effect of order of operations.

In each contextual problem, students evaluate two different answers, one correct and one incorrect. The problems involve numerical expressions of the form $a + b \times c$, $d - e \times f$ and $g - h \div j$, and in each case there is a “fixed” element or cost. The lesson focuses on explaining how the difference between the correct and incorrect answer arises.

A secondary aim is to demonstrate the need for conventions about the order of operations. Discussion of brackets is briefly addressed towards the end of the lesson.

Note: For the contextual problems, the numerical expressions have been written to match the verbal presentation, not on the basis of whether they are right or wrong (in terms of the standard conventions). Thus it turns out that the expression for the Temperature problem is “wrong”, as can be demonstrated by using a standard calculator.

Students' mathematical experiences

Students should realise ...

- when multiplication and addition are combined, order matters
- the difference between the two expressions has a systematic explanation: in $8+3\times 4$, adding the 8 and the 3 first means that 8 is multiplied along with the 3
- the need for brackets.

Students should ...

- make and improve explanations.

Assessment and feedback

You could assess

- the quality of students' explanations
- which students rely largely on contextual explanations and which use more numerical explanations
- their understanding of the effect of multiplication on the “fixed” element.

This can be a useful exercise to assess students' arithmetic skills.

Key questions

How could you convince Jo that they are wrong?

How does the difference arise?

Adapting the lesson

You could extend the expressions to involve other combinations of operations (eg, $30 \div 6 \times 3$, or $30 \times 6 \div 3$, or $30 \div 6 \div 3$, or $30 \times 6 \times 3$).

You could encourage students to invent their own similar problems. If students have difficulty doing this, you could suggest they adapt some of the problems considered in the lesson.

A task that focusses on brackets is given in the Revisists section.

Algebra 4A

Lesson

12A**Outline of the lesson (annotated)**

1. Present the above problem to the class and focus attention on the numerical expression.

- Ask the class, “How might the expression $8+3\times 4$ lead to the answers 20 and 44? Which is right?”
- Discuss the difference, 24, between the answers. “What does the 24 mean in the story?”
- “Could you change the story so that Jo is right?” [Jo exits Adventure Land after each ride!]

— At this stage of the lesson, avoid using brackets to clarify the meaning of the expression. The ambiguity is important, because it encourages students to provide explanations.

2. Repeat Step 1 using the Pizza problem.

- Some students might suggest the use of brackets. Acknowledge this but defer discussion: “Brackets would help avoid the problem. Could we talk about it later?”

— Each of these episodes follows the same pattern. As the lesson progresses, the students can take more responsibility allowing you space to observe them.

3. Repeat Step 1 using the Candle problem.

— Here the wrong answer is very obviously nonsensical – the candle cannot get longer!

4. Repeat Step 1 using the Temperature problem.

— You could ask a student to enter the expression into a calculator: How come the calculator is wrong?
The structure of this problem is different to the others. Here the correct answer is to carry out the subtraction first: $(90 - 30) \div 2$.
The formula was often used prior to the widespread availability of computers to convert F to C. You may need to discuss the meaning of F and C.

5. Consider other numerical expressions involving addition (or subtraction) and multiplication (or division).

- Make up an expression such as $3\times 4+8$. Consider what happens when the operations are applied in a different order [ie $\times$ first or $+$ first].
- Explain how the answers differ.
- Discuss the use of brackets.

— Discuss how the hazard of getting different answers for the same expression could be avoided. Try to focus initially on the different meanings underlying the different answers. If no-one suggests using brackets, suggest them yourself: “Would brackets help?”

Algebra 4A

Lesson

12A

Algebraic structure and the laws of arithmetic

In this lesson we examine how a pair of operations (in particular, multiplication and addition) interact. Order matters for this pair because multiplication is distributive over addition. For example,

$$(100+10) \times 3 = 100 \times 3 + 10 \times 3 \text{ rather than } 100 \times 3 + 10.$$

Similarly

$$(x + b) \times a \text{ is equivalent to } ax + ab \text{ rather than } ax + b.$$

Put more concretely, if a collection of objects is multiplied by 4, say, then each element in the collection is multiplied by 4; but if 4 is added to a collection, 4 is not added to each element, just to the collection as a whole.

The use of context

For this lesson, we decided to use problems set in a context, for two reasons. One, it provides a rationale for considering numerical expressions which are not simply evaluated from left to right. Second, it helps to ground students' thinking as they try to make sense of how the operations interact. In the case of the expression $8+3 \times 4$ and the incorrect answer 44, it is both richer and more concrete to say "This is like entering Adventure Land 4 times instead of once", than to say "We should only multiply the 3 by 4, not the 8 as well".

Order and BODMAS

Some students may mention BODMAS or BIDMAS. These mnemonics are very particular to the UK but don't always work because they are oversimplifications. Thus while BODMAS might correctly tell us that $2+3 \times 4$ means $2+12$, not 5×4 , it might lead us to interpret $6 \times 8 \div 4$ as 6×2 rather than the conventional left-to-right interpretation of $48 \div 4$. BODMAS in arithmetic does not appear to help students' understanding of algebra because it encourages students to see the 'open' expressions commonplace in algebra as incomplete and still needing to be 'done'.

We provide a task involving brackets in the Revisits section.

Algebra 4B

Lesson

12B**Think of a number**

Jo is using the function machines in this diagram:

He thinks of a number.

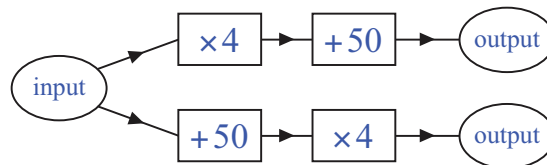
He multiplies his number by 4, then adds 50 to the result.

Jo starts with his original number again.

This time he first adds 50, then multiplies the result by 4.

Think about Jo's final results.

What can you say about them? What's going on?

**Summary**

In this lesson, we examine the effect of changing the order of two operations (a multiplication and an addition). We compare a pair of function machines: $\times 4$, $+50$ and $+50$, $\times 4$ (which always produce a difference of 150). Later we compare $\times 3$, $+10$ and $+10$, $\times 3$ (which always produce a difference of 20).

The prime aim is not to calculate outputs but to examine how multiplication and addition combine (which effectively means, to look at the distributive property of multiplication over addition). To further this, we make use of open and algebraic expressions (some with brackets).

Outline of the lesson

1. Ask the class whether they think $\times 4$, $+50$ and $+50$, $\times 4$ will give the same output for the same input.
 - Each student should try one or two specific inputs. Record their results in a table and note the constant difference of 150 between pairs of outputs.
 - With the class, consider an input of 100.
 - In pairs, then as a class, discuss: "How come the difference is always 150?" Record all their explanations on the board.
 - If no one suggests it, write these open expressions on the board: $100 \times 4 + 50$ and $(100 + 50) \times 4$.
How do these help to explain that the difference is 150?

For example:

$\times 4 + 50$	$+50 \times 4$
100 100 50	100 50 100 50
100 100 50	100 50 100 50

2. Ask the class to compare $\times 3$, $+10$ and $+10$, $\times 3$. Will the outputs have a constant difference?
 - Students discuss the task in small groups and try to come up with a result, a rule and an explanation.
 - Gather a wide selection of explanations.
 - With the class, consider an input of 100, as above.
 - Choose a symbol for the input (eg $?$, Δ , N or x). Write expressions for the outputs, eg $(N \times 3) + 10$ and $(N + 10) \times 3$.
Do these help to explain that the difference is 20?

For example:

$\times 3 + 10$	$+10 \times 3$

3. Use a peer assessment task to explain why order matters.
 - Present peer assessment task P.
 - Ask students to work on the task in pairs.
 - Discuss their ideas.

Algebra 4B

Lesson

12B**Overview****Mathematical ideas**

The ideas in this lesson hinge on the fact that multiplication is distributive over addition:

$$(a + b) \times c = (a \times c) + (b \times c).$$

In this lesson, we help students develop a 'feel' for, and some degree of insight about, the effect of swapping the order of operations. Students are asked to help each other explain, justify and prove their ideas.

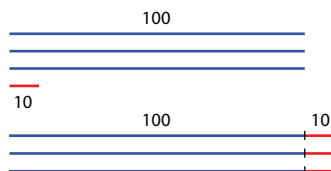
The students are encouraged to work with open expressions in order to reveal structure. This is a precursor for later lessons on algebraic manipulation and brackets.

Students' mathematical experiences

Students evaluate their own and others' explanations.

They may discover some of the following:

- when multiplication and addition are combined, the order matters
- multiplication is distributive over addition
- switching the order of the function machine $\times 4$ then $+50$, changes the output by $50 \times 4 - 50$, ie by $50 \times (4 - 1)$
- we can use open expressions to describe function machines. For example, $\times 3$ then $+10$ can be written as $100 \times 3 + 10$ for an input of 100, or as $3N + 10$ for an input of N .
- we can use visual representations for function machines. For example,

**Assessment and feedback**

Most students' explanations will not be perfect. Some may be "incorrect" in some way. In this lesson, the focus is on peer assessment. We have provided a task which asks students to help another student respond to a teacher's comment. A second task, which can be found in the Revisits section, examines ways of re-wording and improving explanations.

If this is the first time that students have done this, they will find it difficult to get started. Ask them to talk in pairs, then share some of their ideas as a class, then ask them to talk in pairs again.

You might follow up this paired and whole class discussion by asking them to finish the task for homework either individually or in pairs.

Key questions

Why does switching the order produce a different result for addition and multiplication?

Is there a rule for the constant difference?

How can I change 'this' open expression with brackets to an open expression without brackets?

Adapting the lesson

You might want to try other kinds of numbers for the multiplication and addition operations, such as $\times 80$ and $+2$, or $\times 10$ and $+0.5$, or $\times 0.1$ and $+100$. Or you might want to try other operations (such as division and addition, or division and another division), perhaps to see whether there are cases where the order of the operations does not matter.

Note: In this 'function machine' context, the order does not matter in the case of the same operation, eg $\div 12$ followed by $\div 3$ is the same as $\div 3$ followed by $\div 12$, as they are both equivalent to $\div 36$. Put another way, $(a \div 12) \div 3 = (a \div 3) \div 12$. However, $(a \div 12) \div 3$ is *not* the same as $a \div (12 \div 3)$, and similarly $(a \div 3) \div 12$ is not the same as $a \div (3 \div 12)$.

Algebra 4B

Lesson

12B**Outline of the lesson (annotated)**

1. Ask the class whether they think $\times 4$, $+50$ and $+50$, $\times 4$ will give the same output for the same input.
 - Each student should try one or two specific inputs. Record their results in a table and note the constant difference of 150 between pairs of outputs.
 - With the class, consider an input of 100.
 - In pairs, then as a class, discuss: “How come the difference is always 150?” Record all their explanations on the board.
 - If no one suggests it, write these open expressions on the board: $100 \times 4 + 50$ and $(100+50) \times 4$. How do these help to explain that the difference is 150?
 2. Ask the class to compare $\times 3$, $+10$ and $+10$, $\times 3$. Will the outputs have a constant difference?
 - Students discuss the task in small groups and try to come up with a result, a rule and an explanation.
 - Gather a wide selection of explanations.
 - With the class, consider an input of 100, as above.
 - Choose a symbol for the input (eg $?$, Δ , N or x). Write expressions for the outputs, eg $(N \times 3) + 10$ and $(N+10) \times 3$. Do these help to explain that the difference is 20?
 3. Use a peer assessment task to explain why order matters.
 - Present peer assessment task P. Ask students to work on the task in pairs. Discuss their ideas.
- There are likely to be some lively opinions on this. Ask for students' views but don't resolve the question yet.
- The students' data should suggest that the difference is always 150. This raises the questions ‘Why 150?’ and ‘Will this always be true?’. Alert students to these questions, and that you will return to them later.
- Give the class time to think about the outputs, but encourage them to approach this mentally, without any jottings (ie to try to keep everything in mind). It is hoped that the choice of the large (but simple) input will help students see how the operations interact.
- Work through the calculations and write the outputs (450 and 600) on the board. Confirm that the difference is 150.
- The result might be expressed as ‘a constant difference of 20’; the rule as ‘difference = 2×10 ’ or ‘ $(3-1) \times 10$ ’.
- Give lots of students a chance to voice their explanations. You can *then* start to interrogate the explanations, but you will need to judge how much of this you do now and how much you do during the next steps in the lesson.
- Again, give the class time to think about the outputs, but discourage any jottings. Again, it is hoped that the choice of the large (but simple) input will help students see how the operations interact.
- Then work through the calculations with the class and as you do so, write the two open expressions on the board. Students might say ‘do the brackets first’ and so instead of transforming $(100+10) \times 3$ to $(100 \times 3) + (10 \times 3)$, they might change it to 110×3 , which loses the structure.
- Here we move towards algebraic symbolisation. You will need to decide how far to take this, and how closely the expression should match the function machine representation or conventional notation. For example, do we represent the first machine like this $N \times 3 + 10$, or this $3 \times N + 10$, or this $3N + 10$? It is hoped that the algebraic expressions will help students see the structure of the outputs, and so give a *purpose* to algebraic symbolisation.
- In Task P, some students might simply spot that $21 = 3 \times 7$; others might spot that $21 = (4-1) \times 7$ and $21 = (8-1) \times 3$, and some might also have some insight into why this is so.

Algebra 4B

Lesson

12B

Background

Algebraic structure and the laws of arithmetic

In this lesson we again examine how a pair of operations (multiplication and addition) interact. Order matters for this pair because multiplication is distributive over addition.

Explanation and proof

Here we compare outputs of the form $ax + b$ and $(x + b)a$.

Their difference is $b(a - 1)$, which can of course be proved algebraically:

$$(x + b)a - (ax + b) = ax + ab - ax - b = ab - b = b(a - 1).$$

We move tentatively *towards* such a proof in this lesson, but suggest adopting a numerical approach to start with, and to gradually look for rules-with-reasons. The balance of the approach you use will depend on the class and on students' responses during the lesson.

In the case of the outputs for the $\times 3, +10$ and $+10, \times 3$ function machines, where the difference is always 20, students might give a rule/reason like this initially:

In the first machine you add one 10. In the second machine you add 3 lots of 10, so you have added $(3 - 1)$ extra 10s.

Later, students might come up with something like this:

For an input of 100,
the first machine has an output of $100 \times 3 + 10$;
the second has an output of $(100 + 10) \times 3$ or $100 \times 3 + 10 \times 3$.
The difference is $10 \times 3 - 10$, which is $10 \times (3 - 1)$.

Or

for an input of N ,
the first machine has an output of $3N + 10$;
the second has an output of $3(N + 10)$ which is $3N + 3 \times 10$.
The difference is $(3 - 1) \times 10$.

Generalised number and variable

We are working more in the realm of generalised number than variable in this lesson. We don't consider a systematic set of inputs (although that would be a perfectly viable strategy).

Take the function machines $\times 3, +10$ and $+10, \times 3$.

Here we are not primarily interested in the input (which is multiplied by 3 in both machines). Though we have to keep the input in mind, our focus is on the fact that in the first machine we add 10 while in the second we add 10×3 .

Thus, we are trying to adopt a *generic* approach. Though we are considering specific numerical inputs and outputs, we want students to focus on the *structure* of multiplication and addition, not just on their numerical results. We hope students will see the *general in the particular*, especially when we use large (albeit simple) numbers.

[An alternative, more *dynamic*, approach involving the notion of variable, would be to note what happens when the input *changes*. In contrast to *Boat hire*, say, here the two outputs will always

change in identical ways (namely by 2 times the change in the input). This means there is a constant difference between the two outputs. This can be seen nicely on the Cartesian graph.]

Students create algebraic expressions and use brackets

In the lesson we make use of the quasi-variable 100, in open expressions like $100 \times 3 + 10$, and move towards the creation of algebraic expressions like $3N + 10$ and $3(N + 10)$. How far you take this and how much weight you give it will depend on the class and the feedback you get from them. The aim is that students will begin to see that algebraic expressions help reveal structure and allow one to prove rules.

In developing the algebraic expressions, you will need to decide how closely they should mirror the function machines from which they arise and how far you move towards conventional algebraic notation. Thus, for the function machine $\times 3, +10$ you might move from the algebraic expression $N \times 3 + 10$ to $3 \times N + 10$ to $3N + 10$.

The work provides a good opportunity to use brackets and to gain an understanding of how they can be expanded.

Peer assessment, marking and comments

Peer assessment provides opportunities for students to help or advise each other, which can be mutually beneficial. However, students need help and training to do this.

If there is time, we suggest using the peer assessment task P towards the end of the lesson. This is probably best done in small groups, eg in pairs. The task makes the point that we want to go beyond merely generating numerical examples. At the same time, it allows students to consolidate the understanding that they have achieved through the work so far, by presenting pairs of functions that are similar to but slightly different from those considered in the lesson.

A second task, Q, can be found in the Revisits section. Task Q is perhaps more demanding, in that it is presented in a more condensed way and involves a fairly general argument. However, it underlines the importance of trying to understand each other's ideas and of expressing ideas clearly.

You might want to introduce other comments for students to evaluate (ones that you've encountered in this or another class, or ones that you have made up). For example, in Task P one could ask students to evaluate this explanation:

In the first pair you've got 3 extra 7s, in the second pair you've got 7 extra 3s.

Algebra 4B

Lesson

12B

Peer assessment

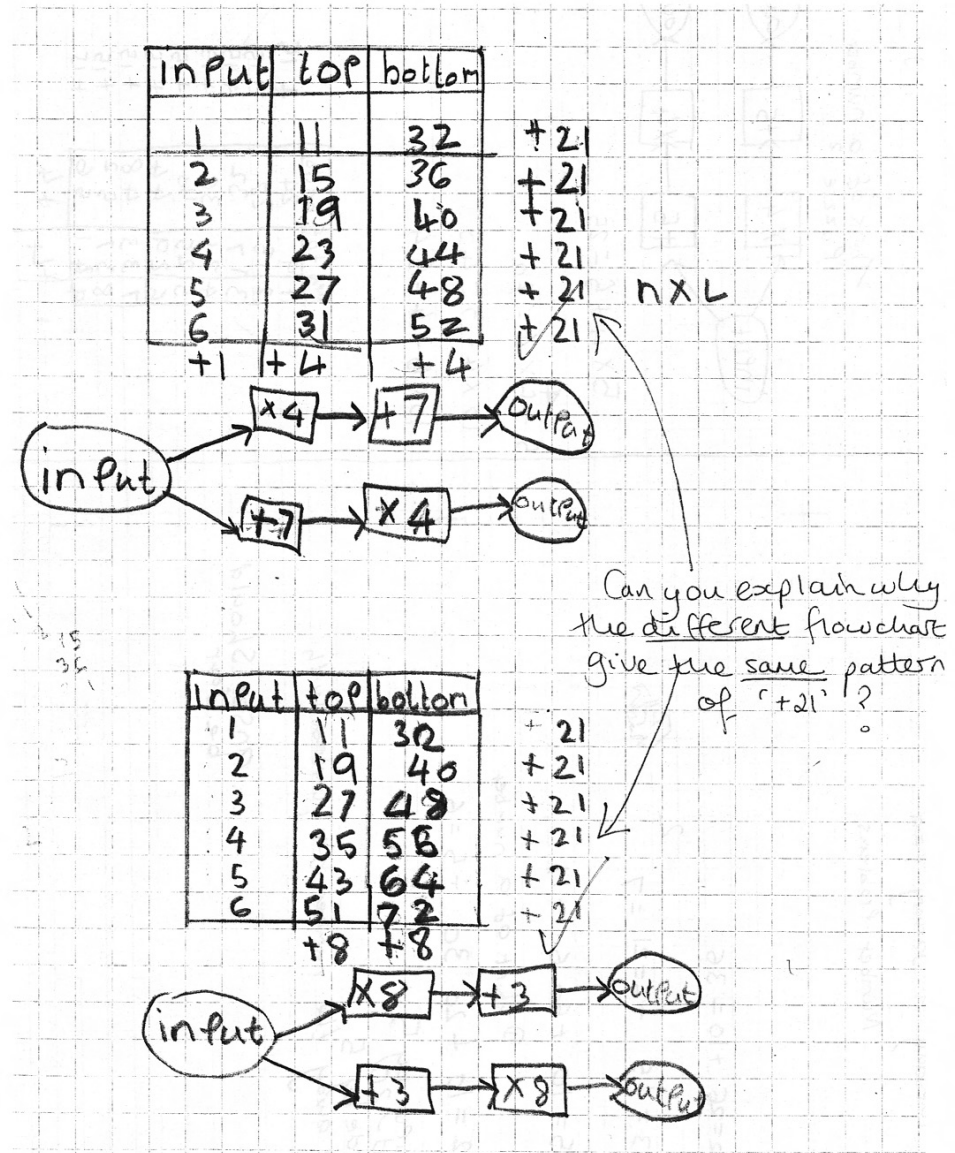
Task P

This work was done by a Year 8 student.

His teacher has written:

Can you explain why the *different* flow charts give the *same* pattern of '+21'?

Look at the student's work. Discuss how you could help him answer the teacher's question.



Lesson

12B

Notes

Algebra 5AB

Mini-assessment

13AB**Larger and larger (1)**

Find the value of v
when $u = 2$.

What happens to v
as u gets larger and
larger?

$$v = 20u - u^2$$

Commentary

This is a variant on Mini-assessment 3AB (Algebra 1AB), “Which is larger, $3n$ or $n + 3$?”. It is relevant to Lesson 13A (*Pedalo hire*, which is a variant on Lesson 3A, *Boat hire*) but is particularly targeted at Lesson 13B (*Emergency kitty*) which involves a quadratic relation. If you decide only to teach Lesson 13A, or to revisit *Boat hire*, you might want to re-use Mini-assessment 3AB, or use the alternative Mini-assessment on the next page.

Here the idea that the value of an unknown can vary is made explicit. On the other hand, coordinating the two terms of $20u - u^2$ may be more complex than comparing $3n$ and $n + 3$. A steady increase in u results in a steady increase in $20u$ whereas u^2 increases more and more rapidly. It may thus be quite challenging to see what happens to their difference.

Students who use an empirical approach, and who work systematically but just with small values of u greater than 2 (such as 3, 4, 5) might notice that v gets larger and larger - but won't necessarily spot that the increase is slowing down, given the quite large and awkward values of v . Students who adopt a more reflective approach might realise that when $u = 20$ the two terms of the expression (20×20 and 20^2) cancel out so that $v = 0$, and thus infer that somewhere between $u = 2$ and $u = 20$ the value of v starts to decrease (see table, right).

u	$20u - u^2$
2	36
3	51
4	64
5	75
6	84
7	91
8	96
9	99
10	100
11	99
12	96
13	91
14	84
15	75
16	64
17	51
18	36
19	19
20	0

Algebra 5AB

Mini-assessment

13AB**Larger and larger (2)**

What happens to v
as u gets larger and
larger?

$$v = \frac{u + 20}{u + 2}$$

Commentary

Some students might think v only ‘works’ for some values of u , eg 1, 4 and 7, where v is a whole number.

Some might ignore u and think that v is always 10.

Some might have the insight to notice that the numerator is always 20 more than the denominator:

Does this mean that v doesn't change?

Some might notice that v gets smaller and smaller:

Can it get ‘as small as you like’?

Note: We are assuming that students will think of u as positive. Interesting things happen either side of the value $u = -2$, but if students raise the possibility of u being negative you might want to suggest they stick with positive numbers ‘for now’.

Algebra 5A

Lesson

13A**Pedalo hire**

Jason is spending the day at a boating lake.

He wants to hire a pedalo for some of the time.

He can hire one from Boss Boats or Perfect Pedalos.

He notes down what they charge.

Whose pedalo should Jason choose?

Cost (in pounds) for h hours

Boss: $3h$

Perfect: $5 + 2h$

**Summary**

As with the original *Boat hire* lesson (Lesson 3A) the pedalo hire problem is used to explore the two algebraic relationships underlying the different hire charges. A variety of representations are used to express the relationships, but this time we start with algebraic expressions rather than everyday language:

- everyday language
- algebraic expressions
- tables of values
- points on a Cartesian graph.

Outline of the lesson1. Display the *Pedalo hire* task.

- Ask students for their immediate response.
- Try to express the hire charges in words.
- Ask students to consider the task further.
- Collect numerical data on the total cost, for different numbers of hours (and listen to students' ideas but don't pursue them yet).
- Record the data 'randomly' on the board, then in a randomly ordered table.
- Represent the data in an ordered table, but first try to prompt the need for this.

4 hours: £12 and £13 For 1 hour Boss is cheaper
3 hours cost £9 and £11
Perfect is cheaper for 6 hours: 18 and 17

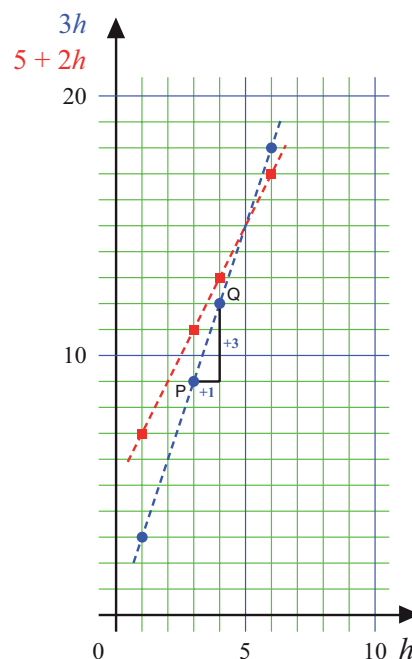
h	$3h$	$5+2h$	h	$3h$	$5+2h$
1	3	7	1	3	7
4	12	13	3	9	11
6	18	17	4	12	13
3	9	11	6	18	17

2. Represent the data as points on a Cartesian graph.

- Ask students to draw or sketch a graph of the data.
- Discuss the graph.
Relate it to the other representations and to the story.
"Are the Boss and Perfect charges ever equal?"

3. Modify the charging rules.

- Think about changing a rule slightly.
Consider how that affects its relative value, the numbers in the table, the shape of its graph, etc.



Algebra 5A

Lesson

13A

Overview

Mathematical ideas

In this lesson, we revisit ideas already encountered, particularly in the *Boat hire* sequence (Algebra Lessons 1A & 1B):

- using letters as variables
- constructing and 'reading' different representations, particularly Cartesian graphs
- interpreting algebraic expressions
- the notion of continuity.

It is worth re-reading Algebra Lessons 1A & 1B before teaching this lesson.

Students' mathematical experiences

Students should discover some of the following

- when $h = 5$ the expressions are equal
- if h increases by 1, then $3h$ increases by 3, but $5+2h$ only increases by 2.

Students might discuss

- differences and similarities between unordered and ordered tables
- different slopes and how these relate to the hourly charges
- continuity, ie whether some or all points on the line fit the relationship.

Assessment and feedback

Reflect on how students have engaged with expression, tables and graphs in previous lessons. This will help you to predict how different students will engage with the *Pedalo hire* problem.

Allow students time to 'read' the algebraic expressions and to re-express them in words.

Some students may draw some 'unconventional' graphs. Challenge the students to consider what is similar and different between these and a more conventional Cartesian graph. Decide whether you need to spend some time with either a group or the whole class teaching them the conventions of drawing a Cartesian graph.

You may need to prompt or challenge some students to 're-tell' each other's explanations.

Key questions

When is Perfect (or Boss) cheaper?

How could we record this more systematically?

What happens to the cost as the number of hours increases?

What happens if you change the rule for charges?

Adapting the lesson

With some classes, you may feel that it is more appropriate to repeat *Boat hire* (Algebra Lesson 1A), either as an alternative to this lesson, or before teaching this lesson but leaving out Lesson 13B (*Emergency kitty*).

For other classes, you may feel that the *Pedalo hire* task can be covered relatively quickly before moving on to Lesson 13B (*Emergency kitty*).

Algebra 5A

Lesson

13A**Outline of the lesson annotated)**1. Display the *Pedalo hire* task.

- Ask students for their immediate response.

Students will tend to say that one or other hire firm is cheaper. Let them voice their opinions but don't discuss their reasons at this stage. The differences of opinion should make the task more engaging for students.

- Try to express the hire charges in words.

Allow plenty of time for this. Do students realise that h is not shorthand for *hours*, but represents a *number* of hours? Do they know/remember the convention that $2h$ means $2 \times h$ and that $5+2h$ means $5+(2h)$, not $(5+2)h$?

- Ask students to consider the task further.

This allows students to think more carefully about their and other students' ideas.

- Collect numerical data on the total cost, for different numbers of hours (and listen to students' ideas but don't pursue them yet).

- Record the data 'randomly' on the board, then in a randomly ordered table.

It is worth recording the data randomly at first, even if students are already used to putting information in order.

- Represent the data in an ordered table, but first try to prompt the need for this.

Ordered tables allow one to bring out the fact that increasing the hire time by one hour, say, increases the hire charge by £3 (Boss) or £2 (Perfect). This can of course be related back to the original algebraic expressions and story, but also later in the lesson to the graphical representations of the relationship.

2. Represent the data as points on a Cartesian graph.

- Ask students to draw or sketch a graph of the data.
- Discuss the graph.
Relate it to the other representations and to the story.
"Are the Boss and Perfect charges ever equal?"

It is more illuminating to use axes with the same scale, even though the resulting line for Boss is then quite steep.

One approach is to solve $3h = 5 + 2h$.

3. Modify the charging rules.

- Think about changing a rule slightly.
Consider how that affects its relative value, the numbers in the table, the shape of its graph, etc.

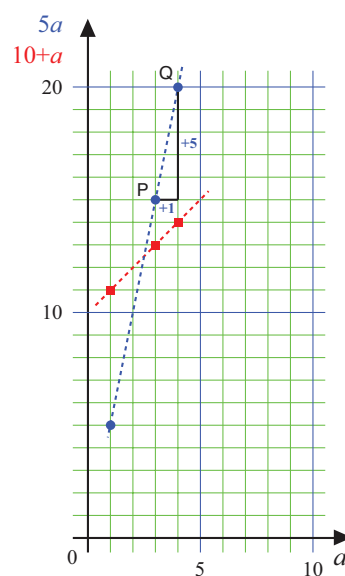
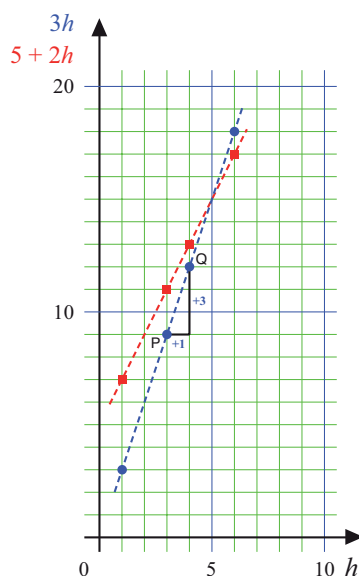
You might want to compare the graph with the original *Boat hire* graph.

What would it have cost Jason Lewis to hire his pedalo from Boss or Perfect?

Man returns from round-the-world pedal boat trip after 13 years

Saturday 6 October 2007 | [C88 comments](#)

Jason Lewis used a 26ft-long wooden pedalo, Moksha, to travel more than 17,000 miles, crossing the Atlantic in 111 days, the Pacific in 178 days, and the Arabian Sea in a relatively short 46 days.



Algebra 5A

Lesson

13A**Background****Students further develop their understanding of 'variable'**

The kind of variable we are learning about (be it called ' h ' or 'the number of hours') has these properties:

- it is a number;
- it can take on lots of different values;
- as it changes in a systematic way, the 'dependent variable' (in this case 'the total hire charge') may also change in a systematic way.

Specifically, students might discover that

- for 'largish' values, such as $h=8$, $3h$ is larger than $5+2h$
- for some 'small' values, like $h=1$, $3h$ is smaller than $5+2h$
- there is a particular value, $h=5$, where the two expressions are equal
- if h is increased by a specific amount, then $3h$ increases by 3 times that amount, and $5+2h$ increases by 2 times that amount.

Students further develop their ability to 'read' various representations

Each of the properties of variable mentioned above can be seen in the three representations (tables, expressions, graph). For example,

- if a value of h in a table goes up by 1, then the corresponding values of $3h$ have a difference of 3 (see the table, right)
- if two points representing $(h, 3h)$ are 1 horizontal unit apart on a graph, they will be 3 vertical units apart (see points P and Q on the graph, right)
- the two sets of points on the graph form a pattern (in this case, they lie on two straight lines), and so help students get a general sense of the relationships, ie a sense of what happens for a range of values rather than just isolated numerical cases
- the patterns (ie the straight lines) also suggest that there is a systematic relationship between h and $3h$, and between h and $5+2h$ and that any further pedalo-hire points that we might calculate will also lie on the corresponding line
- the two lines cross, so there is a strong incentive to find the common point [in this case, $(5, 15)$] and to consider what it might mean in the present pedalo-hire context.

Students interpret algebraic expressions

The problem could have been presented and solved perfectly well without algebraic symbolisation; on the other hand, the expressions do provide a convenient shorthand and the context should help make them meaningful to the students. It also allows us to refer explicitly to variables.

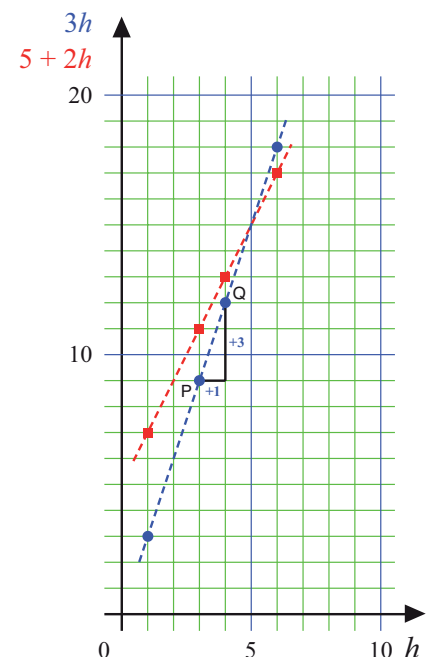
They also give students the opportunity to work informally with symbols. For example, we can compare $3h$ and $5+2h$ by thinking of $3h$ as $h+2h$. We can now use 'matching' to conclude that $h+2h$ and $5+2h$ are equal when $h = 5$.

Students engage with the notion of continuity

The graph, in particular, prompts the question of 'intermediate' values: given that each set of points lies on a line, what about some other points on that line: do they fit the algebraic relationship, and do they fit the pedalo-hire story? And do 'all' points on the line fit the relationship?

Many hire firms charge in whole numbers of hours (so a pedalo used for 2 hours 15 mins, say, would be charged for 3 hours). This results in a step function, represented on a Cartesian graph by a set of horizontal lines.

h	$3h$	$5+2h$
1	3	7
3	9	11
4	12	13
6	18	17



Algebra 5B

Lesson

13B**Emergency kitty**

Dona and Marco are planning a day trip with some classmates. They decide they need an emergency kitty.

Dona says, **What if everyone puts £10 into the kitty?**

Marco says, What if we each pay £5 plus £1 for every other person in the group?

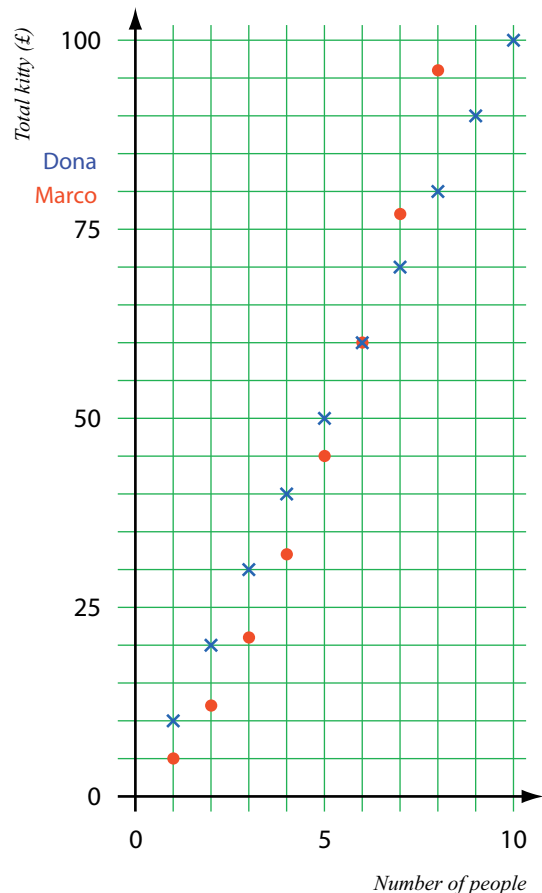
Whose rule would produce the larger kitty?

Summary

As with *Boat hire*, we compare two methods of charging, in this case involving pooled money for a kitty. One of the underlying relations is linear again, but this time one is quadratic: as the number of people increases, the total kitty increases steadily for one method, but ever more rapidly for the other. We again represent the relations in several ways: verbally, algebraically, as numbers in a table and as points on a graph.

Outline of the lesson

1. Display the *Emergency kitty* task.
 - Ask students for their immediate response.
 - Ask students to consider the task further.
 - Collect numerical data on the total kitty, for different numbers of people (and listen to students' ideas but don't pursue them yet).
 - Record the data 'randomly' on the board, then in a randomly ordered table.
 - Represent the data in an ordered table, but first try to prompt the need for this.
2. Represent the rules using symbolic algebra.
 - Ask students to represent the rules as algebraic expressions, eg $10n$ and $5n + (n - 1)n$, or algebraic relations, eg $k = 10n$, $k = 5n + (n - 1)n$.
3. Represent the data as points on a Cartesian graph.
 - Ask students to draw or sketch a graph of the data.
 - Discuss the graph.
Relate it to the other representations and to the story.
"Are Dona's and Marco's charges ever equal?"
4. Modify the charging rules.
 - Think about changing a rule slightly.
Consider how that affects its relative value, the numbers in the table, the shape of its graph, etc.



Algebra 5B

Lesson

13B**Overview****Mathematical ideas**

In this lesson, we extend the use of algebraic expressions to include quadratic relationships and consider:

- how quadratic and linear relationships differ
- the use of different representations, including Cartesian graphs, to compare quadratic and linear expressions.

When represented on a Cartesian graph, the differences between the quadratic and linear expressions are quite subtle. This is a deliberate strategy to enable you to encourage students to consider these differences mathematically rather than just empirically.

Students' mathematical experiences

Students should explain and justify some of the following

- when $n = 6$ the expressions are equal
- if n increases by 1, then $10n$ increases by 10
- the increase in $5n + (n - 1)n$ increases as n increases.

As in previous lessons, such as *Pedalo hire*, we are aiming for the students themselves to notice that systematically ordering data is helpful.

Students should have the opportunity to discuss

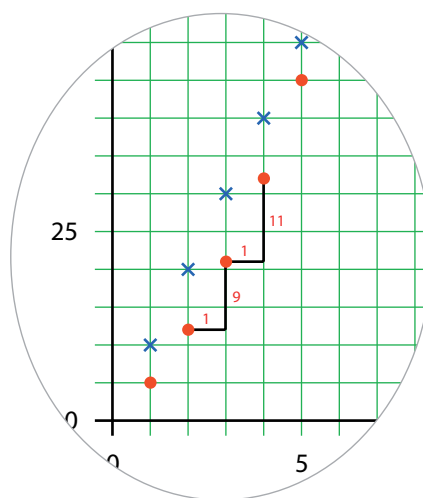
- continuity:

Since the context is discrete, joining the dots does not make sense in terms of the story ("Can there be half a person?"), although, if we allow non-integer values of n , joining the dots can be a helpful way of comparing the algebraic expressions.

Assessment and feedback

Some students may have difficulty with the scales on the Cartesian graph. Address these and other graph drawing skills quickly and directly.

Allow the students time to describe and compare the graphs and the tables. For Marco's rule, it may help some students to sketch on the graph how adding 1 to n (the number of people) makes a greater change to k (the amount of money in the kitty) as n increases.



Encourage students to “re-tell” and comment on each other’s ideas: Are these two explanations the same or different?

Key questions

How can you be sure that $5n + (n - 1)n$ and $4n + n^2$ are equivalent?

Can you make sense of the expressions in terms of the story?

Can you draw a straight line through the data for Marco’s (or Dona’s) rule?

Adapting the lesson

With some classes, you may feel that it is more appropriate to focus on linear expressions at this stage.

With other classes, you may want to follow this lesson up with the two other lessons looking at non-linear relations, 23A (*Garden plot*) and 23B (*Growing flowers*).

Algebra 5B

Lesson

13B**Outline of the lesson (annotated)**1. Display the *Emergency kitty* task.

- Ask students for their immediate response.
- Ask students to consider the task further.
- Collect numerical data on the total kitty for different numbers of people (and listen to students' ideas but don't pursue them yet).
- Record the data 'randomly' on the board, then in a (randomly ordered) table.
- Represent the data in an ordered table, but first try to prompt the need for this.

— Check that students understand Marco's rule, which is quite complex and easy to mis-interpret.

— You might have to make a conscious effort to do this, since we are in the habit of ordering data - for good reason!

2. Represent the rules using symbolic algebra.

- Ask students to represent the rules as algebraic expressions, eg $10n$ and $5n + (n - 1)n$, or algebraic relations, eg $k = 10n$, $k = 5n + (n - 1)n$.

It might be quite a challenge for students to come up with the expression $5n + (n - 1)n$, and then also to simplify it.

Some students might spontaneously come up with $4n + n^2$ and it then becomes an interesting challenge to explain why the two expressions are equivalent *algebraically* and in terms of the *story*.

Regarding the story, if a person pays £5 plus £1 for everyone else, this is equivalent to the person paying £4 plus £1 for everyone including themselves.

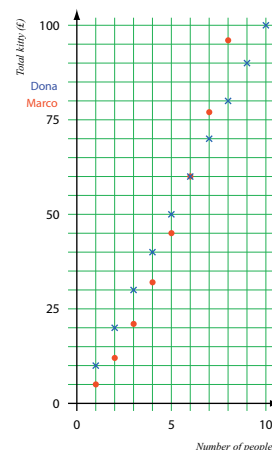
3. Represent the data as points on a Cartesian graph.

- Ask students to draw or sketch a graph of the data.

- Discuss the graph.
Relate it to the other representations and to the story.
"Are Dona's and Marco's charges ever equal?"

In our graph we have used different scales for the axes. This might make the graph easier to read, though not necessarily any easier to interpret.

Note that the non-linear nature of the red dots' path is still not that obvious. However, we can make a virtue of this, in that it pushes us towards constructing a mathematical rather than just an empirical argument about the pattern of the dots.



This can also be solved in terms of the story:

Under Dona's rule everyone pays £10. For a person to pay £10 under Marco's rule would mean paying the flat £5 fee and an extra £1 for 5 more people, so there would be 6 people in all.

It can also be solved algebraically: If $10n = 4n + n^2$, then $10 = 4 + n$ ($n \neq 0$), so $6 = n$.

Other questions: Can we join the dots? If so, what do intermediate points mean, and the point(s) where the line/curve cuts the axes?

Ignoring the story, what happens to $4n + n^2$ when n is negative? Where would the red dots go?

4. Modify the charging rules.

- Think about changing a rule slightly.
Consider how that affects its relative value, the numbers in the table, the shape of its graph, etc.

— For example, "What happens to the blue dots on the graph if we change Dona's £10 to £11?"

Algebra 5B

Lesson

13B**Background****Linear and non linear relations**

This is the first lesson in which we compare a linear relation (or rule or function) with a non-linear relation. Specifically, the relations in this lesson can be expressed as $x \rightarrow 10x$ and $x \rightarrow 4x + x^2$. The latter contains a 'squared' term, hence the name 'quadratic'.

You might want to use the term 'quadratic' in the lesson, but perhaps not make too much of it. We look at quadratic relations in more detail in lessons 23A and 23B (ALG 12A and 12B) and discuss them further in the lesson outlines.

Here, the key idea to get across is that not all relations are linear - and that what is special about linear relations is that they 'change' in a constant way. Thus in our story about Dona and Marco's emergency kitty for their day trip, with Dona's rule, for each extra person joining the day trip, the kitty goes up by £10. In contrast to this, with Marco's rule, the kitty goes up by ever larger amounts.

We can get a good sense of this by putting values in an ordered table and by graphing them. The left hand table, below, is non-ordered, to make the point how powerful it can be to put data in order, and we strongly recommend starting with a non-ordered table in the lesson.

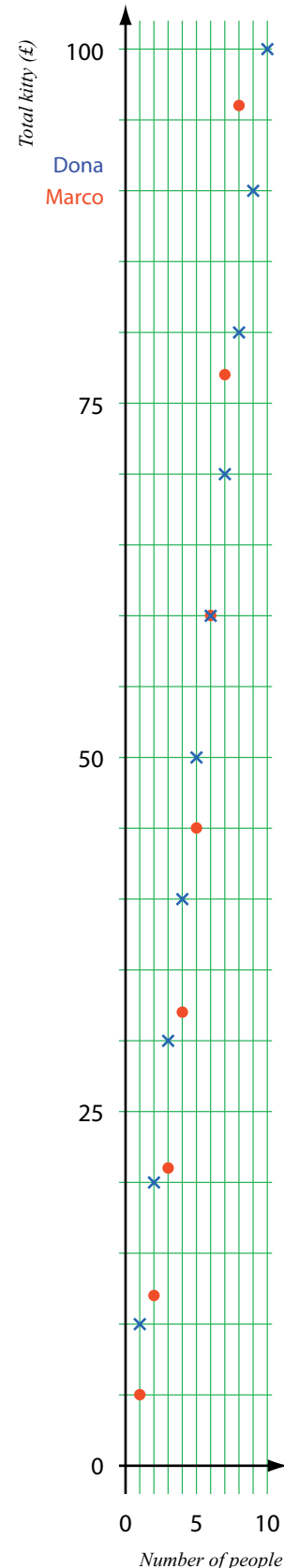
In the ordered table, we can see that as the number of people increases by 1, the value of Dona's kitty increases by £10 each time, while for Marco it increases by ever larger amounts. Moreover, and perhaps not surprisingly, this *increase* in the increase is itself regular: it is 2 each time. It turns out that having a constant 'second order difference' is a property of quadratic relations.

Because the values in our story increase so rapidly, it is not

non-ordered			ordered		
Number of people	Dona's rule	Marco's rule	Number of people	Dona's rule	Marco's rule
4	40	32	1	10	5
8	80	96	2	20	12
3	30	21	3	30	21
7	70	77	4	40	32
5	50	45	5	50	45
1	10	5	6	60	60
9	90	117	7	70	77
6	60	60	8	80	96
10	100	140	9	90	117
2	20	12	10	100	140

easy to produce a graph that shows key features, such as the fact that graph of the quadratic relation is curved. Earlier we had a graph where we attempted to compensate for this by having different scales for the axes. Here, for completeness, we show the graph with the scales the same.

Here the curved nature of the quadratic graph is not compelling, to say the least. However, the graph is still a useful device for focussing our thinking. "Should the graph be curved, or might it really be a straight line? How can we be sure?"



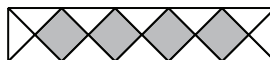
Algebra 6AB

Mini-assessment

14AB**Up the garden path**

Jan has built a garden path.

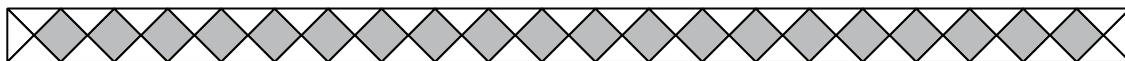
He used 4 diamond-shaped tiles and 12 triangle-shaped tiles:



Kim's path

has the same

pattern:



Kim says, "I used 20 diamond-shaped tiles".

Jan says, "Then you must have used 60 triangle-shaped tiles".

Explain Jan's reasoning. Is it right or wrong?

Commentary

The aims of this starter are to find out whether

- students tend to focus on numerical data to analyse a geometric pattern
- students consider a pattern's geometric properties to discern its structure.

Can students make sense of Jan's reasoning (whether or not it is right)?

Can students see that Jan's reasoning is wrong? What arguments do they use?

Note: If you think students will struggle with this quite complex *Garden path* pattern, use the *Flower beds* version on the next page.

Note: The pattern can be structured in various ways, such as these:



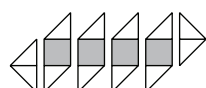
$$t = 2 + 2d + 2$$



$$t = 1 + 2d + 3$$



$$t = 2 + 2d + 2$$



$$t = 2 + 2d + 2$$



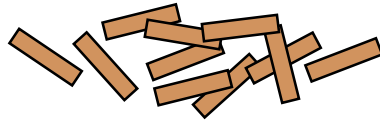
$$t = 4(d + 1) - 2d$$

Algebra 6AB

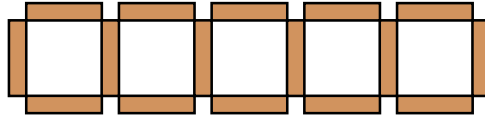
Mini-assessment

14AB**Flower beds**

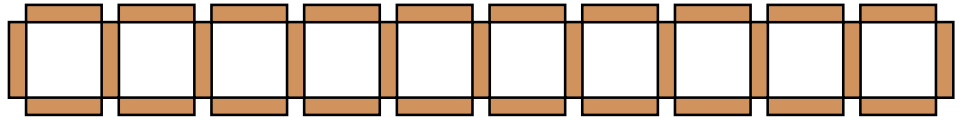
Bob and Carol have lots of 1m long wooden planks.



Bob uses some to make a row of 5 square flower beds:



Carol uses some to make a row of 10 square flower beds:



Carol says, "I've got twice as many flower beds".

Bob says, "Then you must have used twice as many planks".

Explain Bob's reasoning. Is it right or wrong?

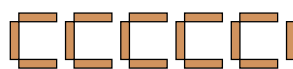
Commentary

The aims of this starter are to find out whether

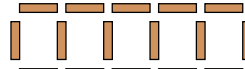
- students tend to focus on numerical data to analyse a geometric pattern
- students consider a pattern's geometric properties to discern its structure.

Can students see that Bob's reasoning is wrong? What arguments do they use?

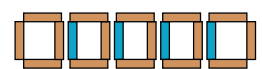
Note: The pattern can be structured in various ways, such as these:



$$p = 3b + 1$$



$$p = b + b + 1 + b$$



$$p = 4b - (b - 1)$$

Algebra 6A

Lesson

14A**Row of tiles**

A single row of grey tiles is surrounded by a double layer of white tiles.

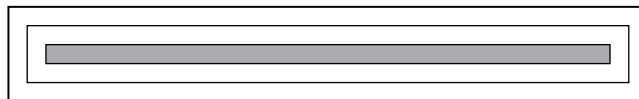
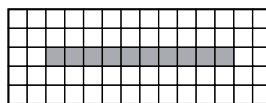
In this pattern there are 10 grey tiles.

Imagine the pattern with 30 grey tiles.

How many white tiles will there be?

Use this sketch to help you.→

Find different ways of calculating the answer.

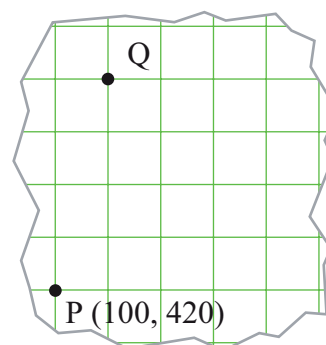
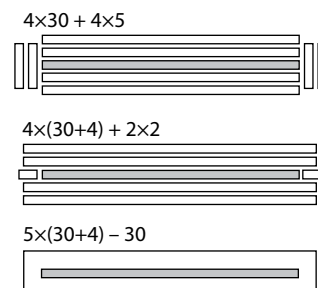
**Summary**

The main purpose of this lesson is to create *equivalent algebraic expressions* for a general relationship (in this case between numbers of grey tiles and white tiles).

It is common practice to find a general relationship by considering several examples. The ‘double layer’ pattern has been chosen because it is easy to describe and to produce, without having to show lots of examples. Put another way, examples of the pattern are not defined by their position in a sequence. This allows us to adopt a *generic* approach in this lesson, where we look for the general in a single, ‘typical’ example, eg a pattern with 30 grey tiles.

Outline of the lesson

1. Present the task to the class, including the schematic diagram of 30 grey tiles.
 - Ask for some quick estimates for the number of white tiles (for 30 grey tiles).
 - Ask students for an efficient way to work out the answer (individually or in groups).
 - Share students’ methods. Try to collect at least three different successful methods.
2. Use open expressions.
 - Discuss how to use open expressions to record the methods.
Write them on the board, eg $4 \times 30 + 4 \times 5$.
 - Compare expressions.
Discuss how one can transform the expressions to show they are equivalent.
3. Create open expressions for 100 and/or n grey tiles.
 - Ask students to work on this individually or in groups.
Share expressions on the board.
4. Possible extension: the ‘next’ pattern.
 - Consider the pattern with 100 grey tiles, say. Ask students about the ‘next’ pattern (ie 101 grey tiles, 424 white tiles). How many *extra* white tiles will it have? Imagine representing the pair of patterns as two points on a graph (points P and Q, right).



Algebra 6A

Lesson

14A**Overview****Mathematical ideas**

In this lesson we use a tile pattern whose structure can be construed in different ways. This allows us to construct equivalent expressions for the structure. It also provides an incentive to check that the expressions really are equivalent, which can be done using the laws of arithmetic to transform one expression into another, eg by expanding brackets and simplifying.

The approach is *generic*, and we are trying to find rules based on a geometric pattern, not on pure number patterns.

Students' mathematical experiences

Students may discover some of the following

- if there are instructions for constructing a geometric pattern, it is sometimes possible to see its structure by looking closely at a single example
- a pattern's structure can be construed in different, equivalent, ways
- expressions can be transformed into equivalent expressions, eg by expanding brackets and simplifying

Key questions

If there are 30 (say) grey tiles, how can we calculate the number of white tiles?

Are there other ways?

Can we write an *expression* for the number of grey tiles?

If we add a white tile, how many grey tiles are needed?

How could we use this open expression $[4(30 + 4) + 4, \text{ say}]$ for 25 white tiles? (Note: the answer is not 'halve all the numbers'!)

Assessment and feedback

The first pattern that students are shown has 10 grey tiles and 60 white tiles. Some students might infer from this that the number of white tiles is always 6 times the number of grey tiles. However, this is a purely numerical rule which ignores the structure of the tile pattern. We have deliberately chosen this 10, 60 pattern to highlight this issue, which you might want to address with your students, and it is worth checking whether any students are approaching the task in this way.

If lots of students are stuck initially, you might need to show them a first rule for finding the number of white tiles (for 30 grey tiles). You might also consider using smaller numbers or a simpler pattern, but be wary of doing this: the geometric structure may not be so obvious with smaller numbers, and a simpler pattern might be rather trivial. It is worth, though, having alternative patterns available as a fallback, eg a single row of n grey tiles surrounded by a single layer of white tiles ($2n+6$). [See page 189 for further examples.]

Adapting the lesson

When transforming open expressions like $4(30 + 4) + 4$ into $4 \times 30 + 4 \times 4 + 4$ and then into $4 \times 30 + 20$, some numbers get combined but some (in this case the 30) do not. Here the 30 stands for the number of grey tiles and is operating as a quasi variable. It might be worth constructing the corresponding algebraic expressions alongside the open expression [eg write $4(n + 4) + 4$ under $4(30 + 4) + 4$].

You could ask students to make up their own patterns - though it can be challenging to devise truly structured patterns that are easy to describe and not dependent only on position in a sequence. In the Revisits section we provide the reverse kind of challenge, of devising a pattern that fits a given rule.

Algebra 6A

Lesson

14A**Outline of the lesson (annotated)**

1. Present the task to the class, including the schematic diagram of 30 grey tiles.
 - Ask for some quick estimates for the number of white tiles (for 30 grey tiles).
 - Ask students for an efficient way to work out the answer (individually or in groups).
 - Share students' methods. Try to collect at least three different successful methods.

————— The choice of a large number like 30 is deliberate. Small numbers don't show the structure so well and can lead to students spotting spurious numerical patterns.

————— You might ask,
 "Will there be more than 100 grey tiles?",
 "More than 200?", etc.

————— If students have difficulty, ask:
 Can you instruct me how to draw the pattern without saying 'draw a double layer'? Or ask:
 How can I work out the number of white tiles without counting every one?
 If students don't spot one of the approaches, try to prompt it by, for example, drawing a 5 by (30+4) rectangle and asking: How could I work out how many tiles in this rectangle should be white?

2. Use open expressions.
 - Discuss how to use open expressions to record the methods.
 Write them on the board, eg $4 \times 30 + 4 \times 5$.
 - Compare expressions.
 Discuss how one can transform the expressions to show they are equivalent.

————— For example,
 $4(30 + 4) + 4$ can be expanded to $4 \times 30 + 4 \times 4 + 4$
 then simplified to $4 \times 30 + 20$

3. Create open expressions for 100 and/or n grey tiles.
 - Ask students to work on this individually or in groups.
 Share expressions on the board.

————— You might decide to go straight to n grey tiles, or only consider the case of 100 grey tiles.
 Again, it is possible to find a variety of (equivalent) expressions for the number of white tiles, eg

for 100 grey tiles:	for n grey tiles:
$4 \times 100 + 20$	$4n + 20$
$4(100 + 4) + 4$	$4(n + 4) + 4$
$5(100 + 4) - 50$	$5(n + 4) - n$

4. Possible extension: the 'next' pattern.
 - Consider the pattern with 100 grey tiles, say. Ask students about the 'next' pattern (ie 101 grey tiles, 424 white tiles). How many *extra* white tiles will it have? Imagine representing the pair of patterns as two points on a graph (points P and Q, right).

————— Every extra grey tile requires 4 extra white tiles. If we plot points on a Cartesian graph, with their coordinates (x, y) representing (number of grey tiles, number of white tiles), then the 'next' point will be '1 across, 4 up'.
 This has direct links with 'gradient' and with the multiplier in the expressions $4 \times 100 + 20$, $4n + 20$, etc.

Algebra 6A

Lesson

14A**Background****Equivalent structure and expressions**

It is often possible to construe (and express) a pattern in different ways, as here for our original pattern with 30 grey tiles:

- 2 rows of 30 white tiles above the row of grey tiles and 2 rows below, plus 10 white tiles at each end [$4 \times 30 + 20$]
- 2 rows of $(30+4)$ white tiles above the row of grey tiles, and 2 rows below, plus 2 white tiles at each end [$4(30+4) + 4$]
- a block of $(30+4)$ by 5 tiles of which 30 are grey [leaving $5(30+4) - 30$ white tiles].

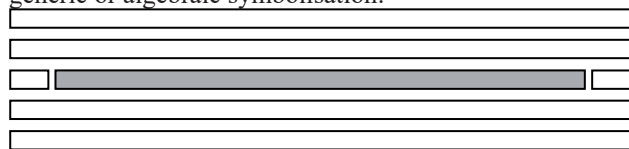
Thus, such patterns provide a fruitful context for working with equivalent algebraic expressions: students have the challenge of finding different ways to structure the pattern, and to express these algebraically; they then have an incentive for using and developing their manipulation skills, in order to check that the expressions are equivalent.

Different representations of structure

Here is a more succinct version of the second structuring listed above:

- 4 rows of $(30+4)$ white tiles, plus 2 white tiles at each end.

We can represent such a structuring in different but equivalent ways. For example, geometrically or by using generic or algebraic symbolisation:



Generic symbolisation: $(30+4) \times 4 + 2 \times 2$

Algebraic symbolisation: $4(n+4) + 2 \times 2$

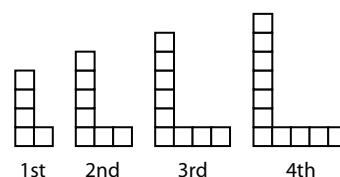
Generalising patterns

Much of the work on generalising that we give students is purely numerical. For example, we present a sequence like 20, 24, 28, 32, ..., and ask them to find a term-to-term and position-to-term rule that fits the numbers (eg, 'add 4' and ' $20 + 4(n-1)$ ' or ' $16 + 4n$ ' respectively).

In this lesson we are concerned with generalising a pattern that has a *context*, in this case a geometric pattern of grey and white tiles. One approach is to turn this into a purely numerical task, by considering specific numbers of grey (and white) tiles, listing them in a table, and then finding a rule that fits the numbers in the table. For some patterns this may be the only viable approach. However, rather than abandoning the context in this way, we can often see a pattern's structure by examining a single, typical example, ie by adopting a *generic* approach, which is what we do in this lesson.

It is worth noting that we often identify members of a geometric pattern by their *position* (rather than by direct, numeric features), as in this growth pattern of L-shapes.

Given such a sequence it is hard to resist counting the numbers of tiles and tabulating the result! And it is difficult to treat such patterns generically, since individual members are not



defined in terms of their own features, but only in relation to their neighbours. (However, one can get round this for the L-shapes, by showing several non-ordered examples, and identifying them by a feature such as their base, which would then allow questions like, "How many tiles are needed altogether for an L-shape whose base consists of 100 tiles?". Note though, that both variables here refer to white tiles, which is not as clear as using different coloured tiles, as in our example).

Generalised number, variable and quasi-variable

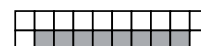
As we are adopting a generic approach in this lesson, ie generalising on the basis of single examples of a pattern rather than a sequence of examples, it could be said that we are working with generalised number here, rather than variable (although we do suggest switching to a variables focus towards the end of the lesson, when we consider what happens as the number of white tiles *changes*).

Examples of other grey → white tile patterns that could be made to work generically (if described clearly enough...)

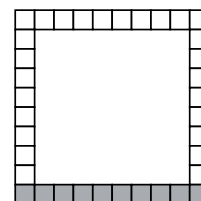
Grey stepping stones



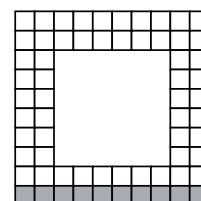
Bridge over a row of grey tiles



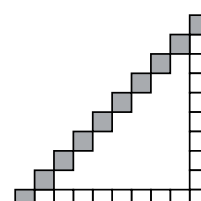
Square of tiles with a grey base



Double square of tiles with a grey base



Isosceles right-angled triangle with a grey hypotenuse



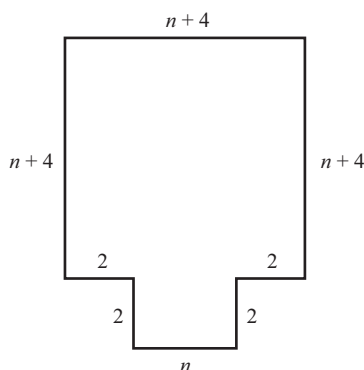
Algebra 6B

Lesson

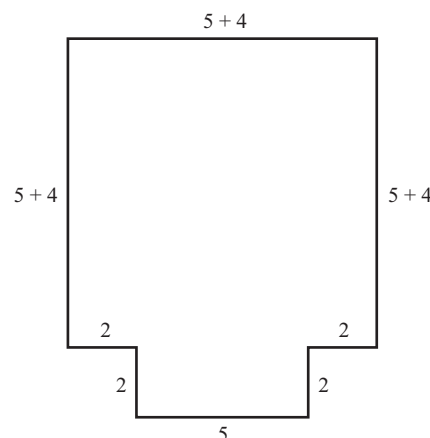
14B**A family of T-shapes**

The drawing below describes a family of T-shapes.

Every value of n produces a different T-shape.



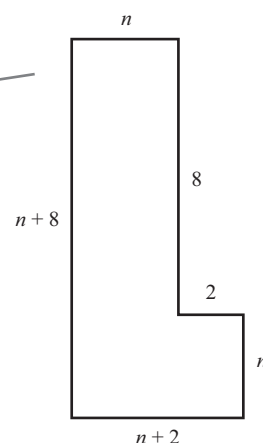
The drawing on the right shows the T-shape for $n = 5$. Find the perimeter of this T-shape.

**Summary**

The purpose of this lesson is to simplify and use *algebraic expressions* and to consider *equivalent* expressions. Students are encouraged to construct expressions for the perimeter of simple rectilinear shapes and to use them to solve problems involving perimeter. Subsequently, students are asked to invent shapes whose perimeters fit a given expression.

Outline of the lesson

- Present the task to the class.
 - Ask for some quick estimates of the perimeter when $n = 5$.
 - Ask students to work out the perimeter (individually or in groups).
 - Check answers (the perimeter is 40 units when $n = 5$).
- Consider the T-shape for $n = 100$.
 - Ask students to imagine what the T-shape looks like for $n = 100$.
 - Ask for some quick estimates of the perimeter when $n = 100$.
 - Ask students to work out the perimeter (individually or in groups).
 - Discuss students' methods.
Discuss the idea of using an expression (or formula).
Find a *simple* expression [namely $4n + 20$, or $4(n + 5)$]
- Consider the T-shape with a perimeter of 30 units.
 - Ask students to find the T-shape whose perimeter is 30 units (ie find the value of n).
 - Discuss students' methods.
Discuss the use of the earlier expression.
- Introduce this family of L-shapes.
 - Ask students to find the perimeter of the L-shape when $n = 5$, $n = 100$, $n = 2.25$
 - Why are we getting the same values as for the corresponding T-shapes?
 - Find an expression for the perimeter. Compare it to the T-shape expression.
- Draw some other families with an equivalent perimeter expression.
 - Find a family of squares.



Algebra 6B

Lesson

14B**Overview****Mathematical ideas**

This lesson is about creating and simplifying algebraic expressions (or formulae) and demonstrating their *utility*. The lesson uses a geometric context (perimeter) and deliberately involves a family of quite complex shapes (with 8 sides). This complexity means that it is fairly tedious to calculate perimeters in a step-by-step manner; however, the perimeter can be expressed algebraically in quite a simple form (eg as $4n + 20$) and so it is more efficient to use the expression to calculate perimeters.

Students' mathematical experiences

Students may discover some of the following

- a diagram of a shape can represent a family of shapes
- calculations can sometimes be done more efficiently by using an algebraic expression (or formula)
- equivalent algebraic expression can be modelled by different looking families of shapes
- equivalent expressions can be created from the same shape by construing it differently
- equivalent expressions can be created by manipulating expressions

Key questions

What does the T-shape for look like for $n = 100$?

Can we write an expression for the perimeter?

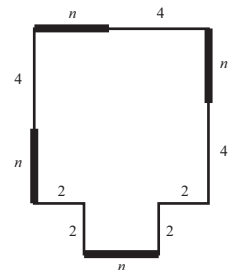
Can we simplify the expression?

How can we use the expression?

Assessment and feedback

Most students will probably not have too much difficulty seeing that an expression for the perimeter of the T-shapes, such as $n+4 + n+4 + n+4 + 2 + 2 + n + 2 + 2 + n+4$, can be written as $4n + 20$ (though some might need reminding of the convention that $4n$ means $4 \times n$).

However, some students might want to simplify this further, eg to $24n$. If so, you should discuss this with the class and consider ways of seeing whether this is true. One way is to test it numerically, eg see what happens to $4n+20$ and $24n$ when $n=3$, say. Another might be to refer back to the T-shape and highlight the four sections of length n units. How can we possibly equate the total perimeter with $24n$?!

**Adapting the lesson**

Some students may be expecting that there should be some 'real life' point to the T-shape context, when there isn't. Encourage students to see the context as a piece of mathematics that can be interesting in its own right. For example, What happens to the T-shape as n gets bigger and bigger? Is the shape still the 'same'? How can we describe it in words? What happens when n gets close to 0? Is there a minimum value for the perimeter? Can n be negative?

Resources

A Geogebra file showing an animation of the T-shape is available on the website, www.iccams-maths.org

Algebra 6B

Lesson

14B**Outline of the lesson (annotated)**

1. Present the task to the class.
 - Ask for some quick estimates of the perimeter when $n = 5$. — Check that students know what we mean by 'perimeter'.
 - Ask students to work out the perimeter (individually or in groups). — Students are likely to do this in a step-by-step manner, which is fine at this stage (eg, going clockwise from top-left: $9+9+2+2+5+2+2+9$). Don't spend time discussing methods here - wait till Part 2.
 - Check answers (the perimeter is 40 units when $n = 5$).

2. Consider the T-shape for $n = 100$.
 - Ask students to imagine what the T-shape looks like for $n = 100$. — This question is to encourage students to visualise what happens as n varies, and to look at the shape in a more holistic way - the shape is a square on top of a (relatively) tiny stem - so the perimeter will be close to $4 \times (100 + 4)$. Thus you might want to ask students to describe the T-shape when $n = 100$, or sketch it on the board.
 - Ask for some quick estimates of the perimeter when $n = 100$. — Some students will probably also tackle this in a step-by-step way, but the large value of n might prompt some to use a more 'compacted' approach, eg by finding 3 lots or 4 lots of $(100+4)$ and then adjusting for the more complex part of the shape along the bottom.
 - Ask students to work out the perimeter (individually or in groups). — Discuss how to produce an expression (or formula) for the perimeter, involving n . Discuss how to simplify this and how the expression (or formula) could have been used to find the perimeter for $n = 100$.
 - Discuss students' methods.
Discuss the idea of using an expression (or formula).
Find a *simple* expression [namely $4n + 20$, or $4(n + 5)$]

3. Consider the T-shape with a perimeter of 50 units.
 - Ask students to find the T-shape whose perimeter is 50 units (ie find the value of n). — The value of n here is not a whole number (7.5). Students who use a general, simplified expression (or formula) for the perimeter ($4n + 20$ or equivalent) are more likely to find this value (even with trial and improvement) than those who adopt a step-by-step approach.
 - Discuss students' methods.
Discuss the use of the earlier expression.

4. Introduce the family of L-shapes.
 - Ask students to find the perimeter of the L-shape when $n = 5$, $n = 100$, $n = 7.5$
 - Why are we getting the same values as for the corresponding T-shapes? — The expressions (or formulae) for the perimeters are equivalent for the two families; they can both be written as $4n + 20$, say. (Geometrically, we can re-arrange the edges of a shape from one family to exactly cover the edges of the corresponding shape from the other family.)
 - Find an expression for the perimeter. Compare it to the T-shape expression.

5. Draw some other families with an equivalent perimeter expression.
 - Find a family of squares. — An expression for the perimeter is $4(n + 5)$; this can be modelled by a square of side $(n + 5)$.
Another, fairly obvious shape with the same perimeter would be an n by $(n + 10)$ rectangle, or $(n + 1)$ by $(n + 9)$, etc (see Fig 1 on page 193).
Equivalent expressions can also be created by construing a shape in different ways (see Fig 2 on page 193).

Algebra 6B

Lesson

14B**Background**
Generalising shapes

Here we are presented with what looks like a single shape, but is actually a family of shapes already defined in general terms. Specifically, we are presented with a figure where the lengths of sides are defined in terms of a variable n . Thus the focus is not on *making* a generalisation but on *interpreting* it, ie on 'seeing' the generality in the given representation.

We start with a family of T-shapes. It can be instructive to try to describe the family in words, and to imagine what the members look like for different values of n , and perhaps to draw several, progressively larger, members of the family.

As n increases, so the shape increases - however, the members of the family are not in proportion. For example, the 'stem' of the T-shape is always 2 units high, whereas the square-shape on top of the stem can vary in height.

Equivalent expressions

Here, equivalent expressions occur in several ways:

- in the process of simplifying an expression (eg, $n+4 + n+4 + n+4 + 2 + 2 + n + 2 + 2 + n+4 = 4n + 20$);
- through construing the perimeter of a given family of shapes in different ways (see Figure 1), and
- through finding 'equivalent' families of shapes, ie ones where the perimeter of a member of one family is the

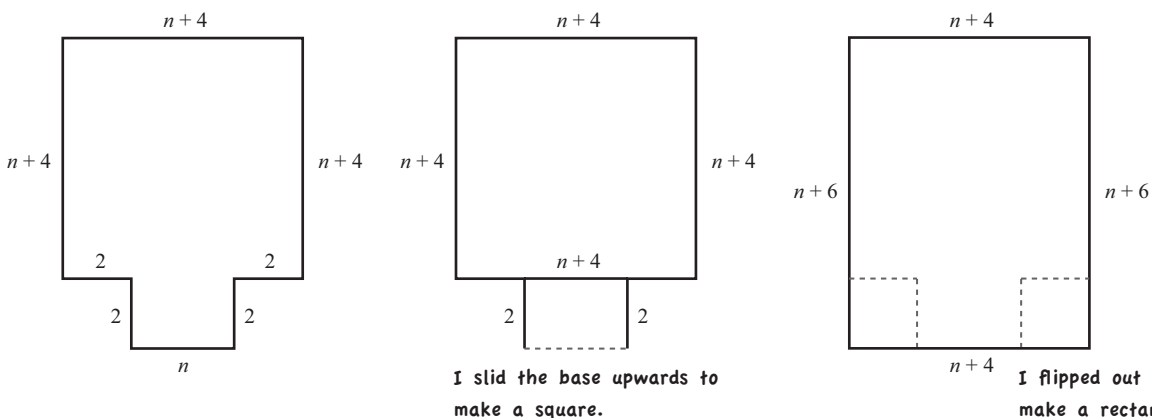
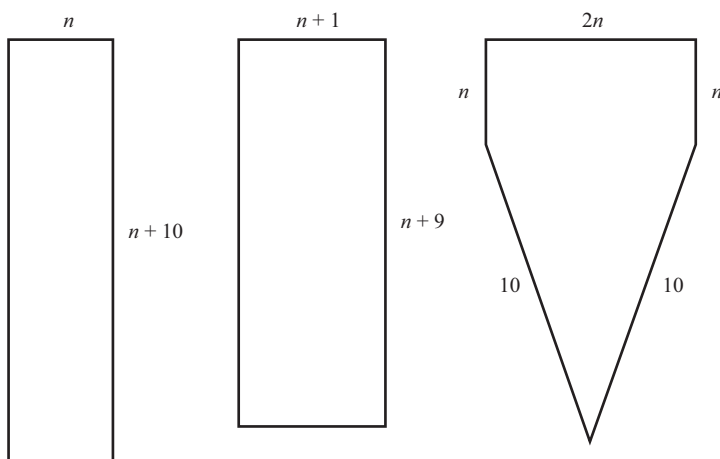
same as the perimeter for the member with the same value of n from another family, for any value of n (see Figure 2).

Generalised number, variable, letter as object

Here, by virtue of using a single figure to represent a family of shapes, we are probably viewing n primarily as a generalised number rather than a variable (though we can readily move between the two).

However, it is also easy to see n as a label for the side of a figure or part of a side (rather than as an unknown number) and so slip into using the *letter as object*. This does not lead to any overt contradictions here as we can painlessly slip back into the correct usage. Thus the letters can, temporarily, be thought of as quite tangible objects (rather than unknown numbers) which are to be collected together.

However, we do need to be careful about using letter as object (as in ' a stands for apples') as this is a move away from the central idea that we are dealing with numbers and because it can lead to pseudo-algebraic statements which are plain wrong (as when the letters d and w are defined as 'number of days' and 'corresponding number of weeks', but are treated as simply standing for 'days' and 'weeks', thus leading to $7d = w$, when the actual relationship between the numbers is $d = 7w$).

**Figure 1****Figure 2**

Multiplicative Reasoning 9AB

Mini-assessment

15AB

Multiplication (1)

$$15 \times 12 =$$

How many ways can you think of
for finding the answer?

Commentary

The main aim of this first multiplication Mini-assessment is to see how readily students can come up with different sorts of expressions for the multiplication 15×12 . For example,

$$10 \times 12 + 5 \times 12$$

$$30 \times 6$$

$$15 \times 4 \times 3.$$

Record the students' methods on the board as fully as possible.

What different ways do the students suggest for doing the calculation?

Can students think of more than one method?

Can students carry out the methods fluently?

You might find it useful to repeat the activity with other calculations, such as 12×25 , 14×16 or 18×15 .

Multiplicative Reasoning 9A

Lesson

15A**Ways of multiplying: A**

Some people are estimating the value of 15×12 .

Astrid says, The answer is more than 144.

Benny says, The answer is less than 400.

Conny says, It's more than 100.

Explain how they might be making these estimates.

$$15 \times 12 =$$

Summary

As with the Year 7 Lesson 1A, this lesson includes a strong assessment element designed to help you plan a *programme* of work on students' understanding of multiplication and on their skills and fluency.

The lesson looks briefly at estimation, which is developed further in lesson 15B. The main focus is on partitioning and a 'compensation' method of multiplying involving factors. The lesson thus builds on the work on arrays in Lessons 2A and 2B. Depending on students' knowledge, each part of the lesson could become a lesson in itself.

In the light of students' responses to the second 15AB Mini-assessment activity, you might want to develop a further lesson on multiplication, this time with the focus on multiplication by a non-integer (see page 199).

Outline of the lesson

1. Consider ways of estimating 15×12 .

- Discuss Task A, above (as a class or in small groups).
"Which estimates are 'good enough'? Can you think of any more?"

2. Consider ways of calculating 15×12 .

- Refer back to the first 1AB starter activity. Discuss students' methods.
Try to identify methods that involve
 - repeated addition
 - partitioning (including long multiplication and the grid method)
 - 'compensation'.
- Discuss *why* the methods work (or don't work).
Find ways to *represent* the partitioning and compensation methods.

— See **page 198** for representations involving

- arrays of dots
- areas of rectangles
- numerical expressions.

3. Consolidation (optional): present Task B, C and D (below).

- Discuss the methods that seem to have been used.
- Ask students to come up with similar ways of finding 15×12 , using partitioning (as in Task B) and compensation (Task C).

Ways of multiplying: B

Fran and Gil are calculating 15×12 .

Fran says,

I get 180, because $72 + 108 = 180$.

Gil says, I get 180, because $75 + 105 = 180$.

What do you think they are doing?

Why does it work?

Ways of multiplying: C

Ken is calculating 15×12 .

Ken says,

The answer is 180 because $30 \times 6 = 180$.

What do you think Ken is doing?

Why does his method work?

Ways of multiplying: D

Clara is calculating 15×12 .

She says,

The answer is 170 because $17 \times 10 = 170$.

What do you think Clara is doing?

Why does her method not work?

Multiplicative Reasoning 9A

Lesson

15A

Overview

Mathematical ideas

This lesson contains a strong assessment component designed to help you plan a programme of work on students' understanding of multiplication and their skills and fluency in multiplying.

The lesson is based on students' responses to the first 15AB Mini-assessment activity, $15 \times 12 =$. We consider the methods and procedures used, and see whether students can provide a rationale for them. We focus on partitioning and on a method based on factors, and provide tasks involving these methods which you might want to use - along with a slide showing a variety of student responses.

Students will be familiar with two special cases of partitioning, namely long multiplication and the grid method in which the numbers are partitioned into units, tens, hundreds, etc. Students might also have drawn on the factor method when multiplying large numbers or decimals. Here we examine why the methods work and make use of various models or representations: rectangular arrays, the area model and algebraic symbolisation.

Students' mathematical experiences

Students have the opportunity to assess what they know

- about multiplication
- about multiplying.

They might realise that

- the value of a multiplication can be estimated in a variety of ways
- a multiplication can be represented in various ways
- multiplications can be carried out in various ways.

Assessment and feedback

The lesson (and Revisits activities) should help you to plan a programme of starter activities and lessons on multiplication.

It should also enable students to assess what they need to know better.

In the light of the first Mini-assessment activity, identify students who could be called upon to explain their methods. Also, decide whether to make use of the supplementary Tasks and the Slide of student responses.

Review Question A in the Revisits section asks students, working in pairs, to assess their knowledge of the multiplication tables.

Review Question B in the same section asks students to assess how good they are at

- estimating
- finding multiplication methods
- calculating
- using multiplication tables.

The later 'quick methods' questions in that section allow you to assess how readily students are able to think about multiplicative expressions without evaluating them.

Key questions

What can you do well when multiplying numbers?

What do you need to do better?

What are the advantages and disadvantages of the various methods?

Adapting the lesson

It is worth visiting this kind of work on a regular basis. It is easy to change the task difficulty by varying the numbers in the expressions (eg by changing $15 \times 12 =$ to $1.5 \times 12 =$ or to $15 \times 17 =$, or by using 3 figure numbers, or fractions or decimals) and it is useful to vary the structure of the expressions (eg by having a longer string of multiplications or by changing the operation to division). Further consolidation, extension and review questions are shown in the Revisits section.

In the light of the first Mini-assessment activity, you might also want to devote some time to repeated addition and to consider how this can be 'condensed'. You might also want to build a lesson on the second Mini-assessment activity, Multiplication (2) [$8 \times ? = 18$], which is discussed on page 199.

Multiplicative Reasoning 9A

Lesson

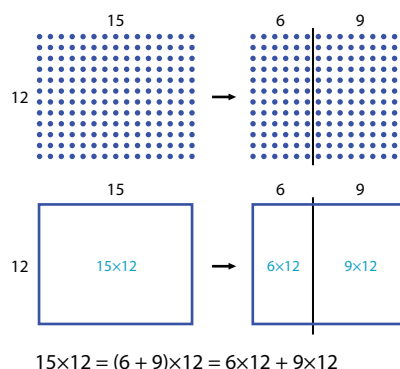
15A**Outline of the lesson (annotated)**1. Consider ways of estimating 15×12 .

- Discuss Task A, above (as a class or in small groups).
“Which methods are ‘good enough’?”
“Can you think of any more?”

2. Consider ways of calculating 15×12 .

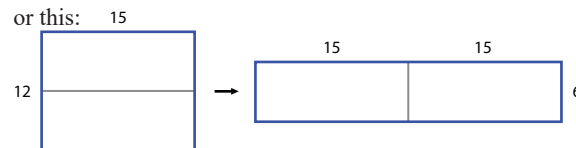
- Refer back to the first 1AB Starter activity. Discuss students' methods. Try to identify methods that involve
 - repeated addition
 - partitioning (including long multiplication and the grid method)
 - ‘compensation’.
- Discuss *why* the methods work (or not). Find ways to *represent* the partitioning and compensation methods.

We can represent partitioning (as in Task B) by using arrays, or the area model, or expressions with brackets:



We can represent ‘compensation’ (as in Task C) like this:

$15 \times 12 = 15 \times (2 \times 6) = (15 \times 2) \times 6$, or this:
 $15 \times 12 = 15 \times 12 \times (2 \times 0.5) = (15 \times 2) \times (12 \times 0.5) = 30 \times 6$;
 or this:



3. Consolidation (optional): present Tasks B, C, D.

- Discuss the methods that seem to have been used.

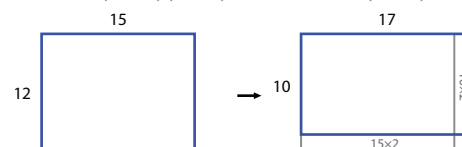
- Ask students to come up with other ways of finding 15×12 , using partitioning (as in Task B) and compensation (Task C).

In Task B, we have chosen unusual partitionings for 15×12 to help students think about what is going on. You could show students the familiar long multiplication and grid method partitionings:
 $(10+5) \times 12$, and $15 \times (10+2)$, and $(10+5) \times (10+2)$.
 “How are these similar to/different from Fran and Gil’s methods? Which are easier? Why?”

Geometrically, we can interpret Task D as changing a 15 by 12 rectangle to a 17 by 10 rectangle (below), ie removing a 15 by 2 strip but only adding a 10 by 2 strip.

Algebraically, $17 \times 10 = (15 + 2)(12 - 2)$
 $= 15 \times 12 - 15 \times 2 + 2(12 - 2)$
 $= 15 \times 12 - 15 \times 2 + 10 \times 2$.

or $(a + 2)(b - 2) = ab - 2a + 2(b - 2)$.



You might want students to try a product like 17×19 , which lends itself to using subtraction as well as addition, as in $(20-3) \times 19$ and $17 \times (20-1)$, etc;
 or seemingly difficult products like 3.5×18 ;
 or ask students to think up challenging products for which they have a simple solution.

Encourage students to use jottings to help them calculate.

Multiplicative Reasoning 9A

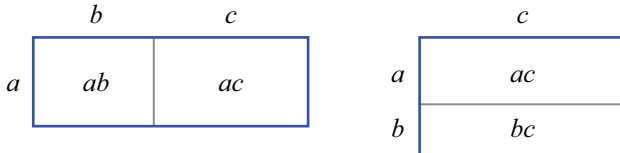
Lesson

15A

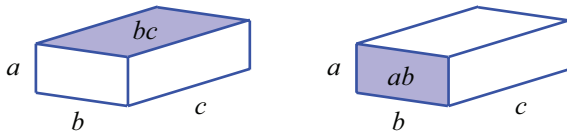
Background

Ways of calculating and the laws of arithmetic

Partitioning rests on the distributive law, namely that multiplication is distributive over addition:



$a(b+c) \equiv a \times b + a \times c$ and similarly $(a+b)c \equiv a \times c + b \times c$. The method based on factors rests primarily on the associative law:



$a \times (b \times c) \equiv (a \times b) \times c$.

The fact that multiplication is commutative [$a \times b = b \times a$] may also come into play.

Students are not expected to know these laws explicitly, though they will know them, to a greater or lesser extent, as 'laws in action'.

The laws don't necessarily apply to other operations and it is easy to mis-apply them, eg by wrongly assuming that $60 \div 13$ is equivalent to $60 \div 10 + 60 \div 3$, or that $20 - (8 - 5)$ is equivalent to $(20 - 8) - 5$.

The underlying laws can be made more explicit by writing numerical expressions out in full. For example, if a student suggests calculating 15×12 by partitioning 15 into $7+8$, we can write this as $7 \times 12 + 8 \times 12$ and as $(7+8) \times 12$.

Do students see that these are equivalent?

If 15×12 is transformed into 30×6 , say, this might have been done by doubling 15 and halving 12, or by changing 12 to 6×2 . Again, it is worth writing these moves out more fully, in these two kinds of ways:

A: $15 \times 12 = (15 \times 2) \times (12 \div 2) = 30 \times 6$, or even
 $15 \times 12 = 15 \times 12 \div 2 \times 2 = 15 \times 6 \times 2 = 15 \times 2 \times 6 = 30 \times 6$.

B: $15 \times 12 = 15 \times (2 \times 6) = (15 \times 2) \times 6 = 30 \times 6$.

Multiplication (2)

This second Mini-assessment activity is designed to see whether students regard a task like this (below), where the multiplier is not a whole number, as purely multiplicative (ie as $8 \times 2.25 = 18$ or $8 \times 2\frac{1}{4} = 18$) or in a more additive way (for example as $8 + 8 + \text{a-quarter-of-}8 = 18$).

Students who are thinking purely in terms of whole numbers may say the task is impossible (since 2 is too small and 3 is too large).

$$8 \times ? = 18$$

Can you find the number?

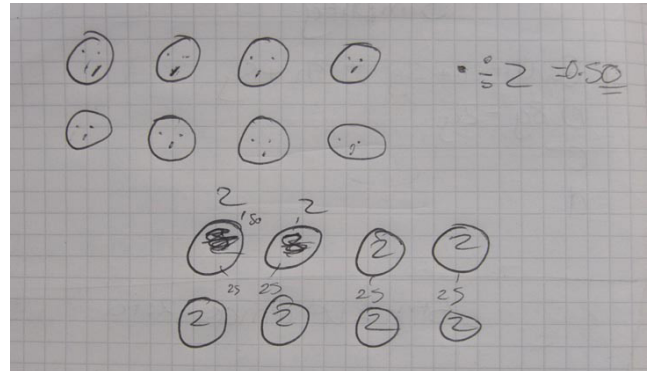
Others may work their way to 18 in this kind of way:

$8 \times 2 = 16$, so I need 2 more.

$8 \times \frac{1}{4} = 2$, so the answer is $2 + \frac{1}{4}$ or $2\frac{1}{4}$.

Students who are thinking multiplicatively might see that the missing number can be found by division, namely $18 \div 8$. In this regard, it is worth asking students, towards the end of the Mini-assessment activity, to solve the task with a calculator: do they use division, or do they stick with multiplication (eg by adopting a trial and improvement approach)?

A very grounded approach based on 'sharing' is illustrated below. Here the student set out to distribute 18 dots amongst 8 circles. She started by assigning 2 dots to each circle, with an extra dot for the first two circles. She then distributed these 2 extra dots more evenly, first by halving them and then by quartering them.



Such a grounded approach is not to be disparaged since it is valid and meaningful to the student. Nonetheless, one would want to help the student move to a more multiplicative approach over time.

In a subsequent lesson, it is worth discussing how to model the task. The area model fits well (though some students might have a shaky understanding of area); the array model fits less well, but produces some nice challenges:

How many rows of 8 dots make 18 dots?

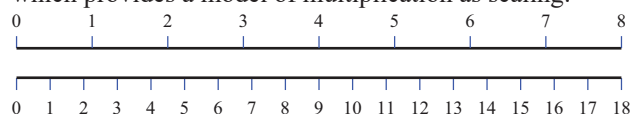
How many dots are there in a row, if 8 equal rows make 18 dots?

It is also interesting to contrast a partitive and a quotative view of division, as in these two examples respectively:

An 18 cm rod is cut into 8 equal pieces. How long is each piece?

In how many 8 cm lengths can an 18 cm rod be cut?

We consider the partitive (or sharing) model in more detail in later lessons. We also consider the double number line, which provides a model of multiplication as scaling:



Here both scales are linear and the 0s line up. Each number on the top line is mapped onto the number directly below by the multiplier $\times 2\frac{1}{4}$.

You might want to vary the numbers in the task: $8 \times ? = 20$ is probably easier because it involves $\frac{1}{2}$; $8 \times ? = 19$ is probably harder because it does not involve a simple fraction.

Multiplicative Reasoning 9B

Lesson

15B**Is this enough?**

Mr Gregg is organising a school trip but has left it a bit late to hire the coaches.

He phones Starr Coaches.

Read the conversation.

Are there enough coaches? Yes or No?

Mr Gregg: Please help! I need lots of coaches for next Saturday.

Mrs Starr: I could let you have 13 coaches. Each can take 43 passengers.

Mr Gregg: Ah! Wait! Let me think! Is this enough for 395 people?

Summary

In this lesson we help students realise more fully that it is not always necessary to perform precise calculations to arrive at a satisfactory answer to a problem. To this end, we present students with a series of tasks that can be solved *mentally* using simple calculations based on rounding.

Outline of the lesson

1. Are there enough coaches?

- Introduce the above scenario to the class, but **don't show the phone conversation yet**. Say that you are going to read through a phone conversation and that as soon as you've finished you want students to provide a Yes/No answer.
- Display the phone conversation and read it through. "Hands up 'Yes'. Hands up 'No'."
- Discuss quick ways of solving the task.
"Why is it not necessary to calculate 13×43 ?"

— Stress that all the tasks in this lesson are to be solved mentally: no written calculations allowed!

2. Swimming pool.

- Repeat Stage 1 for this swimming pool task.
- Invent similar swimming pool tasks that are easy to solve by estimating.
- Invent similar tasks that are *not* easy to solve by estimating. [Optional]

Sophie is in the Oasis swimming pool.
The pool is about 27 m long and she wants to swim 1000 m.
She swims 29 lengths. Is this enough?

3. Using short cuts.

- Present students with some of these estimation tasks. Ask them to look for shortcuts.
[For example, '19 seconds for 1 potato' is about '1 minute for 3 potatoes'.]

Polka Dot Balloons cost 49p each.
Norman has £9.
Is this enough to buy 21 balloons?



On average, Jake takes 19 seconds to peel a potato.
He has 10 minutes to peel 28 potatoes.
Is this enough time?

Flora has 5 litres of orange juice.
Is this enough to fill eighteen 230 ml glasses?



4. Invent other 'short cut' estimation tasks.

- Try to invent other 'short cut' tasks. Include some with different scenarios from those in Stage 3.

Multiplicative Reasoning 9B

Lesson

15B

Overview

Mathematical ideas

This lesson focuses on estimation. The tasks presented can be solved mentally by carrying out simple calculations based on approximations. Estimation and approximation are powerful approaches for generating appropriate solutions to 'real-world' problems as well as for checking calculations. Yet, many students prefer an exact answer produced by using a formal written method or by a calculator. Here we present tasks in ways that encourage estimation and approximation and can be solved using mental methods.

Students' mathematical experiences

Students have the opportunity to

- estimate the solutions to problems
- record mental methods
- consider how different solutions to a problem are related (particularly solutions involving multiplication or division).

They might realise that

- simple multiplicative problems can be estimated in a variety of ways
- some estimates can provide an upper or lower bound to a problem, whilst others do not
- some problems are easier to estimate than others.

Assessment and feedback

You may need to clarify each problem. For example, Mr Gregg needs to know whether the company has enough coaches available to accommodate all the people who are going on the school trip.

Some students may want to calculate exact answers. Encourage them to be bold when estimating.

When calculating mentally, some students may find it helpful to make jottings recording the structure of the problem, their method or of partial / intermediate answers. Encourage them to do this, but discourage them from using formal written methods.

Key questions

Why is it not necessary to calculate the exact answer?

Can you explain your method?

Jo's method involved division, whilst Kai's method involved multiplication. Can you explain why both work?

Adapting the lesson

It is worth visiting this kind of work on a regular basis. Several tasks are provided in the Revisits section that extend and develop the ideas in this lesson.

Multiplicative Reasoning 9B

Lesson

15B**Outline of the lesson
(annotated)****1. Are there enough coaches?**

- Introduce the above scenario to the class, but **don't show the phone conversation yet**. Say that you are going to read through a phone conversation and that as soon as you've finished you want students to provide a Yes/No answer.
- Display the phone conversation and read it through. "Hands up 'Yes'. Hands up 'No'."
- Discuss quick ways of solving the task.
"Why is it not necessary to calculate 13×43 ?"

The simplest calculation to perform here is 10×40 . Both numbers are underestimates so the resulting number of passengers, 400, is an underestimate too. However, this is greater than 395 so we can be sure that 13 coaches is enough.

You might want to return to this task later in the lesson to discuss whether fewer coaches would do. Clearly 10 would suffice ($10 \times 43 = 430 > 395$) but what about 9? Can we resolve this mentally?

2. Swimming pool.

- Repeat Stage 1 for this swimming pool task.
- Invent similar swimming pool tasks that are easy to solve by estimating.
- Invent similar tasks that are *not* so easy to solve by estimating. [Optional]

Sophie is in the Oasis swimming pool.

The pool is about 27 m long and she wants to swim 1000 m.

She swims 29 lengths. Is this enough?

Here $30 \times 30 = 900$ would be an overestimate of the distance Sophie has swum so far. However it is less than 1000, so we can conclude that she has not yet reached her target distance.

Such stories might well arise, unintended, in the previous part. Imagine the *swimming pool* story with '29 lengths' changed to '31 lengths', say. If we still used $30 \times 30 = 900$ to estimate the distance swum, it would be more difficult to tell whether this was an over- or under-estimate, since one number has been rounded down and the other rounded up.

3. Using short cuts.

- Present students with some of these estimation tasks. Ask them to look for shortcuts.
[For example, '19 seconds for 1 potato' is about '1 minute for 3 potatoes'.]

Polka Dot Balloons cost 49p each.

Norman has £9.

Is this enough to buy 21 balloons?



On average, Jake takes 19 seconds to peel a potato.

He has 10 minutes to peel 28 potatoes.

Is this enough time?

Flora has 5 litres of orange juice.

Is this enough to fill eighteen 230 ml glasses?



Norman can buy 2 balloons for £1 (with a tiny amount of money left over). So he can buy 18 balloons with £9 (with some money left over - is this enough to buy another 3 balloons...?).

230 ml is just under one quarter of a litre. So Flora can comfortably fill 4 glasses with 1 litre of juice, and 20 glasses with 5 litres of juice.

4. Invent other 'short cut' estimation tasks.

- Try to invent other 'short cut' tasks. Include some with different scenarios from those in Stage 3.

Multiplicative Reasoning 9B

Lesson

15B

Background

Seeing the need to estimate

Students' difficulties with estimation tasks can stem in part from their not seeing the need to estimate, and believing instead that a proper solution to a mathematics problem involves a detailed calculation. Thus it is important to get students into the habit of estimating. In turn this means finding tasks where students can appreciate the purpose of estimating and see that performing full-blown calculations is not always necessary.

Finding suitable contexts

It is also important to check that students are sufficiently familiar with the context in which a task is set. We have found students who seem to lack a sound knowledge of how units of distance, say, or time, are related, or who don't have a good sense of how far it is from home to school (perhaps because they travel by car or bus), or how long their lessons last (perhaps because they use their phone to tell the time). And they may have little experience of ascertaining best buys, or knowledge about the cost of a bus fare or a pint of milk. Of course, other students will know these things.

Estimation and approximation

An estimation is a rough or educated guess of an unknown quantity or the outcome to a problem and can often be carried out mentally. It is useful to estimate the outcome of a problem in order to check whether our more exactly calculated answer is reasonably close to what we would expect. It is important that students do develop the habit of estimating before working out a more exact solution to the problem.

For many 'real-life' problems, a rough mental estimate or approximation can be a sufficiently precise solution for the context. For example, in the first task presented in this lesson, Mr Gregg can work out that Starr Coaches have enough coaches available by working out that 10 coaches can carry at least 10×40 people or that 400 people can be easily accommodated by 10 coaches.

Often school mathematics approximation is dealt with procedurally in terms of rounding (e.g., "round up if the number ends in 5 or more, otherwise round down"). Judging the appropriate precision for a sensible answer to a problem requires a different level of reasoning. For example, in the second task presented, if Sophie swims complete lengths of the pool, she cannot swim exactly 1000m. But she may decide that 27 lengths, or 999m, is sufficiently close. If she wanted to swim 1500m, say, she could either complete 51 lengths, which is 21m less than 1500m, or 52 lengths which is 8m more than 1500m.

Estimation can also be a useful way of 'decluttering' a

problem in order to 'see' the structure. Like many 'real-life' problems, the estimation tasks presented in this lesson involve 'tricky' numbers. By approximating these to numbers that can be tackled mentally, it is possible to see how the problems presented in this lesson can be modeled multiplicatively and then solved using multiplication, division or a combination of the two.

Revisits

It is worth tackling estimation tasks on a regular basis, and we include some more tasks in the Revisits section. This also includes some pure tasks, to underline the importance of checking calculations.

Multiplicative Reasoning 9B

Lesson

15B

Notes

Multiplicative Reasoning 10AB

Mini-assessment

16AB**Free lentil soup**

The soup tin says,
 50% extra free.
 George says,
 That means one third
 or about 33% of the
 soup is free.
 Can both statements be right?

**Commentary**

The aim of the Mini-assessment is to assess students' understanding of percentage change (in particular, the importance of determining which of several given quantities represents 100 % and of being able to switch between them).

Do students focus just on the whole tin (eg, "It is not 50% because the green line is not half way down")?

Do students agree that both statements can be true?

What kinds of argument do students use? For example,
 Do they make use of decimals or fractions?

Do they consider a numerical example?

Do they make use of the double number line?

Multiplicative Reasoning 10A

Lesson

16A**Extra flowers**

Mandy bought this bunch of 10 flowers.

The label says, “25% extra free”.

Mandy says,

That means $2\frac{1}{2}$ flowers
were free.

How is that possible?

Explain how Mandy is wrong.

Find as many explanations as you can.

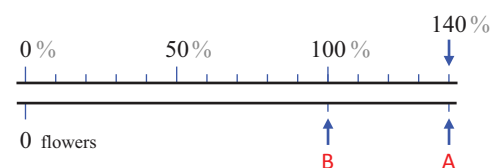
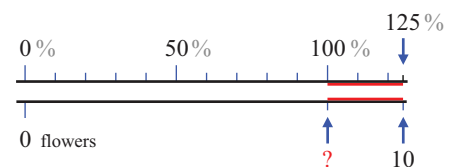
**Summary**

Here we use the double number line (DNL) to represent percentages and percentage change. As with the *Free lentil soup* Starter, a key idea is to consider which quantity is denoted by 100%.

Percentage change is commonly expressed additively, although the underlying relationship is multiplicative (eg, “25% extra” is a scaling of $\times 1.25$). You might want to explore this idea with some students.

Outline of the lesson

- Discuss the task.
 - Ask students to discuss the task in small groups.
 - Discuss their solutions.
- Use the double number line to model the task.
 - Present this DNL. “How does it relate to the story?”
 - Ask students how the DNL could be used to solve the task.
- Use the double number line for other stories.
 - Present this DNL (right). Say, “It’s for another flowers story, this time with 40% EXTRA FREE”. Ask for pairs of numbers for A and B that work.
 - “How can the DNL be used to find the pairs?”
- Use the double number line to invent other stories.
 - Ask students to sketch a DNL and use it to invent flowers stories for different ‘percent extra free’.
 - Discuss and solve some of the stories.
- Re-phrase Mandy’s story.
 - Present this problem.
 - Discuss how a DNL can be used to reconcile the two versions of the story.



Dave is looking at the 10 flowers that Mandy bought.
He says,
20% of the flowers were free.
Is Dave right?

Multiplicative Reasoning 10A

Lesson

16A

Overview

Mathematical ideas

When describing a percentage increase (or decrease), we need to be clear about which quantity we are taking as our 'base' or 'whole', ie which quantity is to be denoted by 100%. In this lesson, we use discrete quantities (in particular, numbers of flowers) which makes this need for clarity more apparent by triggering seeming contradictions (like '2½ flowers') that have to be resolved.

We see how the DNL can be used to keep track of what is going on.

Students' mathematical experiences

Students might discover some of the following

- when considering a percentage change, we need to know which quantity is taken to be 100%
- '25% extra' means '25% of the original number extra', not '25% of the resulting number extra'
- a DNL can help us see numerical relationships more clearly.

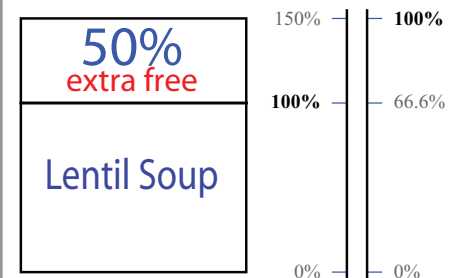
Assessment and feedback

The *Free lentil soup* Mini-assessment should give you some idea of how readily students will cope with the initial *Flowers* task - although students are likely to find *Flowers* more difficult as it has not been structured for them. If students get stuck, or regard the task as impossible, remind them of the Mini-assessment.

How well can students 'read' the DNL in Stage 2 of the lesson? How well can they use it?

How fluent are students at using a *sketch* of a DNL for their stories in Stage 4?

Can students sketch a DNL for the *Lentil Soup* task, as here?



Key questions

When the shop say '25% extra' it means 25% extra of *what* amount?

What percentage of a normal bunch of flowers is my bargain bunch?

What percentage of my bargain bunch of flowers is a normal bunch?

Adapting the lesson

You can make the tasks easier or more challenging by changing the numbers of flowers and/or the percentages. It might help students to vary the context, eg number of oranges/pencils/lightbulbs in a pack. You could also have another look at the *Free lentil soup* Mini-assessment, which provides an opportunity to consider continuous quantities.

You might also want to try the **percentages puzzles** in the Revisits section at some point.

Multiplicative Reasoning 10A

Lesson

16A**Outline of the lesson (annotated)**

1. Discuss the task.

- Ask students to discuss the task in small groups.
- Discuss their solutions.

— “It doesn’t make practical sense to sell half a flower, so it must be wrong to find 25% of 10. But why?”

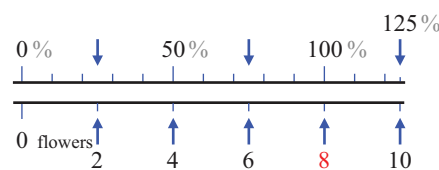
2. Use the double number line to model the task.

- Present this DNL. “How does it relate to the story?”

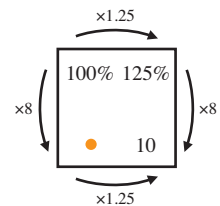
— Bring out the idea that 100% refers to the usual number of flowers. The bunch of 10 flowers is 125% of this.

- Ask students how the DNL could be used to solve the task.

— The DNL can be marked off like this:



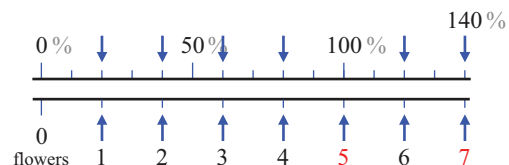
Some students might realise that the additive increase of ‘25% extra’ can be expressed multiplicatively as $\times 1.25$ or $\times 125\%$. Note, however, that here $\times 1.25$ is a scaling *along* rather than *between* the number lines:



3. Use the double number line for other stories.

- Present this DNL (right). Say, “It’s for another flowers story, this time with 40% EXTRA FREE”. Ask for pairs of numbers for A and B that work.
- “How can the DNL be used to find the pairs?”

— The DNL looks like this, in the case where $A = 7$, $B = 5$.



The pair (A, B) can be any multiple of $(7, 5)$, ie (A, B) is $(7n, 5n)$, where n is any positive whole number.

4. Use the double number line to invent other stories.

- Ask students to sketch a DNL and use it to invent flowers stories for different ‘percent extra free’.
- Discuss and solve some of the stories.

— The smallest solution can be found by equating the highest common factor of 100% and the ‘extra %’ with 1 flower.

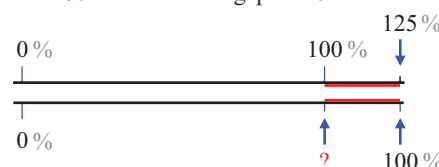
5. Re-phrase Mandy’s story.

- Present this problem.
- Discuss how a DNL can be used to reconcile the two versions of the story.

Dave is looking at the 10 flowers that Mandy bought.
He says,
20% of the flowers were free.
Is Dave right?

This task is similar to the *Lentil Soup Starter*. We can use a DNL like this (below).

‘?’ = 80% and the ‘red gap’ is 20% on the lower line.



Multiplicative Reasoning 10A

Lesson

16A

Background

Denoting a quantity by 100%

When describing a percentage increase (or decrease), we need to be clear about whether we are taking the original quantity or the changed quantity as our 'base' or 'whole', ie which quantity is being denoted by 100%.

In the case of Mandy's flowers, the number in her bargain bunch (10) is 25% more than the usual number (8), ie 125% of the usual number.

On the other hand, the usual number (8) is 20% less than the bargain number (10), ie 80% of the bargain number.

The DNL (or in this case, *triple* number line) can be used to keep track of what is going on:

When we say



V is 25% more than U,

we mean

V is 25% of U more than U,

not

V is 25% of V more than U,

which gives a different result.

For the quantities 10 and 8, say,

- 10 is 25% more than 8
- 10 is 125% of 8
- 10 is 1.25×8 .

Similarly,

- 8 is 20% (not 25%) less than 10
- 8 is 80% (not 75%) of 10
- 8 is 0.8×10 .

Percentages puzzles

See the Revisits section.

Multiplicative Reasoning 10B

Lesson

16B**How sweet?**

The information below is about the mass of sugar in breakfast cereals.

Which cereal tastes a. sweetest b. least sweet?



8 % sugar



0.35 g of sugar
per 1g of cereal



19.8 g of sugar
in a 450 g pack



11.1 g sugar
in a 30 g portion



$\frac{1}{6}$ sugar

Summary

In this lesson students apply their knowledge of multiplicative relations by using fractions, decimals, percentages and rate to compare the proportion of sugar (by weight or mass) in various breakfast cereals.

Outline of the lesson

1. Compare the amount of sugar in five breakfast cereals.
 - Present the above task, *without* and then *with* the numerical data. Each time, ask for some quick responses.
 - Let students *discuss* the task; ask for more considered responses.
2. Put the five cereals in order, from sweetest to least sweet.
 - Discuss this ordering task in small groups, then as a class. Try to draw out a variety of methods by considering pairs of cereals.

3. Compare cereals given the amount of sugar in a pack.
 - Present this information (right). “Which is sweetest?” Ask students for their spontaneous methods. (Try to draw out a variety of methods.)
 - “How could we use fractions, decimals or percentages?”



60 g sugar in
a 750 g pack

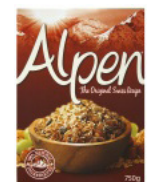


19.8 g sugar in
a 450 g pack

4. Compare cereals given the amount of sugar in a portion.
 - Present this information (right). “Which is sweetest?” Ask students for their spontaneous methods. (Try to draw out a variety of methods.)
 - “How could we use fractions, decimals or percentages?”



5.1 g sugar in
a 30 g portion



10.4 g sugar in
a 45 g portion

5. Compare the sweetness of cereals using a Cartesian graph.
 - Use points on a Cartesian graph to relate mass of cereal (in a pack, portion, or any other convenient amount) and mass of sugar (in that amount). [Use the information given in Stages 1, 3 and 4.]
 - “How can we use the points to compare sweetness?”
 - “How might we express sweetness as a *number*?”

Multiplicative Reasoning 10B

Lesson

16B**Overview****Mathematical ideas**

In this lesson we have chosen a context where the multiplicative nature of the relations is relatively obvious (see page 213). So the focus here is not so much on *discerning* the multiplicative nature of the relations than on students *using* their knowledge of such relations.

Specifically, students use fractions, decimals, percentages and rate as conceptual tools to compare the 'sweetness' of various breakfast cereals. In the process, of course, students have the opportunity to consolidate and develop their understanding of these concepts.

The Mini-assessment activity (*Free lentil soup*) and the preceeding lesson (*Extra flowers*) should provide some useful information for this lesson, at least as far as percentages are concerned.

Students' mathematical experiences

Students may realise some of the following

- that 'sweetness' is concerned with relative not absolute amounts of sugar
- we can compare sweetness by comparing the mass (or weight) of sugar in identical amounts of cereal
- we can measure sweetness by finding the proportion of sugar in any given amount of cereal
- we can represent the proportion of sugar as a fraction, decimal, percentage or rate.

Assessment and feedback

The Mini-assessment activity (*Free lentil soup*) and Lesson 16A (*Extra flowers*) should provide some useful information for this lesson, at least as far as percentages are concerned:

- how well are they understood
- how fluently do students use them
- do students relate them to decimals or fractions?

Key questions

How can we 'measure' sweetness?

If there is less sugar in a pack of cereal A than in a pack of cereal B, could A be sweeter than B?

Adapting the lesson

Depending on information gained from the preceeding Mini-assessment and Lesson, and on the methods that come up in Stage 1 of this lesson, you might want to spend more time on one or other way of representing sweetness: ie fractions or decimals or percentages or rate. You can change the emphasis by carefully choosing which pair of cereals to focus on, in light of the particular kind of information given for each cereal. You might want to focus more on estimating sweetness or more on deriving precise measures.

You might decide to focus on cereals that are similar or very different in sweetness. You could use information about other cereals, by searching on the internet or by asking students to bring labels from cereal packets. You could also compare other ingredients, eg calories or sodium, and you might on another occasion look at some other intensive quantity, such as the *value for money* of different size packs of the same cereal.

Multiplicative Reasoning 10B

Lesson

16B**Outline of the lesson (annotated)**

1. Compare the amount of sugar in five breakfast cereals.
 - Present the above task, *without* and then *with* the numerical data.
Each time, ask for some quick responses.
 - Let students *discuss* the task; ask for more considered responses.

— This first stage of the lesson is exploratory, to give you a sense of what methods students favour and how demanding they find the comparisons. If students come up with lots of ideas it is worth devoting time to them. On the other hand, if students are struggling, you might want to go more quickly to stages 2 and 3, which involve simpler, more structured comparisons.
2. Put the five cereals in order, from sweetest to least sweet.
 - Discuss this ordering task in small groups, then as a class. Try to draw out a variety of methods by considering pairs of cereals.

— Considering the cereals in pairs reduces the complexity of the task and allows you to focus on particular quantities.
3. Compare cereals given the amount of sugar in a pack.
 - Present this information (right). “Which is sweetest?”
Ask students for their spontaneous methods.
(Try to draw out a variety of methods.)
 - “How could we use fractions, decimals or percentages?”

— For example,

 - 60 g is roughly 3 times 19.8 g, but 750 g is not even twice 450 g
 - 150 g packs would contain about 12 g and 7 g of sugar.

— Students might come up with formal methods, eg

 - the proportion of sugar in cornflakes is $\frac{60}{750} = \frac{6}{75} = \frac{2}{25}$
 - the proportion of sugar in cornflakes is $60 \div 750 = 0.08 = 8\%$.

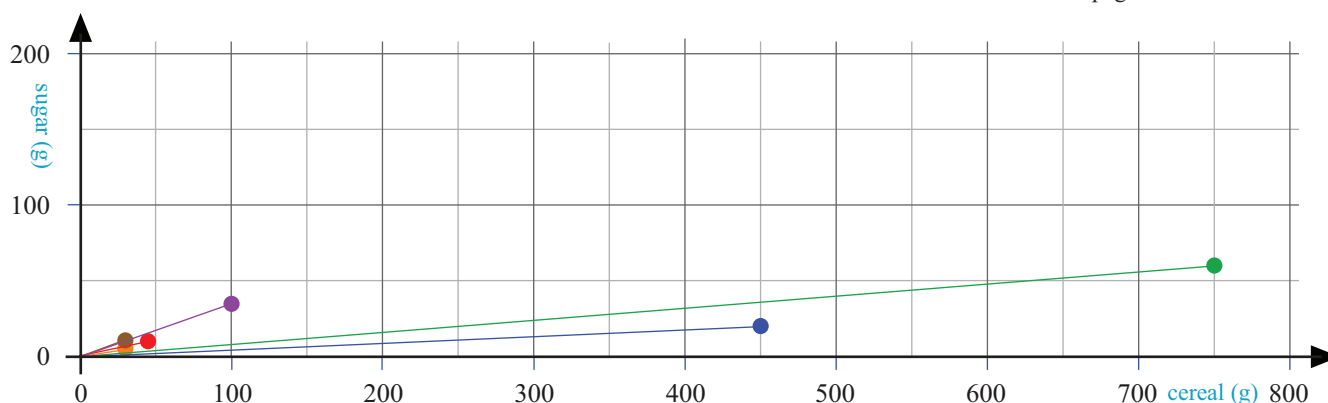
However, encourage estimates and informal methods too:

 - 60 is a bit less than one tenth or 0.1 or 10 % of 750
 - 60 of 750 is the same as 60+20 of 750+250, or 8 of 100 ie 8 %
 - 20 in 450 is about 40 in 900 or 4 in 90, so a bit more than 4 %.
4. Compare cereals given the amount of sugar in a portion.
 - Present this information (right). “Which is sweetest?”
Ask students for their spontaneous methods.
(Try to draw out a variety of methods.)
 - “How could we use fractions, decimals or percentages?”

— For example,

 - 10.4 g is roughly 2 times 5.1 g, but 45 g is not twice 30 g
 - a 45 g portion of cornflakes would contain 5.1 g + half of 5.1 g of sugar, which is less than 10.4 g
 - 15 g portions would contain about 2.5 g and 3.5 g of sugar
 - 90 g portions would contain about 15.3 g and 20.8 g of sugar
 - a 60 g portion of cornflakes would contain about the same amount of sugar as the 45 g Alpen portion.
5. Compare the sweetness of cereals using a Cartesian graph.
 - Use points on a Cartesian graph to relate mass of cereal (in a pack, portion, or any other convenient amount) and mass of sugar (in that amount).
[Use the information given in Stages 1, 3 and 4.]
 - “How can we use the points to compare sweetness?”
 - “How might we express sweetness as a *number*?”

— The scale on the graph below can’t accommodate the given information for *Sugar puffs* (0.35 g per 1 g), so we have plotted ‘35 g per 100 g’. The scale is also not well suited to the ‘sugar per portion’ information (eg for *Frosties*), so it is worth asking students about finding equivalent values to plot. It might help to present the fuller information shown in the table on page 213.



Multiplicative Reasoning 10B

Lesson

16B

Background

The sugar-in-cereals context

The context in this lesson is somewhat different from those we have used before. It involves a recipe, which is a context where multiplicative relations are relatively easy to discern. However, it differs from the recipes we have met before, ie spicy soup (*Mini Ratio Test*) and potato pancakes (Lesson MR 4B), in that it is not relating people and an ingredient; rather it concerns an ingredient (sugar) and its relation to an entity (a breakfast cereal) *of which it is a part*. This means that students are less likely to be drawn to the addition strategy here, since this would clearly not preserve proportions: imagine adding 5 g, say, of sugar to a portion of cornflakes - this would automatically add 5 g to the portion itself, but it would clearly make the portion sweeter since what has been added is exclusively sugar.

Intensive and extensive quantities

'Sweetness' is an *intensive* quantity, like for example density, speed, greyness, value for money. Intensive quantities involve a proportional relation between *extensive* quantities like mass, volume, distance, time, price.

You might at some stage want to devise tasks similar to the ones in this lesson, but using a different intensive quantity.

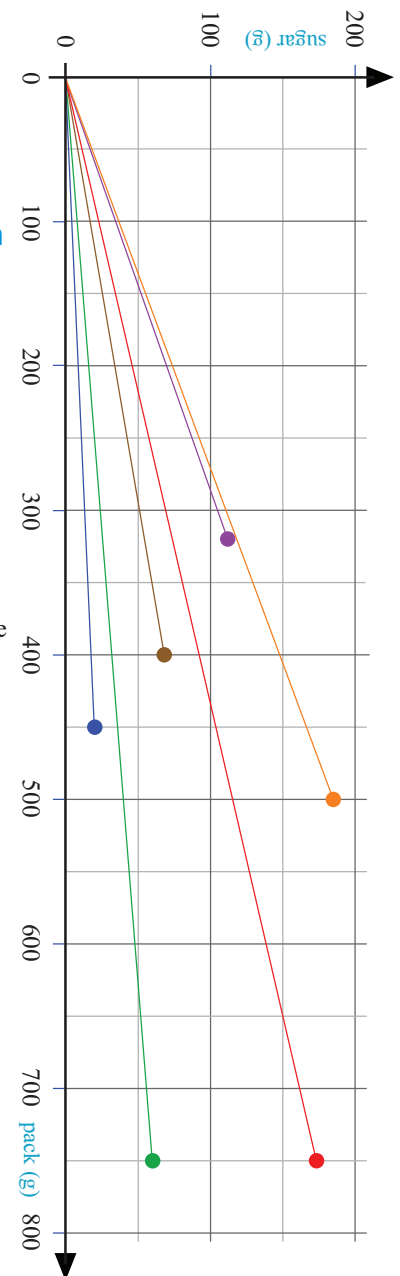
Breakfast cereal data

The first table below provides a variety of information about the cereals considered in the lesson: portion size, pack size, and various rates of sugar: per 100 g, per portion, per pack. The cereals' relative sweetness can be read off the 'sugar per 100 g' column.

You might want to present students with the second, incomplete, version of the table so that they can think about and derive some of the data for themselves (eg for the graphing task in Stage 4).

Using a Cartesian graph

The points on this graph represent (mass of pack, mass of sugar in pack). Sweetness is represented by the *gradient* of the line joining each point to the origin. The gradients should be the same as for the graph on page 212. [Why?] By plotting the data for the full pack, this graph is likely to be clearer and more accurate.



product	sugar per 100g	portion size (g)	sugar per portion	pack size (g)	sugar per pack	other
Alpen original	23.1	45	10.4	750	173.3	
Cornflakes	8	30	2.4	750	60	8 %
Frosties	37	30	11.1	500	185	
Special K	17	30	5.1	400	68	1/6
Sugar puffs	35	30	10.5	320	112	0.35 g per 1 g
Weetabix	4.4	37.5	1.7	450	19.8	

product	sugar per 100g	portion size (g)	sugar per portion	pack size (g)	sugar per pack	other
Alpen original		45	10.4	750		
Cornflakes		30		750	60	8 %
Frosties		30	11.1	500		
Special K		30	5.1	400		1/6
Sugar puffs		30		320		0.35 g per 1 g
Weetabix		37.5		450	19.8	

Multiplicative Reasoning 11AB

Mini-assessment

17AB**Collecting drips**

Ask students to try this task for homework.

Homework: Collecting Drips Name Date

Water is dripping steadily from the ceiling of a cave.
Meg uses a measuring cylinder to catch the drips.
After 4 minutes the cylinder has collected 15 ml of water.

1. Meg draws this double number line:

Add more numbers and more marks to the double number line; use these to *estimate*

a. roughly how much water collects in 11 minutes

b. roughly how long it takes to collect 100 ml of water.

2. Meg draws this ratio table:

4	19
15	•

a. What is Meg trying to find?

b. Explain how Meg could use a calculator to find the answer.
.....
.....

page 2

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Commentary

Here we revisit the double number line (DNL) and the ratio table.

How readily do students make sense of

the DNL representation of the situation, and can they use it effectively to estimate solutions?

the ratio table representation of the situation, and can they use a calculator method to find a solution?

NOTE: Students need to do ‘well’ on this Mini-assessment if they are to try Lesson MR-11A - see page 220 of the MR-11A lesson notes.

Multiplicative Reasoning 11AB

Mini-assessment

17AB

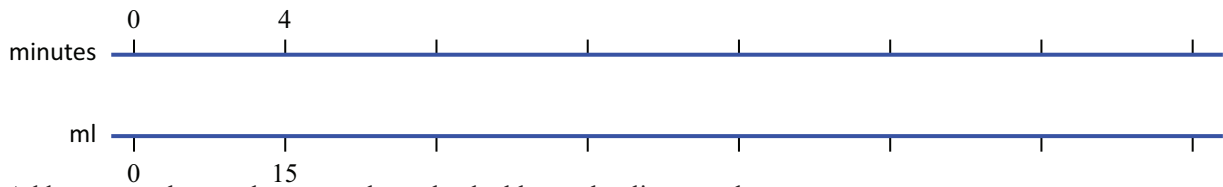
Homework: Collecting Drips Name Date

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1. Meg draws this double number line:



Add more numbers and more marks to the double number line; use these to *estimate*

- roughly how much water collects in 11 minutes
- roughly how long it takes to collect 100 ml of water.

2. Meg draws this ratio table:

4	19
15	•

- What is Meg trying to find?
- Explain how Meg could use a calculator to find the answer.
.....
.....

Multiplicative Reasoning 11A

Lesson

17A**Cheesecake**

A plastic tub contains 125 g of cheesecake.

The label on the lid tells us what a 100 g portion of the cheesecake contains.

Mia writes this ratio table.

22.2g	100g
•	125g

What information is she trying to find?

NUTRITION	
Typical Values	per 100g
Energy kJ	1465
Energy kcal	360
Protein	3.1 g
Carbohydrate	36.9 g
of which sugars	30.2 g
Fat	22.2 g
of which saturates	12.5 g
Fibre	0.5 g
Sodium	0.11 g
Equivalent as salt	0.30 g

**Summary**

For this lesson it is important that students have coped 'well' (see page 220) with the 17AB Mini-assessment homework. The lesson involves finding the weight (or mass) of ingredients in a 125 g portion of cheesecake, using information in a table showing weight per 100 g. As with the Mini-assessment, students interpret and make use of information given in ratio tables and on double number lines. Various methods for finding values are considered, but the focus is on the idea of *scaling* which provides a method that is efficient and general.

Outline of the lesson

- Interpret information given in ratio tables.
 - Display and discuss the information on the cheesecake label.
 - Show Mia's ratio table. What is she trying to find out?
 - Show Ned's ratio table. What is he trying to find out?
 - Show Olive's ratio table. What is she trying to find out?
- Use Mia's ratio table.
 - Estimate* the number that Mia is trying to find.
 - Find ways of calculating Mia's number.
Use her ratio table to record the various methods.
- Represent information on a double number line (DNL).
 - Show this DNL (right).
"Here is another way of representing Mia's information. Could we also use it for Ned's and/or Olive's information?"
 - What names could we give to the lines?

Mia

22.2g	100g
•	125g

Ned

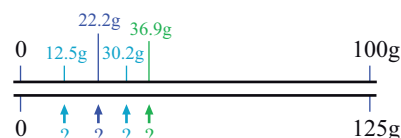
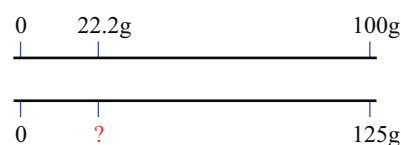
30.2g	100g
•	125g

Olive

22.2g	100g
•	80g

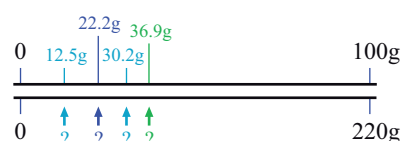
- Use a DNL to find amounts in the 125 g portion.

- Mia wants to convert all these 'per 100 g' amounts to 'per 125 g' amounts:
Find a quick calculator method that maps the numbers on the top line onto the bottom line.



- Use a DNL to find amounts in other portions.

- A bumper-size cheesecake weighs 220 g. Mia wants to convert all these 'per 100 g' amounts to 'per 220 g' amounts:
Find a quick calculator method that maps the numbers on the top line onto the bottom line.
- Pete wants a portion that contains only 10 g of Fat. Sketch a DNL similar to Mia's and explain how it could be used to find the weight of the portion, and the weight of other ingredients that it contains.



Multiplicative Reasoning 11A

Lesson

17A

Overview

Mathematical ideas

In this lesson we use a context that involves scaling-up 'nutrients' in a cheesecake, from amounts per 100 g to amounts per 125 g. We consider various methods but home-in on the powerful and general method of scaling (ie using the scale factor $\times 1.25$).

We start by interpreting information given in some ratio tables and then use the ratio table to record methods for scaling values. We then see how the same information can be represented on a double number line, and use the DNL to bring out the scaling method.

Students' mathematical experiences

Students gain experience of

- interpreting information in a ratio table
- using a ratio table to record methods
- interpreting information on a double number line
- seeing the double number line as a model of scaling.

Assessment and feedback

The Mini-assessment homework task (*Collecting drips*) should give you a good idea of how comfortable students are with 'reading' a DNL and ratio table, and how readily they can construct a formal, calculator method. This lesson gives students the opportunity to explore these ideas further and to consolidate them. However, if the task indicates that students still have a very weak understanding of the DNL we suggest deferring the *Cheesecake* lesson until further work has been done on the DNL - eg by using a variant of Lesson MR-3A (*Westgate Close*), 4A (*Converting Pounds to Leva*) or 8A (*Stretched ruler*). We discuss the homework task further on page 220.

You might want to use the completed homework task as a peer assessment exercise - either by asking pairs of students to comment on each other's work, or by selecting some responses for a whole-class discussion.

Key questions

What is Mia trying to find?

How could we use a calculator to scale-up the quantities?

Adapting the lesson

Students might be more comfortable with a DNL which is labelled and numbered like the 'vertical' DNL on page 219, especially if they use rated addition ("125 g is 100 g + a quarter of 100 g") and like to 'skip along' the number lines. Thus you need to be aware (and might need to acknowledge to the students) that there are alternative forms of the DNL. In this lesson we have chosen a form that allows us to scale-up the weights of *all* the nutrients in the original 'per 100 g' table to their weights in the full 125 g portion. In effect, our DNL is a two-row table containing all these pairs of values (of course, the question then is, how are the two rows related, which is what we address in Stages 4 and 5). Note that as one moves *along* the lines (or the rows), the numbers change by different amounts in the different lines - so the addition strategy can't apply (see also Section 2 on page 4 for a comment about the difference in values *between* lines).

Numerically, the multiplier that maps 100 g onto 125 g (or any other portion size) is relatively easy to find. It might therefore be useful at some stage to replace the 100 with another number. We give an example in the Revisits section.

Multiplicative Reasoning 11A

Lesson

17A**Outline of the lesson (annotated)**

1. Interpret information given in ratio tables.

- Display and discuss the information on the cheesecake label.
- Show Mia's ratio table. What is she trying to find out?
- Show Ned's ratio table. What is he trying to find out?
- Show Olive's ratio table. What is she trying to find out?

— Mia: the amount of Fat in a 125 g portion
 Ned: the amount of Sugar in a 125 g portion
 Olive: the amount of Fat in an 80 g portion.

2. Use Mia's ratio table.

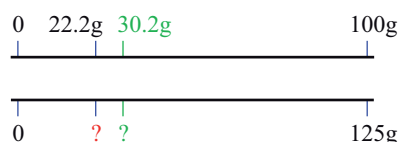
- *Estimate* the number that Mia is trying to find.
- Find ways of calculating Mia's number.
 Use her ratio table to record the various methods.

— eg, "A quarter more, so about 5 g more, so about 27 g".

— See page 4. Students might use rated addition ('a quarter more'), scaling ($\times 1.25$), or the unitary method.

3. Represent information on a double number line (DNL).

- Show this DNL (right).
 "Here is another way of representing Mia's information. Could we also use it for Ned's and/or Olive's information?"



— We can show Ned's information (above), but not Olive's. We would need to add another number line (from 0 to 80 g) for Olive, or draw a 'vertical' DNL - see page 219.

- What names could we give to the lines?

— eg, *Weight of Nutrients in a 100 g portion* and *Weight of Nutrients in a 125 g portion*.

4. Use a DNL to find amounts in the 125 g portion.

- Mia wants to convert all these 'per 100 g' amounts to 'per 125 g' amounts:
 Find a quick calculator method that maps the numbers on the top line onto the bottom line.

The multiplier $\times 1.25$ (or $\times 1\frac{1}{4}$) maps *any* number on the top line to the number directly below it.

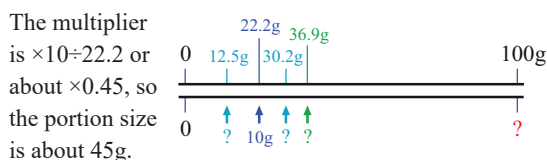
(Of course, we can find *single* pairs of values, such as 22.2 g and 27.74 g, by working along the lines:
 eg, $100 \times 0.222 = 22.2$, $125 \times 0.222 = 27.74$).

Note: because we are starting from a 100 g portion, the $\times 1.25$ multiplier is fairly easy to find. To make the work more challenging (and more illuminating), you could ask students to imagine that the numbers in the original Nutrition table are 'per 90 g' portions, say, rather than 'per 100 g'.

5. Use a DNL to find amounts in other portions.

- A bumper-size cheesecake weighs 220 g. Mia wants to convert all these 'per 100 g' amounts to 'per 220 g' amounts:
 Find a quick calculator method that maps the numbers on the top line onto the bottom line.
- Pete wants a portion that contains only 10 g of Fat. Sketch a DNL similar to Mia's and explain how it could be used to find the weight of the portion, and the weight of other nutrients that it contains.

— Here the multiplier is $\times 2.20$ or $(\times 2\frac{1}{5})$.



The multiplier is $\times 10 \div 22.2$ or about $\times 0.45$, so the portion size is about 45g.

Multiplicative Reasoning 11A

Lesson

17A

Background

The double number line (DNL) and scaling

We introduced the double number line in Lesson MR-3A, *Westgate Close*. As with *Westgate Close*, we have chosen a 'real life' context here, but this time one that is not geometric. Thus there is not such a direct connection between the representation (the DNL) and what is being represented: the weight (or mass) of ingredients of a cheesecake.

The DNL promotes the idea of scaling, and it is hoped that this will help students see that they can find the amount of sugar, say, in the 125 g tub of cheesecake, by scaling the amount in a 100 g portion by $\times 1.25$. At the same time, the cheesecake context probably lends itself more readily than *Westgate Close* to viable additive strategies like "add on a quarter (or 0.25 or 25%)". Don't be surprised, therefore, if students use such strategies, and don't discourage their use. However, give students the chance to see links between such methods and scaling, and the chance to appreciate that scaling offers an equivalent, and more efficient, method. Ratio tables are of particular use here, as they can help students focus on the calculations that they could use.

Calculators can also help with this. The DNL encourages students to estimate, but once we move on to trying to calculate a desired amount, the awkward 'real life' number prompts the use of calculators. In turn, it is likely to be easier to use a calculator to perform a scaling (which is a single operation) than to perform some of the more informal methods like 'find a quarter of the given amount of sugar, add it to the given amount of sugar'.

Moving along the double number line

To help students appreciate the fact that the DNL embodies scaling, it is worth asking students to imagine what happens to corresponding values on the double line *as one moves steadily along the line* - what changes and what stays the same? This can help counter the inappropriate 'addition strategy' since the difference between corresponding values changes as one moves along the line. "So what *does* stay the same?"

Approximation and Estimation

In this lesson, we deliberately use 'tricky' decimal numbers, which have arisen in an actual nutrition advice table for consumers. Partly, we do this to reflect the way numbers are used in 'real life' contexts. Additionally, we deliberately want to encourage students to develop an intuitive 'feel' for the multiplicative relationships. The numbers 'force' the students to use approximation and estimation strategies. The context also enables students to discuss how 'exact' the amount needs to be. The 'exact' amount of fat in the cheesecake is 27.75 g (22.2×1.25). However, giving this amount to 2 decimal places overstates the level of precision in the nutrition table.

There is an 'error' in the table, in that the given quantities 1465 kJ and 360 kcal are not consistent. One kilogram

calorie (kcal) is approximately equivalent to 4.184 kJ and, hence, 360 kcal is equivalent to just over 1500 kJ. Perhaps the typesetter misread 350 kcal and inserted 360 kcal instead, or perhaps a rather crude conversion factor of 4 was used on a number later rounded to 360.

Recording methods on a ratio table

The use of the scalar relationship $\times 1.25$, between 100 g and 125 g, can be recorded like this (below, left).

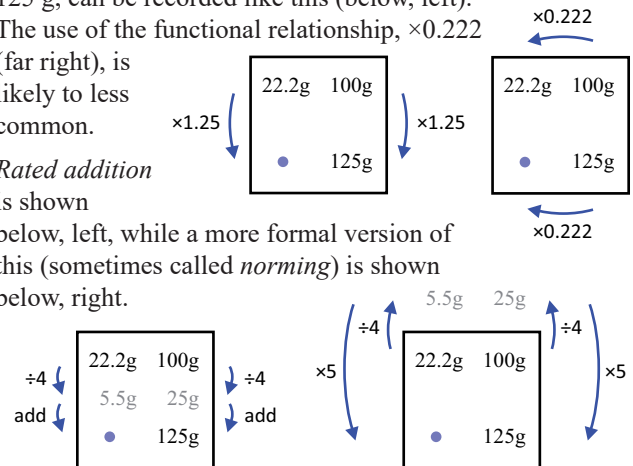
The use of the functional relationship, $\times 0.222$

(far right), is likely to be less common.

Rated addition

is shown

below, left, while a more formal version of this (sometimes called *norming*) is shown below, right.



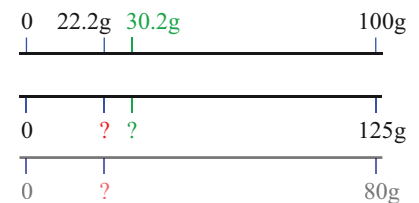
The *unitary method* (which can be thought of as a more general version of *norming*) is shown here. Of course, $\div 100 \times 125$ is equivalent to scaling by $\times 1.25$.

Triple number line and 'vertical' DNL

In Stage 3 of the lesson, we can show Olive's information by adding a third (grey) number line.

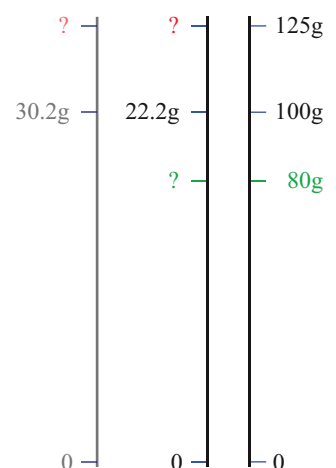
We can also show Olive's (and Mia's) information by

drawing a 'vertical' DNL through the numbers in Olive's (or Mia's) ratio table. This time the third (grey) line is for Ned.



Here the lines represent weight of portion, of fat, and of sugar (from right to left).

You might well find that some students will draw a DNL of this sort, albeit with the lines horizontal.

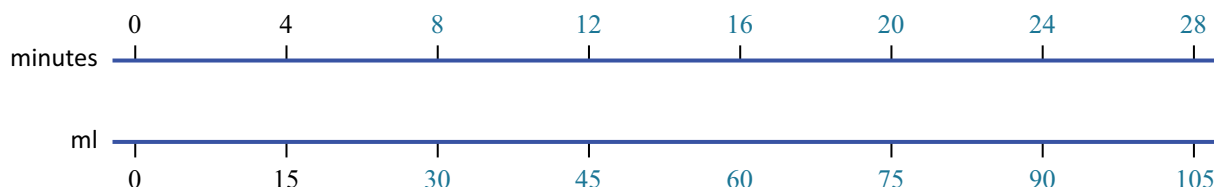


Multiplicative Reasoning 11A

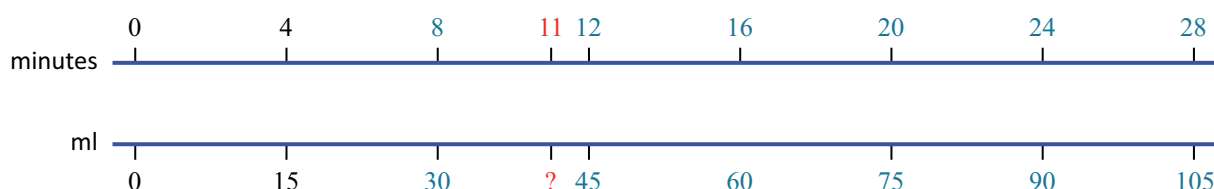
Lesson

17A**The 17AB Mini-assessment homework task****The double number line**

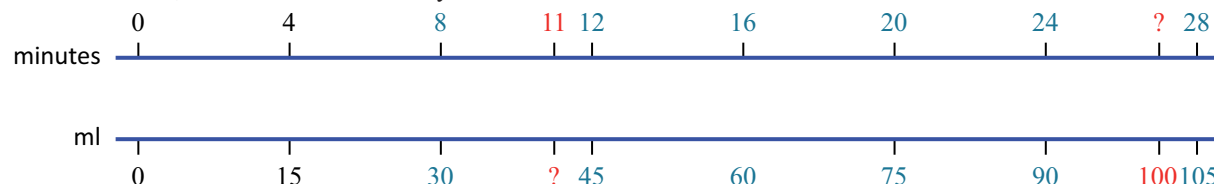
The homework task asks students to number the scales of a double number line. At this stage of the sequence of MR lessons, we would expect students to know that the scales are *linear* on a standard DNL and therefore be able to number the scales correctly (leaving aside arithmetic errors), as shown below. If a substantial number of students are unable to do this, we suggest deferring the *Cheesecake* lesson until more work has been done on the DNL - eg by using a modified version of Lesson MR-3A (*Westgate Close*), MR-4A (*Converting Pounds to Leva*) or MR-8A (*Stretched ruler*).



Students are asked to use the DNL to estimate the amount of water collected in 11 minutes. Here we would expect students to be able to mark '11' in roughly the right place on the *minutes* scale (below). However, some students might resort to the addition strategy and argue that, for example, the amount of water is about 43 ml (ie $44-1$, because $11 = 12-1$) or 33 ml (ie $30+3$, because $11 = 8+3$). Again, if a substantial number of students do this, we suggest deferring the *Cheesecake* lesson.



Similarly, some students might argue that the time taken to collect 100 ml of water is, for example, about 23 minutes (ie $28-5$, because $100 = 105-5$) or 34 minutes (ie $24+10$, because $100 = 90+10$). However, as these numbers lie outside the 24-to-28 interval, fewer students are likely to do this.

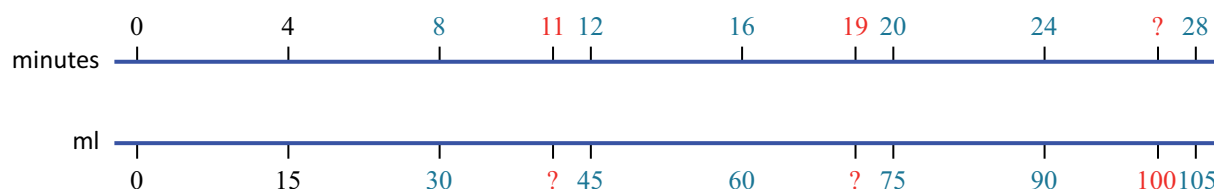
**The ratio table**

The latter part of the homework task concerns the ratio table. Students' responses to this will inform the way you teach the *Cheesecake* lesson, but, in contrast to the DNL responses, we don't suggest that you consider deferring the lesson on the basis of these responses.

Students are first asked to interpret a ratio table. This is an uncommon form of question: students are usually asked to *create* a ratio table in order to solve a given problem, rather than to interpret the nature of a problem *from* a given table. However, for this straightforward context, one would expect most students to be successful.

Next, and last, students are asked to think of how a calculator could be used to find the missing number in the ratio table. The given numbers have been chosen in such a way that the multipliers that relate 4 to the other numbers are not whole numbers: thus we have $4 \times 4.75 = 19$ and $4 \times 3.75 = 15$. A prime aim of the *Cheesecake* lesson is to find multipliers of this sort and so it is interesting to see how many students think in terms of $\times 4.75$ or $\times 3.75$ in this final part of the homework task. It is likely that some students will instead go back to the DNL, either to estimate the missing number, or to help them structure an additive response. (They can of course also use the DNL to find one or other multiplier). Here the addition strategy is likely to give an answer of 74 ml or 63 ml. On the other hand, a rated-addition response such as the following would lead to the correct answer: " $75 \text{ ml} - \frac{1}{4} \text{ of the } 15 \text{ ml interval} = 75 \text{ ml} - 3\frac{3}{4} \text{ ml} = 71\frac{1}{4} \text{ ml}$ ".

4	19
15	•



Lesson

17A

Notes

Multiplicative Reasoning 11B

Lesson

17B**Soup****Carl makes a spicy soup for 4 people.**

He uses 14 ml of tabasco sauce.

Dan makes the same soup for 10 people.

How much tabasco sauce should he use?

**Queue****Meg is in a very slow lunch queue. Liz joins the queue.**

When Liz has been in the queue for 4 minutes, Meg has been in the queue for 14 minutes.

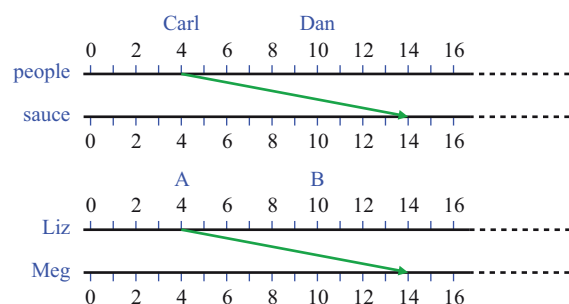
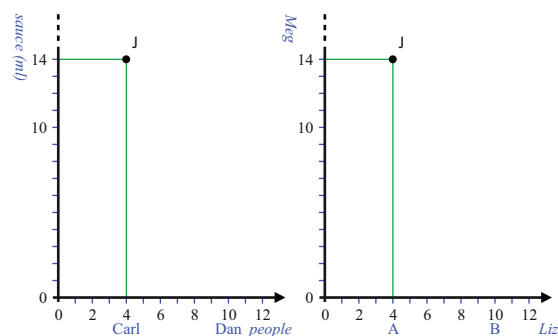
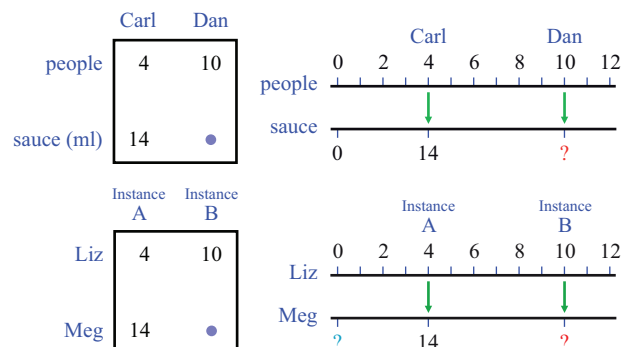
When Liz has been in the queue for 10 minutes, how long has Meg been in the queue?

Summary

In this lesson we look at several models/representations of numerical relations (table, double number line [DNL], Cartesian graph, mapping diagram). We compare them for a multiplicative and an additive relation (ie relations of the form $y = ax$ and $y = x + b$ respectively).

Outline of the lesson

- Show the above tasks and ask students to solve them.
 - Let students discuss the **Soup** and **Queue** tasks in pairs.
 - Discuss their solutions.
- Use a table and DNL for the multiplicative task (Soup).
 - Show these representations of the Soup task. Discuss how we can use them to solve the task.
- Use a table and DNL for the additive task (Queue).
 - Show these representations of the Queue task. Discuss how we can use them to solve the task.
- Compare how the table and DNL represent the multiplicative and additive tasks.
 - Discuss how the models show the multiplicative and additive nature of the relations.
- Use a Cartesian graph for the two tasks.
 - Look at the two graphs. Point J shows the given information for the Soup and Queue tasks.
 - Add a point K to each graph, to show each task's solution. Compare the pairs of points. What patterns do they make?
- Use a mapping diagram for the two tasks.
 - Look at the two mapping diagrams. The arrow shows the given information for the Soup and Queue tasks.
 - Add a second arrow, to show each task's solution. Compare the two mapping diagrams.
- Invent similar problems.
 - Ask students to invent pairs of problems similar to Soup and Queue but in fresh contexts.



Multiplicative Reasoning 11B

Lesson

17B

Mathematical ideas

In this lesson we look at another task involving a multiplicative relation, this time in the context of a recipe, as in the *Mini Ratio Test* and Lesson MR-4B (*Potato pancakes*). However, we also use a task involving an additive relation so that we can bring out the differences between the relations and highlight their properties. We do this by using various representations that we have met before: the (ratio) table, DNL, Cartesian graph and mapping diagram. This also allows us once more to compare the representations themselves and to consider their characteristics.

Students' mathematical experiences

Students gain experience of

- interpreting and using tables, DNLs, graphs
- how multiplicative and additive relations are modelled differently (eg, by a table, or by a DNL, etc)
- the properties of multiplicative and additive relations, and how they differ.

Key questions

Is the relation multiplicative or additive?

How do the relations differ?

Assessment and feedback

The Mini-assessment homework (*Collecting drips*) and Lesson MR-11A (*Cheesecake*) should give you a good idea of how well students can 'read' a ratio table and DNL. In this lesson we ask something more: can they discern specific features of these (and other) representations, and how the features differ for multiplicative and additive relations.

Which students see both tasks as additive or both as multiplicative? How does this relate to their understanding of multiplicative relations?

How fluent are students in reading the various representations and in using them?

Adapting the lesson

The focus of the lesson is on comparing relations and seeing how various representations can be used to model them. It is thus worth making sure that by the end of Stage 1 of the lesson all students understand the two tasks well enough to be able to solve them by at least one method.

For the subsequent stages of the lesson, you will want to consider a range of student methods, but this need not be exhaustive, in contrast to Lesson MR-11A (*Cheesecake*). The focus is on how the multiplicative and additive relations differ and how the representations show this.

You might not reach Stage 6 of the lesson and it is perhaps not as important as the earlier Stages. However, it is worth finding time, in this or a later lesson, to consider the mapping diagram: it may not be a very efficient representation, but it is illuminating to consider how it relates to the DNL and Cartesian graph: we can transform it into a DNL (when the relation is multiplicative) by stretching (or shrinking) one of the axes so that the mapping arrows become parallel, and into the Cartesian graph by rotating an axis and replacing each arrow by a point.

Multiplicative Reasoning 11B

Lesson

17B**Outline of the lesson (annotated)**

1. Show the above tasks and ask students to solve them.
 - Let students discuss the **Soup** and **Queue** tasks in pairs.
 - Discuss their solutions.

— Some students who realise that the Soup task is multiplicative, might think that the Queue task is multiplicative too.

2. Use a table and DNL for the multiplicative task (Soup).
 - Show these representations of the Soup task.
Discuss how we can use them to solve the task.

— As in the *Cheesecake* lesson, we could have used an alternative DNL here (see page 225).

— Students might see that $4 \times 2.5 = 10$ (so $14 \times 2.5 = 35$), or that $4 \times 3.5 = 14$ (so $10 \times 3.5 = 35$).
Or they might use rated addition, norming or the unitary method.

3. Use a table and DNL for the additive task (Queue).
 - Show these representations of the Queue task.
Discuss how we can use them to solve the task.

— Students might see that $4 + 6 = 10$ (so $14 + 6 = 20$), or that $4 + 10 = 14$ (so $10 + 10 = 20$).

4. Compare how the table and DNL represent the multiplicative and additive tasks.
 - Discuss how the models show the multiplicative and additive nature of the relations.

— The multiplicative nature of the Soup task is perhaps shown most clearly by the fact that the scales of the two number lines are *different*, though they are both linear and the zeros coincide.
By contrast, the scales for the Queue number lines are the *same*, but the zeros do *not* coincide.

5. Use a Cartesian graph for the two tasks.
 - Look at the two graphs. Point J shows the given information for the Soup and Queue tasks.
 - Add a point K to each graph, to show each task's solution. Compare the pairs of points.
What patterns do they make?

— See page 225. For Soup, the straight line through J and K goes through the origin, and its gradient is 3.5; for Queue, it cuts the y axis at 10 and its gradient is 1.

 - What do 3.5, the origin, 10 and 1 signify?
 - What do points on the line through J and K represent?

6. Use a mapping diagram for the two tasks.
 - Look at the two mapping diagrams. The arrow shows the given information for the Soup and Queue tasks.
 - Add a second arrow, to show each task's solution. Compare the two mapping diagrams.

— See page 225. You might want to point out that in a mapping diagram the two axes are identical.
For Soup the mapping arrows are not parallel - why? (They meet at a point on a line through the two zeros.)
For Queue the arrows are parallel - why?

7. Invent similar problems.
 - Ask students to invent pairs of problems similar to Soup and Queue but in fresh contexts.

— Or you could give students problems to explore and model whose underlying structure is of the form $y = ax + b$, or $y = ax^2$, or $y = a - x$, or $y = a \div x$.

Multiplicative Reasoning 11B

Lesson

17B

Background

Multiplicative and additive relations

In this lesson we compare multiplicative and additive relations. By *multiplicative* we mean relations of the form $y = ax$ (which mathematicians call *linear*); by *additive* we mean relations of the form $y = ax + b$ (which mathematicians call *affine*), but with $a = 1$.

Over-generalising multiplicative relations

Students who are getting to grips with multiplicative relations may well over-generalise their occurrence and thus conclude that the Queue task is multiplicative. This should be seen as a sign of progress, rather than a matter of concern - students will have little difficulty seeing that the Queue task is additive once it is pointed out to them.

The purpose of using a multiplicative and an additive task in this lesson is to provide a contrast which will illuminate the nature of each kind of relation.

Using alternative DNLs

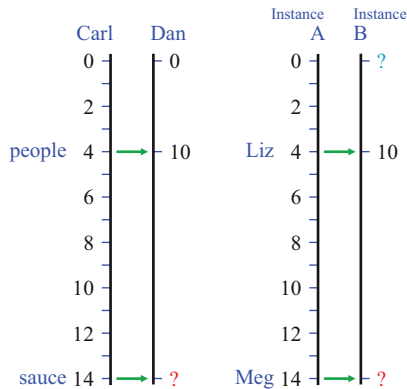
We could have used these alternative DNLs for the Soup and Queue tasks (Stages 2 and 3 of the lesson).

For Soup, this DNL (below, near right) could be used to represent (and find)

the amount of other ingredients in Carl and Dan's soups.

For example, if we were told that Carl's soup needed 2 small onions, we could find the number of small onions needed for Dan's soup.

Note that we are, in effect, treating 'number of people' as an ingredient here!

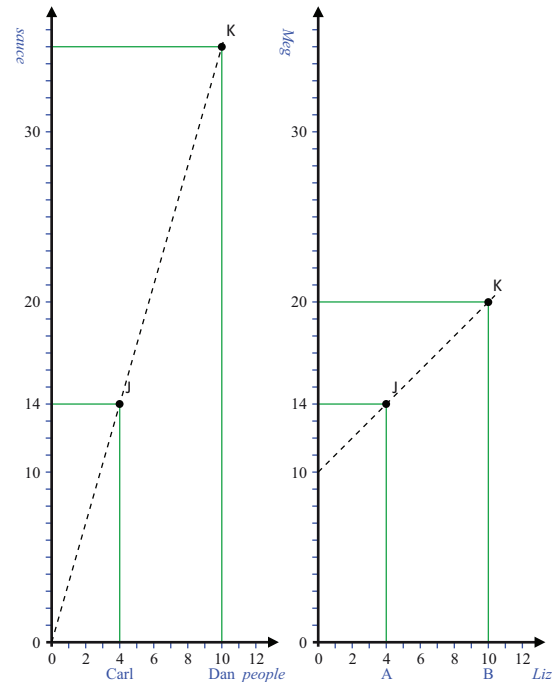


For Queue, this DNL (above, far right) could be used to represent (and find) how long other people have been waiting in the queue at Instance A and Instance B.

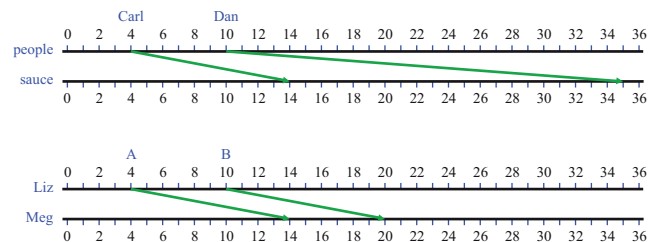
Of course, students might use other methods, in particular rated addition, norming or the unitary method for the Soup task.

Using Cartesian graphs

The graphs below (for Stage 5 of the lesson) show the points J and K. As can be seen, the straight line through J and K goes through the origin for Soup, with a slope

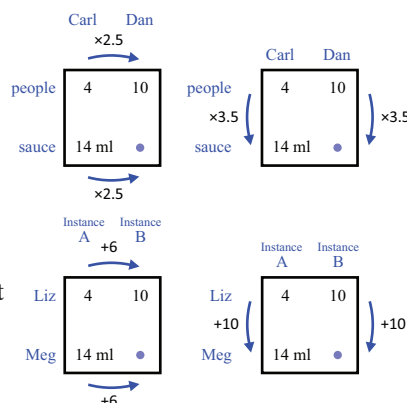


of 3.5, and through (0, 10) for Queue, with a slope of 1. They thus show the multiplicative mapping $x \rightarrow 3.5x$ and additive mapping $x \rightarrow x + 10$, respectively.



Recording methods on a (ratio) table

These tables show the *scalar* and *functional* relations, respectively, for the two tasks. Students are more likely to opt for a scalar relation, if the numbers are not more difficult.



Using mapping diagrams

The mapping diagrams below are for Stage 6 of the lesson. As can be seen, the two arrows diverge for Soup but are parallel for Queue.

[The distance between the axes has been chosen to match the DNLs. However, the mappings would be clearer if the lines were further apart. It might then also be more apparent that the arrows for Soup meet on the vertical line through the two zeros - this point can be thought of as a centre of enlargement with scale factor 3.5.]

Lesson

17B

Notes

Algebra 7AB

Mini-assessment

18AB**Changing expressions
(but not really)**

Can you make this
expression simpler?
Can you make it
more complicated?

$$10h - 2 + 2h + 8$$

Commentary

The aim of this Mini-assessment is to get a sense of how well students can manipulate algebraic expressions.

Can students make the expression simpler? What rules/reasons do they use?

Can they make the expression more complex?

Do they see the changed expressions as *equivalent*?

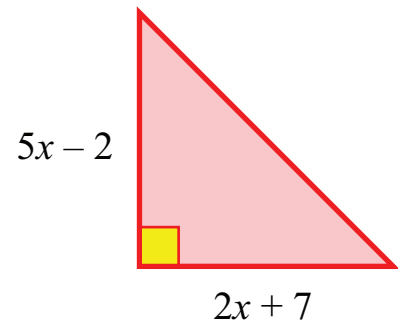
Algebra 7A

Lesson

18A**Growing triangle**

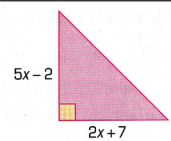
What happens to the triangle
as x changes?

For what value of x is the height
of the triangle twice its base?

**Summary**

This lesson explores a task from a section on algebraic manipulation in a popular textbook*. Our version treats the problem dynamically, and we recommend that you use the accompanying GeoGebra file.

In this right-angled triangle the height is twice its base.
What is the value of x ?

**Outline of the lesson**

1. Display Task 1.

- Ask students to discuss Task 1, above, in pairs.
- Discuss how the triangle changes as x changes.
 - What does the triangle look like when x is 2, or 20, or ... ?
 - What happens to the base when x increases by 1 ... by 1 again ...
 [On the GeoGebra file, you can use the slider to change the value of x , but use this sparingly.]

2. Task 2: For what value of x is the height twice the base?

- Ask students to discuss Task 2, above.
Record different methods on the board. These should include:
 - trial and improvement
 - formal algebra
 - sketches or diagrams.
 If no-one suggests one of these, provide an example yourself. (And if a student suggests using a graph, sketch this on the board too.)
- What are the advantages and disadvantages of these different methods?

3. Inventing new problems.

- Ask the students to invent similar problems of their own. They could begin by suggesting alternative questions for the given triangle, such as:
 - For what value of x are the height and base the same?
 - For what value of x is the height 3 times the base?

4. Problems in the textbook.

- Ask the students to look at similar problems in a textbook:
 - What happens to the triangle as x changes?
 - Can you solve the problem using more than one method?
 - What are the advantages and disadvantages of the different methods?

[* From Framework Maths 8C, page 191]

Algebra 7A

Lesson

18A**Overview****Mathematical ideas**

This lesson gives students a further opportunity to compare expressions as the unknown varies, this time involving two operations (multiplication and addition or subtraction). This is supported by the use of a geometric representation (a 'dynamic' triangle) for the expressions.

Students are encouraged to use and compare a range of approaches: visualisation, trial and improvement, algebraic symbolisation and manipulation, and sketches (eg a line or block diagram).

Students' mathematical experiences

Students might discover some of the following

- as x changes, so the size *and* shape of the triangle changes (the transformation is *not* an enlargement)
- depending on x , the triangle can have a larger base than height, equal base and height, or a smaller base than height
- as the triangle gets bigger and bigger, the shape changes less and less (height : base approaches 2.5 : 1)
- the height changes at a faster rate than the base (and this is related to the multipliers $\times 5$ and $\times 2$, ie to the coefficients of x)
- the difference between the height and base is $3x - 9$ (and the difference is 0 when $x = 3$)
- the ratio of height to base is $5x-2 : 2x+7$ (and this gets closer to $5x : 2x$, ie $5 : 2$ or $2.5 : 1$, as x gets very large).

Assessment and feedback

Do students realise that the triangle can have very different shapes, depending on x ?

Can students differentiate between the idea of size changing and shape changing?

Do students think of x (and the base and height of the triangle) as changing in a systematic way, or simply as taking on different values?

Key questions

Is the height always, sometimes, or never bigger than the base?

What methods can we use for Task 2?

Adapting the lesson

It is worth spending a lot of time on Task 1, exploring the way the triangle changes with x , and finding ways of comparing the base and height (numerically, by scrutinising the algebraic expressions, by constructing generic expressions, and by using the GeoGebra file).

Students might need help to construct and solve a formal relationship like $5x-2 = 2(2x+7)$. Don't rush this and link it to other methods and representations (eg trial and improvement, line diagrams, generic relationships).

Resources

A Geogebra file showing an animation of the triangle is available on the website, www.iccams-maths.org

Also a pdf file showing the triangle for $x = 1$, $x = 2$ and $x = 3$.

Algebra 7A

Lesson

18A**Outline of the lesson (annotated)**

1. Display Task 1.

- Ask students to discuss Task 1, above, in pairs.
 - Discuss how the triangle changes as x changes.
 - What does the triangle look like when x is 2, or 20, or ... ?
 - What happens to the base when x increases by 1 ... by 1 again ...
- [On the GeoGebra file, you can use the slider to change the value of x , but use this sparingly.]

— As x changes the triangle changes size and shape. The distinction between *size* and *shape* is quite subtle.

— Don't use the slider too soon or too extensively. Students will be more engaged and gain more insight if they make predictions and argue them through.

When you do use the slider you may find that part of the triangle goes off the screen as you increase its size - make a virtue of this, by asking students to think about what might be happening.

- Task 2: For what value of x is the height twice the base?
- Ask students to discuss Task 2, above. Record different methods on the board. These should include:

— Don't rush on to Task 2. It is worth spending a lot of time on Task 1.

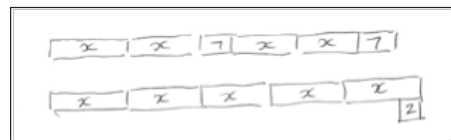
• trial and improvement

x	base $2x+7$	height $5x-2$	
5	17	23	too small
10	27	48	too small
15	37	73	too small
20	47	98	too big
17	41	83	too big
16	39	78	✓

• formal algebra

$$5x - 2 = 2(2x + 7)$$

• sketches



- If no-one suggest one of these, provide an example yourself. (And if a student suggests using a graph, sketch this on the board too.)
- What are the advantages and disadvantages of these different methods?

— Trial and improvement can give students a sense of what happens as x varies (eg, as x increases by 5, the base and height increase by 10 and 25 respectively - why?).

Formal algebra can be very powerful, as students develop methods that are meaningful and efficient.

A sketch can help students to see (and simplify) the relationship between the expressions for base and height.

2. Inventing new problems.

- Ask the students to invent similar problems of their own. They could begin by suggesting alternative questions for the given triangle, such as:
 - For what value of x are the height and base the same?
 - For what value of x is the height 3 times the base?

— The height can never be 3 times the base. As x increases the ratio height : base approaches 2.5 : 1. Why?

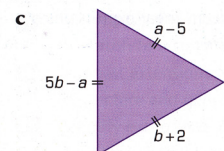
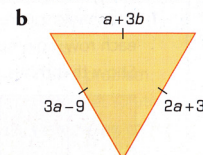
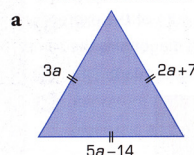
3. Problems in the textbook.

- Ask the students to look at similar problems in a textbook:
 - What happens to the triangle as x changes?
 - Can you solve the problem using more than one method?
 - What are the advantages and disadvantages of the different methods?

Exercise A5.8

1 Each of these are equilateral triangles:

- Find the perimeter for each in terms of a and b .
- Find the actual length of each side and the values for a and b .



Algebra 7A

Lesson

18A**Background****Students further develop their understanding of 'variable'**

Here students compare two relatively complex expressions in a single variable, x . The geometric context, of a 'growing' triangle, emphasises the fact that x can vary. It also supports the idea of x changing in a continuous, systematic way and of thinking about how this affects the size and shape of the triangle. This is made more vivid and engaging by the fact that x has a differential affect on the base and height.

Using formal algebra

In Task 2, students have the opportunity to use formal algebra (amongst other methods) to find a particular value of the variable, x , in the given expressions for the height, $5x-2$, and base, $2x+7$, of the triangle.

In this task, the height of the triangle is twice the base, which can be expressed as $5x-2 = 2(2x+7)$. However, deriving this equation is not a trivial step. The given expressions are quite abstract in themselves and some students will struggle to think of them as representing numbers (rather than instructions of some kind) and, furthermore, numbers that are related (one is twice the other). This step should therefore not be rushed (or pushed too hard) and it should be linked carefully to other methods and representations.

You might want to re-interpret the relation geometrically - for example as representing an *isosceles* right angled triangle with height $5x-2$ and base $2(2x+7)$. It might help initially to think of the length of the base as $2x+7 + 2x+7$.

The solution of $5x-2 = 2(2x+7)$ should also not be rushed. Don't focus only on a formal approach ('do the same to both sides'). Spend time on a variety of strategies, such as 'matching' (which can be supported by the use of a line-diagram) and trial and improvement.

Using trial and improvement

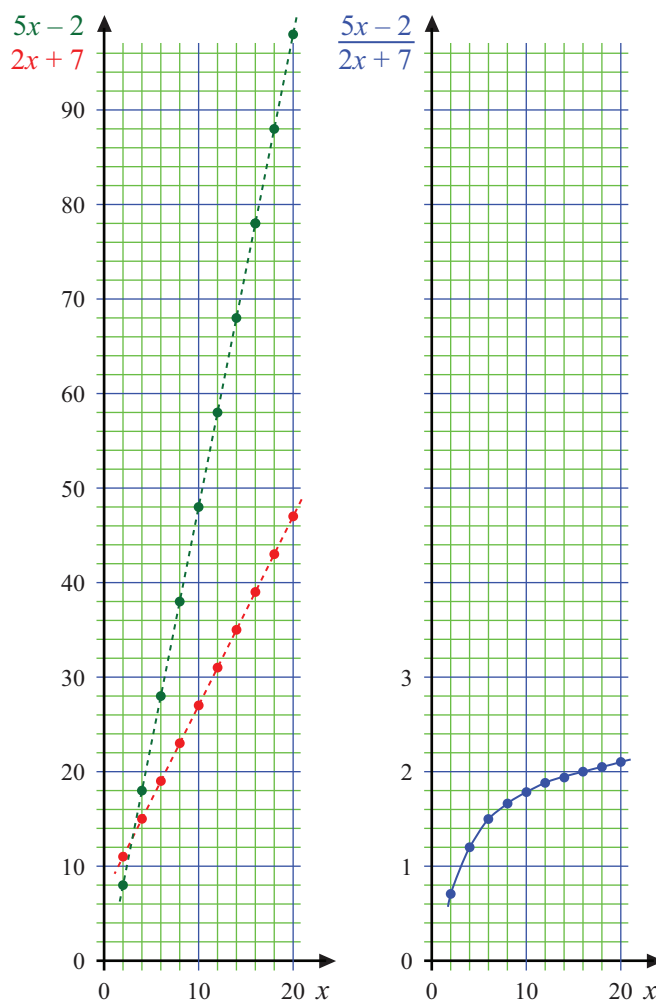
The solution to Task 2 is $x=16$. This is some way away from being a 'small' number, which suggests that students who use trial and improvement are not likely to hit upon this by chance in just a few steps. The situation is made more difficult by the fact that the variable appears in two expressions so that it is not always easy to see what the overall effect will be of choosing a larger or smaller value of x . One virtue of this is that it will encourage students to look for more efficient methods (in particular the formal method of creating an equation and transforming this in such a way that the variable appears just once). Students should also be encouraged to make use of their earlier geometric explorations, which may already have lead them to produce a triangle where the height looks to be about twice the base, and where they may have sensed the fact that as x increases, the height increases more rapidly than the base.

Using graphs

It can be illuminating to graph the values of the expressions (although there are practical difficulties - thus in the first graph below, we have kept the scales of the two axes the same, in order better to be able to interpret the slopes, but this means one of the lines is very steep [because of the multiplier $\times 5$ in $5x-2$]).

The first graph shows that as x increases, the height (ie $5x-2$) increases more rapidly than the base (ie $2x+7$). It also suggests that the height and base are equal for some, small value of x (actually $x=3$) and that the height is twice the base for a value of x near 20 (actually $x=16$).

Another way to compare the height and base is to take their ratio. This is shown in the second graph (where the scale on the y axis has been expanded to show more detail). For small values of x , the ratio is less than one (and so the height is less than the base). The ratio is 2 for $x=16$ (the solution to Task 2) and continues to increase as x increases, but more and more slowly it seems ...



Algebra 7B

Lesson

18B**Equating expressions**

Think about the values of these expressions.

For what value of x are they the same?

A. $3x + 50$

B. $5x + 10$

Summary

Here we compare two expressions in x . We compare the expressions as x *varies* and consider the *specific* value of x for which the expressions are the same. Thus the work provides further exploratory work on solving equations in one unknown, where the unknown appears on both sides.

We start with an empirical approach, but encourage students to analyse what is going on rather than engage in a large number of trials. We then consider a matching approach, as an informal introduction to the idea of ‘balancing’ equations or ‘doing the same to both sides’.

Outline of the lesson

- Compare the expressions $3x + 50$ and $5x + 10$.
 - Present the two expressions.
Ask students,
“Which is larger?”
“Could they ever be the same?”
 - Ask students to find the value of x for which the expressions are the same ($x=20$). Let them use whatever method they like.
 - Discuss students’ methods.
- Compare the expressions $3x + 50$ and $5x + 10$ using ‘trial and scrutiny’.
 - Tell the class that we are going to look at two methods for equating the expressions, starting with ‘trial and inspection’.
 - Choose a value of x , say 10. Ask the class to find the values of the expressions for $x = 10$. Write the results on the board in a table.
 - Ask students to *predict* what happens if x is now increased by 1. Does the difference increase or decrease, and by how much?
 - Check the predictions and add the results to the table.
 - Scrutinise* the table. “How much do we need to change x for the difference to be 0?” Check that this happens when $x = 20$.
- Compare the expressions $3x + 50$ and $5x + 10$ by ‘matching’.
 - Write $3x + 50$ as $3x + 10 + 40$ [or as $x + x + x + 10 + 40$].
Write $5x + 10$ as $2x + 3x + 10$ [or as $x + x + x + x + x + 10$].
Ask, “When are they the same?” - When $2x = 40$, ie when $x = 20$.
 - Confirm the solution by substituting 20 into the expressions:
 $60 + 10 + 40$ and $40 + 60 + 10$.
- Repeat Stages 2 and 3 for another pair of expressions in x .

x	A. $3x + 50$	B. $5x + 10$	A – B
10	80	60	20
+1	+3	+5	-2
11	83	65	18
+9			-18
20	110	110	0

Algebra 7B

Lesson

18B**Overview****Mathematical ideas**

Here we compare two-step expressions, similar to those in Lesson ALG-7A. However, this time we are working in a 'pure' context and the approach is less exploratory, with a tighter focus on algebraic symbolisation and on two specific methods: 'matching' and an analytic form of 'trial and improvement'.

Students' mathematical experiences

Students might discover some of the following

- as x changes, so the values of the expressions change, at different rates
- when x is close to zero, the coefficient of x has relatively little effect
- when x is large, the coefficient of x has the dominant effect
- when comparing expressions, we can ignore the parts that have the same effect.

Assessment and feedback

After Stage 4 of the lesson, ask students to generate an easy and a hard example, and to explain why their examples are easy and hard. This will help develop self-assessment skills.

Key questions

Which expression is larger when x is 'small' (eg, 0.2) ?

Which expression is larger when x is 'large' (eg, 100) ?

Why do the values of the expressions 'cross over'?

Adapting the lesson

You might want to represent the expressions geometrically, as in Lesson ALG-7A, eg as the lengths of the base and height of a triangle or rectangle; or graphically, as points on a Cartesian graph, or as moving points on a number line (or two parallel number lines - see the GeoGebra file `racings-expressions.ggb`).

At some stage, or in a later lesson, you might want to move towards a more formal method by shifting the focus from finding the solution for when a pair of expressions are equal, to finding pairs of modified expressions with the same solution (see the GeoGebra file `modifying-expressions.ggb`).

Algebra 7B

Lesson

18B**Outline of the lesson (annotated)**1. Compare the expressions $3x + 50$ and $5x + 10$.

- Present the two expressions.
Ask students,
“Which is larger?”
“Could they ever be the same?”
- Ask students to find the value of x for which they are the same ($x=20$). Let them use whatever method they like.
- Discuss students' solutions and methods.

2. Compare the expressions $3x + 50$ and $5x + 10$ using ‘trial and scrutiny’.

- Tell the class that we are going to look at two methods for equating the expressions, starting with ‘trial and inspection’.
- Choose a value of x , say 10. Ask the class to find the values of the expressions for $x = 10$. Write the results on the board in a table.
- Ask students to *predict* what happens if x is now increased by 1. Does the difference increase or decrease, and by how much?
- Check the predictions and add the results to the table.
- *Scrutinise* the table. “How much do we need to change x for the difference to be 0?” Check that this happens when $x = 20$.

Some students might resist this, as “we’ve already found the answer”. Emphasise that we are interested in looking at methods - we don’t really care about the answer!

It can also be illuminating to represent the values of the expressions, and the difference, on a graph (see below). Where on the graph does the difference become 0?

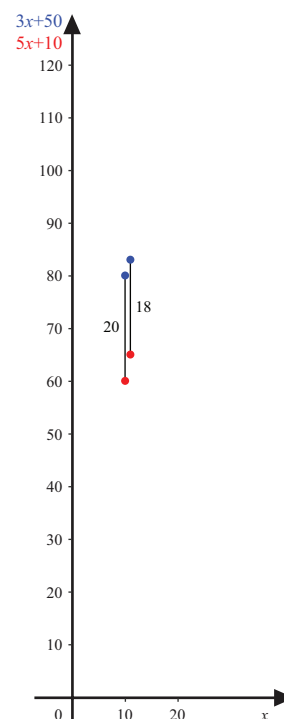
Expression A increases by 3, expression B increases by 5, so the difference decreases by 2.

As x increases by 1, the difference decreases by 2. So if we increase x by another 9, the difference will decrease by 2×9 and become 0.

You might want to transform $2x+3x+10$ further, into $3x+10+2x$, to make the match easier:

A. $3x + 10 + 40$

B. $3x + 10 + 2x$

3. Compare the expressions $3x + 50$ and $5x + 10$ by ‘matching’.

- Write $3x + 50$ as $3x + 10 + 40$ [or as $x+x+x+10+40$].
Write $5x + 10$ as $2x + 3x + 10$ [or as $x+x+x+x+10$].
Ask, “When are they the same?”
- When $2x = 40$, ie when $x = 20$.
- Confirm the solution by substituting 20 into the expressions:
 $60 + 10 + 40$ and $40 + 60 + 10$.

You could use the same two expressions, with one of them tweaked slightly, eg with $3x+50$ changed to $3x+51$, or to $4x+50$. “How does this change the solution?”

Or, “Could we change both expressions so that the solution stays the same?”

Or, “Could we change expression A so that A and B are *never* the same?”

Algebra 7B

Lesson

18B**Background****Using trial and improvement analytically**

In this lesson we use a refined form of trial and improvement, where we consider a minimal number of trials but look not just at the result but at how the result is changing. For linear expressions like the ones that we are considering, this allows us to work out precisely the value we are looking for rather than undertake further trials. It also focusses attention on how the coefficients of x affect the values of the two expressions.

It is worth pointing out that we have deliberately chosen to equate expressions that *both* vary (in effect, equations where x appears 'on both sides'). In this situation it is not clear how to improve a trial, without undertaking further analysis, since a change in the trial value will change both expressions.

Using matching

We also use matching in this lesson to compare expressions, as a way of developing the idea of 'balancing' equations, by 'doing the same to both sides'.

Matching works well for the expressions that we are considering here, as they can easily be broken down into smaller, 'self-standing' additive parts. Thus, if we are equating $3x+50$ and $5x+10$ we can break these into $3x+10+40$ and $3x+10+2x$ and so reduce the task to equating 40 and $2x$.

However, we do need to be sure that the parts we are 'matching off' are self-standing. If our original expressions had been, say, $2(3x+50)$ and $7(5x+10)$, we could not simply reduce this to equating $2(40)$ and $7(2x)$. Similarly, these two equations are not equivalent:

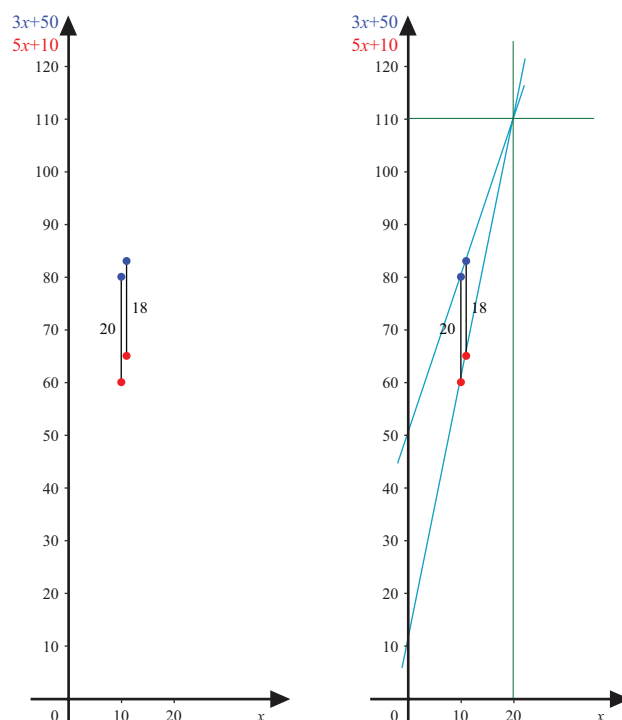
$$\frac{1}{3x+50} = \frac{1}{5x+10} \quad \text{and} \quad \frac{1}{40} = \frac{1}{2x}$$

Using graphs

The first of the two graphs, near right, focusses on the differences in the original two expressions for $x = 10$ and $x = 11$. It represents information from the table used in Stage 2 of the lesson. This paired-down and rather unorthodox Cartesian graph may seem puzzling at first, but it can be helpful in providing a fresh view on what graphs are about. After due consideration, it can be seen that it provides a small slice of the standard straight line graphs of the two expressions, as shown on the second graph, far right.

The GeoGebra file `modifying-expressions.ggb` provides a dynamic version of this second graph. Here the coefficient of x and the constant term in our original expressions can both be varied, but in such a way that the graphs of the expressions still intersect at the point $(20, 110)$. The file is worth exploring for yourself and with colleagues, but it should be used judiciously with students - it contains a lot of ideas ...

A second Geogebra file, `racing-expressions.ggb`, is simpler. It represents the values of the original expressions on two horizontal number lines. The values can be changed by means of a slider. The scales for the number lines have deliberately been left blank, to encourage students to think about what is going on. The file could be used at the end of Stage 1 of the lesson, ie after students have had an opportunity to think about the values of the expressions.



Algebra 8AB

Mini-assessment

19AB**Expressions mini test**

Ask students to work through this mini test, sometime before starting on Algebra Lessons 8A and 8B.

This can be done during another lesson, or for homework.

19AB Mini-assessment Expressions mini test

Name

Find the value of these expressions when $h = 6$.
Work through them in any order you like.

A. $7 + \frac{4h + 1}{5}$

B. $(4h - 3) \div 7$

C. $\frac{3h + 2}{5}$

D. $1 + 4h$

E. $\frac{7 + 4(h + 1)}{5}$

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Commentary

The aim of this Mini-assessment is to get a sense of how well students can evaluate algebraic expressions.

Which expressions do students find difficult?

Do their difficulties stem from a lack of familiarity with notation conventions, or from something deeper?

As an extension activity, you could ask students to *transform* the expressions into something equivalent. This is likely to be much more challenging.

[It is likely that students will be able to evaluate quite complex expressions like these (and get pleasure from being able to do so) even before having a clear understanding of how the operations interact and of how the expressions can be transformed.]

19AB Mini-assessment Expressions mini test

Name

Find the value of these expressions when $h = 6$.
Work through them in any order you like.

A. $7 + \frac{4h + 1}{5}$

B. $(4h - 3) \div 7$

C. $\frac{3h + 2}{5}$

D. $1 + 4h$

E. $\frac{7 + 4(h + 1)}{5}$

19AB Mini-assessment Expressions mini test

Name

Find the value of these expressions when $h = 6$.
Work through them in any order you like.

A. $7 + \frac{4h + 1}{5}$

B. $(4h - 3) \div 7$

C. $\frac{3h + 2}{5}$

D. $1 + 4h$

E. $\frac{7 + 4(h + 1)}{5}$

Algebra 8A

Lesson

19A**Checking with numbers**Task 1 Look at this relation between d and e .

Express it in words.

$$e = d + 4$$

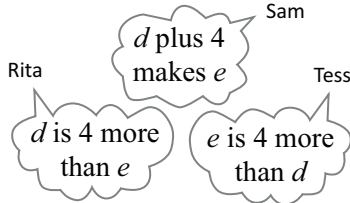
Summary

In this lesson we look at common errors that occur in interpreting, writing and manipulating algebraic expressions and relations. We consider specific values of the relations (together with tables and graphs) to get a better feel for them. And we test our attempts to write relations by using simple numerical checks.

Outline of the lesson

1. Express an algebraic relation in words.

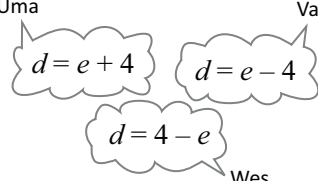
- Ask students to express the relation $e = d + 4$ in words. Write some of their statements on the board. Try to include statements like these:
- Go back to the algebraic relation. Ask for values of d and e that make it true. Write correct pairs (eg $d = 6, e = 10$) in a table.
- Describe the pairs of values in words.
- Use pairs of values to test students' statements. For example, using $d = 6, e = 10$:
 Rita: '6 is 4 more than 10'? False
 Sam: '6 plus 4 makes 10'? True
 Tess: '10 is 4 more than 6'? True.



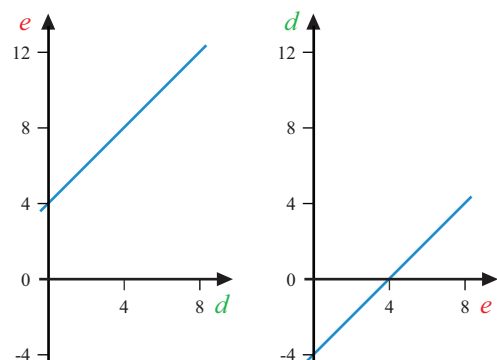
d	e
6	10
1	5
20	24
21	25

2. Transform the relation.

- Ask students to write $e = d + 4$ in the form $d = \dots$ (ie make d the subject of the equation). Write some of their suggestions on the board.
- Use the pairs of values found earlier for d and e to test students' algebraic statements.


3. Consider the graphs of $e = d + 4$ and $d = e - 4$.

- Ask students to *think* about the Cartesian graph of $e = d + 4$. What will it look like?
- Ask students to *sketch* the graph.
- Ask students to *think* about the graph of $d = e - 4$. How will it compare to the graph of $e = d + 4$?



4. Use steps like the above to explore Task 2 (shown on page 241).

Algebra 8A

Lesson

19A**Overview****Mathematical ideas**

Here we look at algebraic expressions, ie expressions about numbers, and test whether they are correct by substituting specific numerical values. Students need to be aware that the expressions *are* numbers, that the equals sign conveys the idea that the expression to the left of the equals sign has the same value as the expression to the right, and that there is not always a direct correspondence between the symbolic and verbal forms of an expression.

We consider simple rules (or relations or equations) in two variables and underline the fact that the relations are general by considering tables of values and Cartesian graphs. We transform the rules by changing the subject and compare the corresponding graphs [although we don't pursue the fact that the graphs are reflections of each other in $y = x$].

There is the opportunity to consider the general nature of rules (Task 2), the confusion that sometimes arises when a letter is used to stand for an object rather than a number (Task 3), and the need to accept the lack of closure of expressions (Task 4).

Students' mathematical experiences

The students should ...

- realise that algebraic expressions are numbers*
- realise that we can test algebraic relations by substituting specific values
- realise that we need to be careful when translating between words and symbols.

* At school level, letters are mostly used to represent numbers. They can of course represent other kinds of element, such as sets or transformations, which gives rise to other algebras.

Assessment and feedback

Do students know that an algebraic relationship is a general relationship between sets of numbers?

You might like to try the Assessment task on page 241 (at the start of the lesson, or before the lesson, eg as a homework).

Do students know that we can represent an algebraic relationship verbally, by a table of values, or by a graph?

Key questions

Does the relationship fit this pair of values?

Does the rule work for other pairs of values?

Does m stand for 'months' or 'number of months'?

Adapting the lesson

We have chosen rather simple expressions and relations here because they are easier to 'visualise' and to compare with their inverses. However, this does make the expressions seem rather 'purposeless' - why would one bother to express the relations algebraically?! Thus you might want to use more complex expressions. However, it might suffice to remind students of a lesson like ALG-2A (*Pyramids*), where we make effective use of quite complex expressions formed from simple expressions like the ones here. The same thing occurs in Lesson ALG-9B (*Ages and ages*).

The Geogebra file E=D+4-mapping-diagram.ggb represents the given relation $e = d + 4$ as a rather unusual mapping diagram. It can be quite challenging (and illuminating), for us and our students, to consider how this relates to the standard Cartesian graph.

You might want to start with Task 2, 3 or 4 rather than Task 1.

Algebra 8A

Lesson

19A**Outline of the lesson (annotated)**

1. Express an algebraic relation in words.

- Ask students to express the relation $e = d + 4$ in words.
Write some of their statements on the board.
Try to include statements like these:
- Go back to the algebraic relation.
Ask for values of d and e that make it true.
Write correct pairs (eg $d = 6$, $e = 10$) in a table.
- Describe the pairs of values in words.
- Use pairs of values to test students' statements.
For example, using $d = 6$, $e = 10$:
Rita: '6 is 4 more than 10'? False
Sam: '6 plus 4 makes 10'? True
Tess: '10 is 4 more than 6'? True.

- Note: we use 'algebraic' here to mean 'expressed in algebraic symbols'. Strictly, a general relation expressed verbally is also algebraic.
- Some students might read '+4' as a description of d (it is 4 more than ...) rather than as an operation on d .
- You might want to display the responses given by 'Rita', 'Sam' and 'Tess'.
- This is to emphasise that the relation is general, ie it holds for a range of values. It also makes the relation more concrete and visible: it can be seen that e is 4 more than d , for each pair of values.

2. Transform the relation.

- Ask students to write $e = d + 4$ in the form $d = \dots$ (ie make d the subject of the equation).
Write some of their suggestions on the board.
- Use the pairs of values found earlier for d and e to test students' algebraic statements.

- Some students might interpret 'changing the subject' as simply a matter of exchanging the variables, ie $e = d + 4$ becomes $d = e + 4$.

3. Consider the graphs of $e = d + 4$ and $d = e - 4$.

- Ask students to *think* about the Cartesian graph of $e = d + 4$. What will it look like?
- Ask students to *sketch* the graph.
- Ask students to *think* about the graph of $d = e - 4$.
How will it compare to the graph of $e = d + 4$?

- Conventionally, we interpret $e = d + 4$, as e being a function of d , and we represent d , the independent variable, on the x axis.
Similarly, for $d = e - 4$, we show e on the x axis.
- The key idea is that the first graph clearly shows that e is bigger than d (in fact, 4 bigger, and for any value of d); the second shows that d is smaller than e (4 smaller).
You might want to point out that if d and e were equal, the graph would be a 45° line through the origin (ie $y = x$). For the first relation that line is translated upwards (by 4 units); for the second it is translated downwards.
Some students might realise that we are dealing with inverse functions and that the graphs are reflections of each other in $y = x$. However, you might not want to pursue these ideas here.
- The tasks are shown on page 241.

4. Use steps like the above to explore Task 2 (shown on page 241).

Algebra 8A

Lesson

19A**Background****Some common errors in using algebraic symbolisation to express numerical relations.**

It is easy to make mistakes when considering algebraic expressions and relations, by attempting to translate between words and algebraic symbols in a too direct way - and by not being sufficiently aware of the fact that the expressions are pure numbers and that the equals sign is used to state that values are the same. This is addressed in Task 1.

Task 2 (below) involves the further issue of generalising from a specific case and without considering structure.

Two further tasks (3 and 4) are shown in the Revisits section. Task 3 involves the use of letters as objects instead of numbers.

Task 4 examines the propensity to 'close' expressions, so that they look more like 'answers' or numbers.

Task 2

A column of white tiles is covered by a 'bridge' of grey tiles.

Lee wants to write a formula, $g = \dots$, for the number, g , of grey tiles when you know the number, w , of white tiles.

Lee draws this pattern and says, "The rule is $g = 3w$ ".

Show that Lee is wrong.

Find the correct formula.

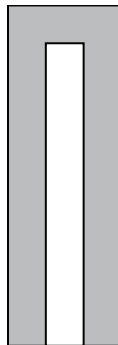
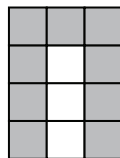
Commentary:

Lee's rule works for his particular example, $w = 3, g = 9$. However, the rule does not express the structure of the pattern (eg 'There are twice as many grey tiles as white, with an extra 3 on top'). So Lee's rule does not work for other examples of the pattern (eg, $w = 10, g = 23$).

Outline:

- Express Lee's rule in words (eg, 'There are three times as many grey tiles as white tiles').
- Express the structure of the pattern in words (eg, 'The number of grey tiles is 3 times the number that is 1 more than the number of white tiles, minus the number of white tiles'). Show that these verbal rules don't match Lee's rule in words.
- Find other numerical values for the pattern. Show that they don't fit Lee's rule.
- Find a correct formula, eg $g = 2w + 3$, or $g = 3(w + 1) - w$, or $g = 2(w + 1) + 1$. Check the formula fits the verbal rules and the numerical values.
- Write the formula as $w = \dots$. Check it fits the numerical values.
- Sketch a graph for each formula.

3 white, 9 grey



Algebra 8B

Lesson

19B**Expressions and function machines**

- Task 1** Look at this expression:
What is its value when $e = 4$?

$$\frac{3e + 11}{2} - 7$$

Summary

Here we focus on expressions in one variable involving several operations. We consider the conventions used to write such seemingly complex expressions and how these convey the order in which the operations are to be carried out. We use the notion of a function machine to help make the order explicit.

We evaluate an expression and explore what happens when we shuffle the numbers or shuffle the operators. Finally we try to find the value of the variable for a specific value of the expression, ie we consider ways of solving the resulting equation (ie an equation where the variable appears just once, but in a complex expression).

Note, however, that in this lesson we leave to one side the more demanding issues of how operations interact, and how a complex expression can be transformed into equivalent expressions.

Outline of the lesson

1. Evaluate a complex expression.

- Present the expression in Task 1.
Ask students to evaluate the expression for $e = 4$.
Write some of their answers on the board.
- Ask students to solve Task 2a. Discuss.
Ask students to solve Task 2b. Discuss.

Task 2

- Estimate the value of the expression for $e = 100$.
- Swap two numbers in the expression, eg 3 and 2: $\frac{2e + 11}{3} - 7$
For $e = 100$, which swops have a large effect on the value, and which have relatively little effect?

2. Use function machines for expressions.

- Present the function machine in Task 3.
Ask students to find the output (4.5) for an input of 4.
Discuss why the expression and function machine are equivalent.
- Present the function machine in Task 4.
Ask students to find the output (19) for an input of 4.

Task 3

This function machine is another way of writing the Task 1 expression:

INPUT $\rightarrow \times 3 \rightarrow +11 \rightarrow \div 2 \rightarrow -7 \rightarrow$ OUTPUT

Use a calculator. Check the output for an input of 4.

Task 4

Here we have shuffled the operations from Task 1.

What is the output when the input is 4 ?

INPUT $\rightarrow +11 \rightarrow \times 3 \rightarrow -7 \rightarrow \div 2 \rightarrow$ OUTPUT

3. Write function machines as expressions.

- Ask students to solve Task 5.
Write some of their expressions on the board.
Show how the expression can be built-up step by step.
Use standard notation but make links with students' notation.
- Ask students to solve Task 6a. Discuss.
Ask students to solve Task 6b. Discuss.

Task 5

Write the function machine in Task 4 as an expression.

Use e for the input.

$$\frac{3(e + 11) - 7}{2}$$

Task 6

- Estimate the value of the new expression for $e = 100$.
- Swap two numbers in the expression, eg 3 and 2: $\frac{2(e + 11) - 7}{3}$
For $e = 100$, which swops have a large effect on the value, and which have relatively little effect?

4. Create an equation and find ways to solve it.

- Give a value to the expression in Stage 3, eg 25.
Estimate the value of e in the resulting equation.
Ask students to solve the equation in at least 2 ways (eg the 'cover up' method and trial and improvement).

$$\frac{3(e + 11) - 7}{2} = 25$$

Algebra 8B

Lesson

19B**Overview****Mathematical ideas**

Here students gain experience of evaluating algebraic expressions involving several operations. They thus need to be able to discern the 'scope' of each operation, ie that part of an expression to which a specific operation applies. This involves being able to read notation such as the fraction bar and the precise positioning of this and other operation symbols. It also involves knowledge of the conventional order in which operations are to be performed, for example that we tend to perform operations from left to right but that multiplication and division have precedence over addition and subtraction.

In this lesson we use expressions involving quite a large number of operations, in the belief that this will only marginally increase the difficulty of deciphering them while greatly increasing students' satisfaction in doing so. However, it is important to bear in mind that evaluating an expression is likely to be more straightforward than transforming it (by for example expanding brackets and simplifying). Thus success at this does not necessarily mean that students have a good understanding of how operations interact or how expressions can be transformed.

Students' mathematical experiences

The students should ...

- gain experience of evaluating quite complex expressions
- realise that expressions can often be evaluated in a straightforward, step-by-step manner
- realise that for (some) expressions we can use a function machine to record the order of operations
- gain further experience of solving equations in different ways, including 'reversing' a function machine.

Assessment and feedback

Do students know what the fraction bar means? And that its 'scope' depends on the terms that are immediately above and below it?

What do students know about the conventions for determining the order of operations? (For example, the left to right rule, the various precedence rules, the use of brackets including the fraction bar?)

Key questions

To evaluate the expression, what operation do you do first, ... second, ... last?

In the expression, which numbers have the greatest effect when e is large, or when e is small (close to zero)?

Adapting the lesson

Depending on the class, you might want to use slightly simpler or more complex expressions.

Or you might want to explore further the effect of changing the order of the four given operators $\times 3$, $+11$, $\div 2$, -7 .

Interestingly, the resulting expressions are all equivalent to expressions of the form $1.5e + k$, so their relative values are the same for any value of e . This is in contrast to expressions we've looked at in other lessons, such as $3n$ and $n+3$.

Thus, when we represent them on a Cartesian graph, they produce parallel lines (ie lines that don't intersect).

Algebra 8B

Lesson

19B**Outline of the lesson (annotated)**

1. Evaluate a complex expression.

- Present the expression in Task 1.
Ask students to evaluate the expression for $e = 4$.
Write some of their answers on the board.
- Ask students to solve Task 2a. Discuss.

Ask students to solve Task 2b. Discuss.

——— Check that students know what the fraction bar means.

——— A quick/rough estimate will do, eg 150.

——— The wording here is deliberately vague. Swopping 3 with 11, and 11 with 2 both have a 'large' effect (though it would be quite a challenge to determine which is actually the larger). Swopping 11 with 7 has a small effect. Other swops (eg 3 with 7) will be somewhere in between ...

2. Use function machines for expressions.

- Present the function machine in Task 3.
Ask students to find the output (4.5) for an input of 4.
Discuss why the expression and function machine are equivalent.
- Present the function machine in Task 4.
Ask students to find the output (19) for an input of 4.

——— The (simple) key to evaluating the expression is to 'work outwards' from e , ie to start with the operation most directly acting on e . Interestingly, the algebraic notation helps here, since it is obvious that the 3 is 'closer' to e than the 11. This is not obvious in an arithmetic statement like $3 \times 4 + 11$, where students would have to know the convention that $\times$ takes precedence over $+$.

3. Write function machines as expressions.

- Ask students to solve Task 5.
Write some of their expressions on the board.
Show how the expression can be built-up step by step.
Use standard notation but make links with students' notation.
- Ask students to solve Task 6a. Discuss.
Ask students to solve Task 6b. Discuss.

——— So write e ,
then $e + 11$,
then $3(e + 11)$,
then $3(e + 11) - 7$,
and finally the expression $\frac{3(e + 11) - 7}{2}$.

4. Create an equation and find ways to solve it.

- Give a value to the expression in Stage 3, eg 25.
Estimate the value of e in the resulting equation.
Ask students to solve the equation in at least 2 ways
(eg the 'cover up' method and trial and improvement).

——— If you want to push students away from simple trial and improvement towards something more analytic (eg the cover up method), choose a value for the expression (eg 26) such that e is not a whole number.

Algebra 8B

Lesson

19B**Background****A single string of operators**

Note that in this lesson we are only considering expressions that are composed of a single 'string' of operators, ie that can be represented as a function machine with no branches. Such expressions can be evaluated in a step by step way, with no need to 'store' intermediate calculations, in contrast to an expression like this:

$$3e + 11 - \frac{7}{2}$$

Note also that this means that any resulting equations (as in Step 4) can be solved in a step by step manner, eg by 'reversing' the corresponding function machine.

Self inverse operators

We have also not used self inverse operators such as 'subtract from 7' or 'divide into 2' as occur in this expression. It is not so simple to represent such expressions with a function machine or to understand how to run the

$$7 - \frac{2}{3e + 11}$$

function machine in reverse.

Transforming expressions

We could transform our original expression in various ways and thereby perhaps simplify it, as in these two examples:

$$\frac{3e + 11}{2} - 7 = 1.5e + 5.5 - 7 = 1.5e - 1.5$$

$$\frac{3e + 11}{2} - 7 = \frac{3e + 11 - 14}{2} = \frac{3e - 3}{2}$$

This involves an understanding of the ways the various operations interact (for example that division is right-distributive over addition/subtraction). This is likely to be more demanding than simply evaluating the expression, which is what we concentrate on in this lesson.

Solving equations

We can solve the equation in Stage 4 in various ways. *Trial and improvement* works quite well here: the solution is a fairly small whole number ($e = 8$), and feedback is very direct (if the left hand side of the equation is too small/big, then e is too small/big). However, it can involve a lot of steps and is thus not likely to be efficient.

The *cover up method* would work like this:

'something' divided by 2 is 25, so 'something' is 50;
'something' minus 7 is 50 so 'something' is 57;
3 times 'something' is 57, so 'something' is 19;
'something' plus 11 is 19, so 'something' is 8.

This has close parallels with 'undoing' the operations by running the corresponding function machine in reverse:

INPUT $\rightarrow +11 \rightarrow \times 3 \rightarrow -7 \rightarrow \div 2 \rightarrow$ **OUTPUT**

8 $\leftarrow -11 \leftarrow \div 3 \leftarrow +7 \leftarrow \times 2 \leftarrow$ **25**

It also has parallels with applying these inverse operations, one by one, to both sides of the equation. It thus prepares the ground for a formal approach:

$$\begin{aligned} \frac{3(e + 11) - 7}{2} &= 25 \\ 3(e + 11) - 7 &= 50 & [\times 2] \\ 3(e + 11) &= 57 & [+7] \\ e + 11 &= 19 & [\div 3] \\ e &= 8 & [-11] \end{aligned}$$

We can also use *matching*, by making the right hand side of the equation look progressively more like the left, as shown below. This too has parallels with the formal approach.

25

$$\frac{50}{2}$$

$$\frac{57 - 7}{2}$$

$$\frac{3 \times 19 - 7}{2}$$

$$\frac{3 \times (8 + 11) - 7}{2}$$

Algebra 9AB

Mini-assessment

20AB**Taxi!**

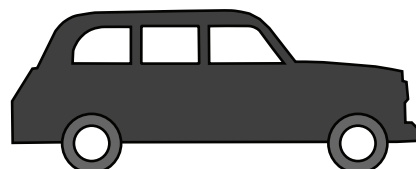
Ace Taxis uses this expression to work out the charge (in £s) for a journey of d miles.

What does it charge for a 5 mile journey?

Dora Cabs charges the same as Ace Taxis for a 5 mile journey, but more for longer journeys.

Write a possible expression for its charges.

$$8 + 1.2d$$

**Commentary**

The aim of this Mini-assessment is to get a sense of how well students evaluate, interpret, construct and compare simple algebraic expressions.

Note: If students struggle with the Taxi context, use the T-shirts version on the next page.

Can students work out what Ace Taxis charges for a 5 mile journey?

What difficulties do students have?

Can students explain what the terms 8, 1.2 and d mean in the context of the story?

Can students find an expression for the charges made by Dora Cabs?

Having found one expression, how readily can they find others?

Note: If students struggle with the first part of the Mini-assessment, you might want to leave this second part for another occasion.

Algebra 9AB

Mini-assessment

20AB**T-shirts**

Capital T-shirts sells personalised T-shirts.
It uses this expression to work out the price (in £s)
for a T-shirt with a name that has d letters.

What is the price of a T-shirt
with a name that has 5 letters?

My T-shirts charges the same as Capital T-shirts when the
name has 5 letters, but more for longer names.

Write a possible expression for its prices.

$$8 + 1.2d$$

**Commentary**

The aim of this Mini-assessment is to get a sense of
how well students evaluate, interpret, construct and compare simple algebraic expressions.

Can students work out what Capital T-shirts charges for a T-shirt with a 5 letter name?
What difficulties do students have?

Can students explain what the terms 8, 1.2 and d mean in the context of the story?

Can students find an expression for the prices charged by My T-shirts?
Having found one expression, how readily can they find others?

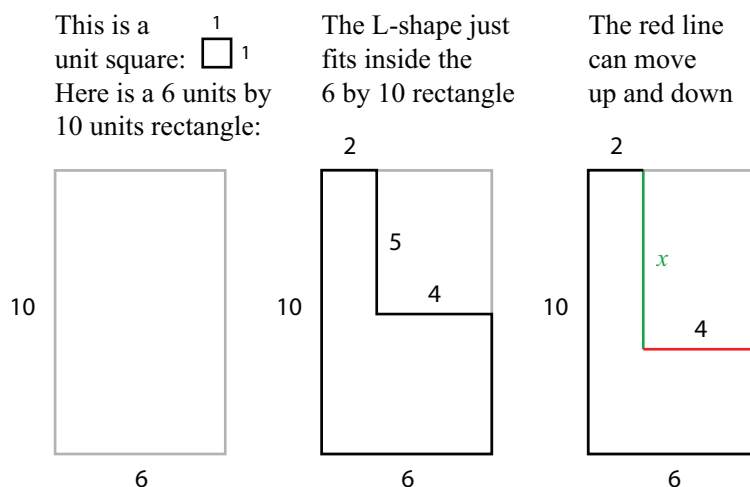
Note: If students struggle with the first part of the Mini-assessment, you might want to leave this second part for another occasion.

Algebra 9A

Lesson

20A**A family of L-shapes**

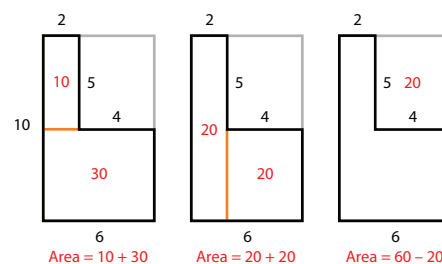
1. Look at the first diagram.
How many *unit squares* would cover the rectangle?
2. Look at the second diagram.
How many *unit squares* would cover the L-shape?
3. Look at the third diagram.
Find the L-shape that covers exactly half the rectangle.
What is its value of x ?

**Summary**

The lesson involves a family of L-shapes which we consider dynamically and statically (to bring out the notions of *variable* and *specific unknown*). We check students' understanding of area and then look for a specific L-shape whose area is half of an enclosing rectangle. We first consider the area geometrically. Then we express it algebraically which allows us to compare equivalent expressions and to explore ways of solving a linear equation (eg by trial and improvement, by the cover up method, by matching or by balancing).

Outline of the lesson

1. Find the area of the 6 by 10 rectangle (Task 1).
 - Discuss the area of the rectangle. Why is it 6×10 unit squares?
2. Find the area of a composite shape (Task 2).
 - Ask students to find the area of the L-shape (second diagram).
 - Discuss methods, including these 'split and calculate' methods:
 $2 \times 5 + 6 \times 5$, $2 \times 10 + 4 \times 5$, $6 \times 10 - 4 \times 5$.
3. Consider the family of L-shapes and the specific case (Task 3).
Use a geometric approach.
 - Explain and explore the dynamic nature of the third diagram.
 - Ask students to find the specific L-shape that covers half the rectangle.
What is the value of x for this L-shape? ($x = 7.5$)
 - Discuss students' methods.
4. Reconsider Task 3. Use an algebraic approach.
 - Find a way to express the area of the L-shape in terms of x .
Equate the expression to 30 and discuss ways of solving the equation.
5. Find equivalent expressions for the area.
 - Find alternative expressions in x for the area of the L-shape.
Consider why they are equivalent.



Students might produce the equation
 $2x + 6(10 - x) = 30$
 (which is relatively complex) or
 $20 + 4(10 - x) = 30$
 (which is almost as complex) or
 $60 - 4x = 30$
 (which is relatively simple).

Algebra 9A

Lesson

20A**Overview****Mathematical ideas**

Questions involving algebraic expressions for the areas (or perimeters) of composite shapes appear in lots of textbooks. The context lends itself to the construction of equivalent expressions, which we do here. We also explore the meaning of such expressions by considering them dynamically, and we make the work more ‘purposeful’ by imposing a constraint: “When is the area of the L-shape half that of the surrounding rectangle?”

Area provides a useful model for the distributive law, and so might enhance students’ understanding of ‘expanding brackets’.

Students’ mathematical experiences

The students should ...

- get a sense of the dynamic nature of the L-shape, ie how its shape and area vary as x varies
- realise that we can count a shape’s area by covering it in unit squares
- realise that we can split the L-shape into rectangles, in different ways
- realise that we can calculate a shape’s area by splitting it into rectangles
- realise that we can use algebra to represent the area, and calculate later
- realise that we can manipulate algebraic expressions by using algebraic rules or by referring to shapes
- realise that we can get equivalent algebraic expressions.

Assessment and feedback

Check students’ understanding of ‘area’ and their knowledge and understanding of the area rule for a rectangle.

Try to keep all students engaged - encourage them to ask questions and to come up with their own ideas and methods - but encourage them also to discuss and make sense of each other’s ideas including algebraic ideas: as with other lessons, we are interested in methods rather than in the answer to a specific problem.

Key questions

Can you describe the L-shape when $x = 4$, say?

Which value of x gives the ‘nicest’ L-shape, do you think?

What happens to the L-shape’s area as the red line moves up and down? Tell me more ...

How can we show that these two expressions are equivalent?

Adapting the lesson

You might want to do some preparatory work on area (the idea of ‘covering rectangles with unit squares’) and on using algebraic expressions to represent area.

A possible extension/homework task would be to consider ways of splitting the L-shape into two parts (eg two rectangles or two trapezia). “For what value of x do the two parts have the same area?”

Algebra 9A

Lesson

20A**Outline of the lesson (annotated)**

1. Find the area of the 6 by 10 rectangle (Task 1).

- Discuss the area of the rectangle. Why is it 6×10 unit squares?

Most students will know that we can find the area by multiplying 6 by 10. However, they may not be sure why. Try to tease out the idea that we normally measure area in unit squares. We want to know **how many unit squares are needed to cover the rectangle**, which can be done with 10 rows of 6 unit squares or 6 columns of 10 unit squares.

2. Find the area of a composite shape (Task 2).

- Ask students to find the area of the L-shape (second diagram).
- Discuss methods, including these 'split and calculate' methods: $2 \times 5 + 6 \times 5$, $2 \times 10 + 4 \times 5$, $6 \times 10 - 4 \times 5$.

Some students might use a counting method, eg by drawing the shape on squared paper. Others might split the shape into rectangles and calculate the area of each - focus on this approach as it is needed for the algebraic approach in Part 4.

3. Consider the family of L-shapes and the specific case (Task 3). Use a geometric approach.

- Explain and explore the dynamic nature of the third diagram.
- Ask students to find the specific L-shape that covers half the rectangle. What is the value of x for this L-shape? ($x = 7.5$)

Ask questions like

"How big/small can the area be?"

"What does the L look like when $x = 9$?"

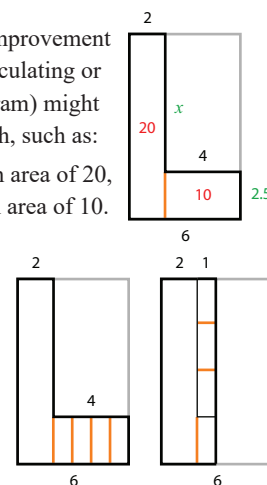
Encourage students to estimate first, ie to visualise the value of x in some way.

- Discuss students' methods.

Students might use trial and improvement by sketching L-shapes and calculating or counting. Some (see this diagram) might adopt a more analytic approach, such as:

the left hand rectangle has an area of 20, so the other needs to have an area of 10. $4 \times 2.5 = 10$, so $x = 10 - 2.5$

Others might try to adopt a 'cut and paste' approach, as here:



4. Reconsider Task 3. Use an algebraic approach.

- Find a way to express the area of the L-shape in terms of x .
Equate the expression to 30 and discuss ways of solving the equation.

There are several ways of doing this, depending on how one construes the composite L-shape. Some expressions are simpler than others:

$2x + 6(10 - x)$ or $2x + 60 - 6x$ (horizontal cut)

$20 + 4(10 - x)$ or $20 + 40 - 4x$ (vertical cut)

$60 - 4x$ (remove a piece from the surrounding rectangle)

eg, trial and improvement, cover up, matching, balancing.

5. Find equivalent expressions for the area.

- Find alternative expressions in x for the area of the L-shape.
Consider why they are equivalent.

One can link the expressions to the various rectangular shapes or one can use the laws of arithmetic to transform the expressions.

Algebra 9A

Lesson

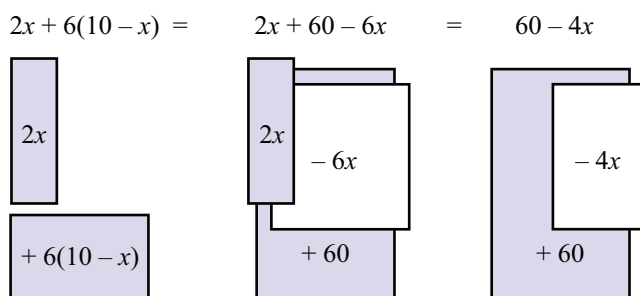
20A**Background**
Relating algebraic expressions to the geometric context

The diagram (right) shows one way of partitioning the L-shape. This leads to an algebraic expression like this for its area: $2x + 6(10 - x)$.

It can be useful (for example, to solve the equation in Step 4 of the lesson) to simplify the expression, by first expanding the brackets:

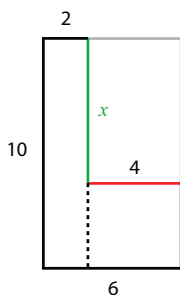
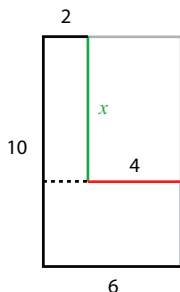
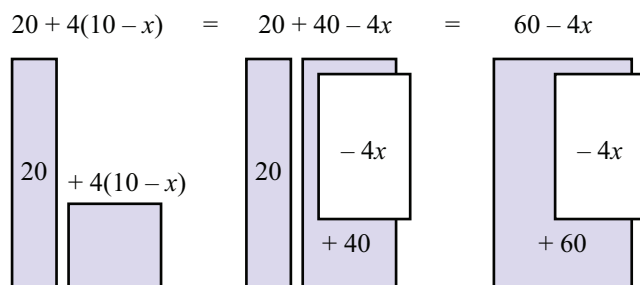
$$2x + 6(10 - x) = 2x + 60 - 6x = 40 - 4x.$$

This can be done (and justified) formally, using the rules of arithmetic (in particular the distributive and associative laws). However, it is also useful to justify the steps geometrically, in this kind of way:



The alternative partitioning (right) leads to another (equivalent) algebraic expression for the area of the L-shape: $20 + 4(10 - x)$.

Again we can simplify the expression by first expanding the brackets, and again we can relate this to the geometric context, as shown below. We again end up with the expression $60 - 4x$, which corresponds to seeing the L-shape as a rectangle with a smaller rectangle taken out.

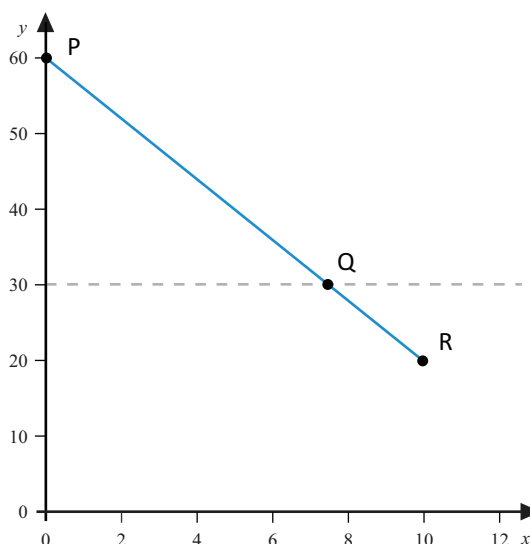
**Graphing the area**

It is interesting to graph the area of the dynamic L-shape.

Here we get a line that slopes downwards. Why?

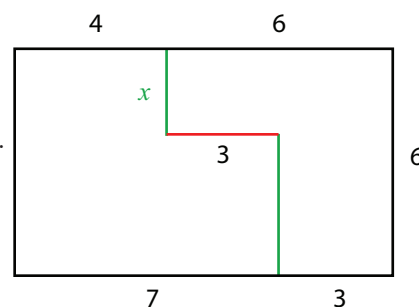
What do the points P, Q and R mean?

Can we continue the line? What could the new points mean?

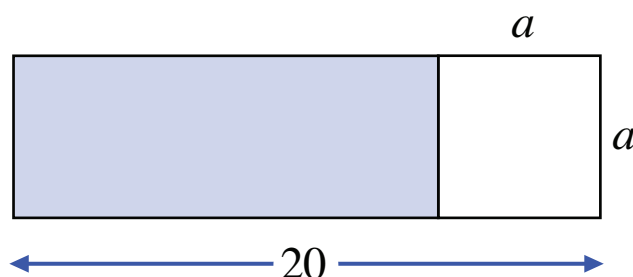
**Other dynamic scenarios**

A task similar to the original L-shapes task, but where the outer rectangle changes, is shown in the Revisits section.

Here (right) is a more complex dynamic scenario that you could use. When do the two regions have the same area?



The task below involves a rather intriguing linear relationship for the perimeter and a quadratic relationship for the area.



Think of the area/perimeter of the shaded rectangle.

What happens as the square of side a gets smaller or larger?

Is there a maximum/minimum?

Use algebraic expressions to help you.

Algebra 9B

Lesson

20B**Using expressions for ages and ages ...**

Task 1 Daisy lives with her father, mother and brother Ed.

Daisy and the others all have a birthday today and their total age is now 100 years.

Ed is 5 years older than Daisy.

It turns out that today Daisy's father has become 5 times as old as Daisy

and Daisy's mother has become 3 times as old as Ed.

How old is Daisy?

Summary

Here we start with a task set in a simple context but involving quite a lot of information. It can be solved by trial and improvement in a straightforward but fairly time-consuming way. It can be solved more efficiently by expressing the information algebraically and solving the resulting equation. Thus it is hoped the task will demonstrate the utility of an algebraic approach.

Outline of the lesson

- Students use their own methods to solve Task 1 (above).
 - Present the above task. Let students solve it (individually or in groups).
 - Confirm that Daisy's age is 8 years. Briefly discuss students' methods.

- Solve Task 1 using algebraic symbolisation.

- Choose a symbol to represent Daisy's age in years, eg d .
Ask students to write the other ages in terms of d [eg $d+5$, $5d$, $3(d+5)$].
Create an equation in d : $d + d+5 + 5d + 3(d+5) = 100$.
- Discuss ways of solving the equation, eg by trial and improvement or by simplifying and using matching, cover up, or balancing.

$$10d + 20 = 100$$

- Solve Task 1 using algebraic symbolisation again.

- Choose a symbol to represent Ed's age in years, eg e .
Write the other ages in terms of e , create an equation and solve it.
- Check that we get the same ages as before.

$$10e - 30 = 100$$

- Solve Task 2 using everyday logic and algebraic symbolisation.

- Find different ways to solve Task 2 (right).
- Why doesn't it work simply to double all the ages?

Task 2 How old will Daisy
be when the total
age is 200 years?

- Possible extensions.

- Create and solve a similar task, perhaps for a very large family.
- Find ways of solving Task 3 (right):
[eg using trial and improvement, a number line, or algebra].
- Investigate the use of a 'uniform steps' numerical approach for the original task: What happens to the total age, as Daisy's age is increased by 1? How does this relate to $10d+20=100$?

Task 3 How old was Daisy
when her father was
9 times as old as her?

Algebra 9B

Lesson

20B**Overview****Mathematical ideas**

As with Algebra Lesson 2A, the emphasis here is on the utility of symbolising. This time we focus particularly on algebraic symbols and consider how they can be used to keep track of the information in a complex task, and how, by manipulating the symbols we can ‘simplify’ the information and solve the task.

It is worth acknowledging that this task about ages is rather contrived: it is unlikely that we would know the total age of the family without already knowing their ages. It also involves a rather limited use of algebra initially, in that we are not discerning any relations, only expressing ones that we’ve been given. Moreover, some of the relations are not general, even though we treat them as such. It is perhaps best to regard the task as a puzzle rather than as a realistic problem.

Students’ mathematical experiences

The students should experience that

- open expressions can convey meaning/structure
- an open expression like $d + 5$ can be an instruction or a result (or both!)
- open expressions can be written in different but equivalent ways
- algebraic symbolisation provides a powerful tool for solving problems.

Key questions

If you knew Daisy’s age (in Task 1),

- what would you need to do to work out Ed’s age (or the father’s age or the mother’s age)?
- how would you write an expression for finding Ed’s age?
- how would you write an expression for Ed’s age?
- what different ways are there for finding the mother’s age?
 - why are these ways equivalent?

If the family’s total age is 100 years, what will it be in 1 year’s time?

Assessment and feedback

As with the *Taxi!* Mini-assessment activity, some students might have difficulties translating between algebraic symbols and words. Thus they might find it challenging to symbolise the statement “Ed is 5 years older than Daisy”. Are they thinking about the numerical relation between the ages? Or do they attempt a ‘word by word’ translation? The latter *might* produce the correct relation $e=5+d$, but could easily produce something like $e+5=d$, or even $e = 5 > d$. Ask students to check their symbolisations by substituting a *value* for Daisy’s age, d . Does the resulting age for Ed, e , fit the story, namely that he is 5 years older?

Check to see whether students who produce an open expression, such as $d+5$ (the correct expression for Ed’s age), are content to leave the expression open. Or do they feel compelled to close it, ie to make it look more like an ‘answer’, by then writing $d5$ or $5d$? Ask students to compare their expressions for Ed’s age ($d+5$) and the father’s age ($5d$).

Some students will express the mother’s age in terms of Ed’s age rather than Daisy’s, eg by writing $3e$. Can they transform this into an expression in d ?

Adapting the lesson

It is worth using generic symbolisation alongside the algebraic symbolisation. For example, if someone tries 6, say, for Daisy’s age in Task 1, you could then show that the total age would be $10 \times 6 + 20$, by treating the 6 generically:

$$\text{total age is } \underline{6} + \underline{6} + 5 + 5 \times \underline{6} + 3 \times (\underline{6} + 5) = \underline{6} + \underline{6} + 5 + 5 \times \underline{6} + 3 \times \underline{6} + 15 = 10 \times \underline{6} + 20.$$

You could then ask, “What number, instead of 6, would give us the total of 100 that we’re looking for?”

When inventing similar tasks, it can be quite challenging (but also quite fun) to ensure that the resulting ages are realistic (eg the parents are roughly the same age, and of the right age to have had the children).

Algebra 9B

Lesson

20B**Outline of the lesson (annotated)**

1. Students use their own methods to solve Task 1 (above).
 - Present the above task. Let students solve it (individually or in groups).
 - Confirm that Daisy's age is 8 years. Briefly discuss students' methods.

Some students might chance upon the answer quickly. Ask them to keep it to themselves (and look for alternative methods) so others have plenty of time to explore the task. You could simplify the task if it turns out to be *much* too difficult. The example below expresses the same ages in a simpler (and general) way:
 $d, d+5, d+32, d+31$ (and total age 100).
2. Solve Task 1 using algebraic symbolisation.
 - Choose a symbol to represent Daisy's age in years, eg d .
 Ask students to write the other ages in terms of d [eg $d+5, 5d, 3(d+5)$].
 Create an equation in d : $d + d+5 + 5d + 3(d+5) = 100$.
 - Discuss ways of solving the equation, eg by trial and improvement or by simplifying and using matching, cover up, or balancing.

Solving a problem for which we already know the answer, reinforces the message that we are interested in methods and ideas, not the answer itself.

The simplified equation $10d + 20 = 100$ can be solved by matching as follows:
 $10d + 20 = 80 + 20$, so $10d = 80$, etc.
 The cover up method involves this kind of argument:
 'something' + 20 = 100, so 'something' = 80;
 $10 \times$ 'something' = 80, so 'something' = 8.
3. Solve Task 1 using algebraic symbolisation again.
 - Choose a symbol to represent Ed's age in years, eg e . Write the other ages in terms of e , create an equation and solve it.
 - Check that we get the same ages as before.

The idea that we can switch the independent variable (from Daisy's age to Ed's) is quite subtle and you might prefer to stick with d as the independent variable and modify the task instead.
4. Solve Task 2 using everyday logic and algebraic symbolisation.
 - Find different ways to solve Task 2 (right).
 - Why doesn't it work simply to double all the ages?

Logic: Every year the total age increases by 4, so it takes another 25 years to get to 200, so Amy will be 8+25.
 Note that we can *not* simply double all the ages, though this doubles the total.
 Algebra: From Task 1, we know that the parents are 32 and 31 years older than Amy, so the general relationship is $d + d+5 + d+32 + d+31 = 200$, or $4d + 68 = 200$.
 Note that we can *not* use the relationship $10d + 20 = 200$.
5. Possible extensions.
 - Create and solve a similar task, perhaps for a very large family.
 - Find ways of solving Task 3 (right): [eg using trial and improvement, a number line, or algebra].
 - Investigate the use of a 'uniform steps' numerical approach for the original task: What happens to the total age, as Daisy's age is increased by 1? How does this relate to $10d+20=100$?

Task 3 brings out the fact that the multiplicative relationships change. It might make a good Starter.

Algebra 9B

Lesson

20B**Background****Context and 'realistic' problems**

The task used in this lesson is perhaps closer to 'real life' than 'Pyramids', but it is still not 'realistic'. The context will make sense to the students, but the task should be regarded primarily as a puzzle, and it is worth making a virtue of this: on one hand, it is quite a nice challenge to find the answer; on the other hand, our prime interest is in developing an understanding of an algebraic approach and an appreciation of its utility.

Specific unknown and variable

Initially, students are likely to solve Task 1 by trial and improvement, ie by trying different values for Daisy's age. It is thus tempting to regard Daisy's age as a variable. Note, however, that some of the relationships concerning Daisy's age are only true for one specific value. Thus we are really only pretending that her age can vary in this Task. What is really being varied is the age of *different* Daisys, from a host of families of 4 that happen to have children called Daisy and Ed and where the same three age relationships hold at this moment in time but where the total age is not necessarily 100. [Note that some of these families may not be biologically possible, or fit social norms.]

Exploring the problem numerically

It can be illuminating to solve Task 1 using an 'incremental' numerical approach, by finding the ages of the family members for a Daisy aged 5, say, and then seeing what happens for a Daisy that is 1 year older. It turns out (and it can be argued logically) that the age of the Eds also increases by 1, and that the age of the fathers and mothers increase by 5 and 3 respectively (as in the table below). Thus the total age increases by 10, which it will do for every increase of 1 in the Daisy age. It is thus easy to see that we need to get to a Daisy aged 8 for the total to be 100. This may also lead students to see that the expression in d for the total age will be of the form $10d + ? = 100$.

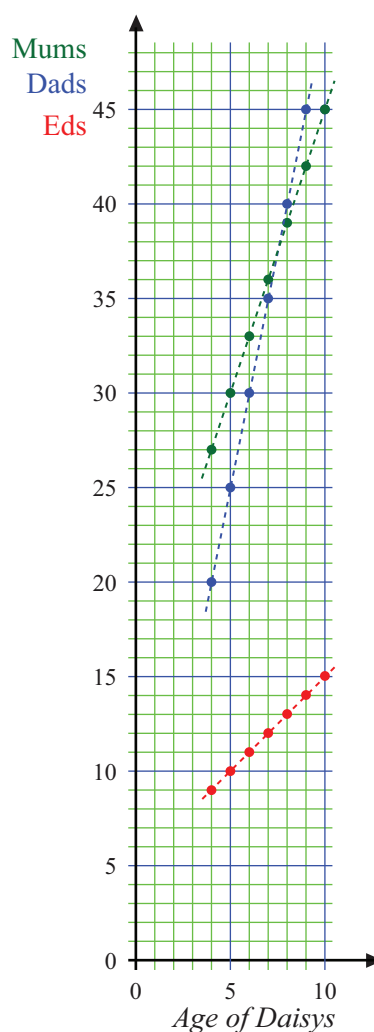
It is also interesting to consider the parents' ages. It can be seen from the table that for a Daisy aged 5, the father is 5 years younger than the mother. However, the gap is narrower for a Daisy aged 6, and it turns out that for a Daisy aged 8, the father is older than the mother. What's happening here, and why?

Daisy	Ed	Dad	Mum	Total
5	10	25	30	70
+1				+10
6	11	30	33	80
+?				+?
?				100

Exploring the problem graphically

It can also be illuminating to graph these and other numerical values against the ages of the Daisys - or to represent them on a dynamic number line (using a software package such as GeoGebra).

- Is there a family where the parents have the same age?
- What happens to the ages and the three lines when $d = 0$, say? What does this mean in terms of the story?!



Lesson

20B

Notes

Algebra 10AB

Mini-assessment

21AB**Brackets**

Compare these two expressions.
Which is larger?

$$7(u + 2) \quad \text{and} \quad 2(u + 7)$$

Commentary

The aim of this Mini-assessment is to see how students compare algebraic expressions with brackets.

Do students focus on multiplication ('multiplication makes bigger'; ' $\times 7$ has a greater effect than $\times 2$ ') ?

Do they evaluate the expressions for a specific value (or several specific values) of u ?

Do they transform the expressions by expanding the brackets?

Do they suggest using graphs or other representations?

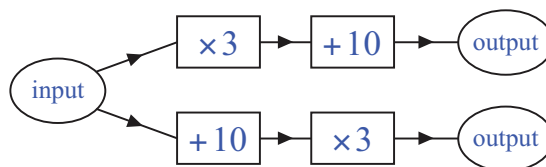
Algebra 10A

Lesson

21A**Think of two operations**

In this pair of function machines, the operations $\times 3$ and $+10$ have been swapped round.

The difference in the outputs is always 20.



Find another pair of function machines

- involving a multiplication and an addition, and
- where the operations are swapped round, and
- where the difference in the outputs is always 20.

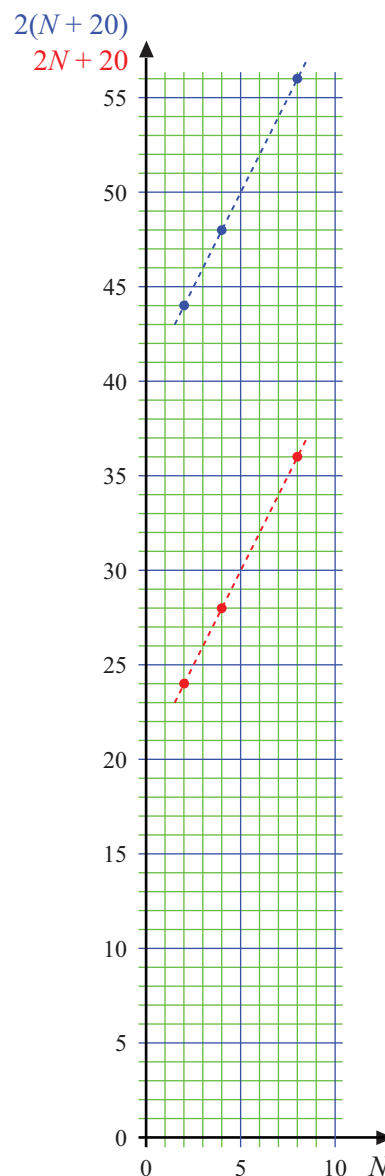
Summary

In this lesson, we again examine the effect of changing the order of two operations (a multiplication and an addition). We look for pairs of function machines where the difference in the outputs is always 20, as with one of the pairs from Algebra Lesson 4B.

The aim is to consolidate ideas first met in Algebra Lessons 4A and 4B and re-visited in other lessons. To further this, we again make use of open and algebraic expressions (some with brackets) and Cartesian graphs.

Outline of the lesson

1. Remind the class of the function machines $\times 3$, $+10$ and $+10$, $\times 3$ from Lesson 4B and the constant difference of 20 in the outputs.
2. Ask the class to find another pair of function machines $\times A$, $+B$ and $+B$, $\times A$ where the difference in the outputs is always 20.
 - Let students work on the task in small groups.
 - List students' results.
3. Choose a correct pair of functions, eg $\times 2$, $+20$ and $+20$, $\times 2$.
Why is the difference in the outputs always 20?
 - Ask the class for reasons.
 - Record using open expressions or diagrams.
4. Use a Cartesian graph to represent inputs/outputs for the chosen pair of function machines.
 - Discuss features of the graph. "Where is the constant difference?"
 - Discuss what the graphs of other correct pairs of functions from Part 2 would look like.
5. Explain why order matters.
 - Ask the students in pairs to explain to a younger student why the order of operations matters, using pictures or stories.
 - Discuss their ideas. Which are the best explanations? Why?



Algebra 10A

Lesson

21A**Overview****Mathematical ideas**

The lesson provides more experience of using 'open' expressions and Cartesian graphs to examine the distributive law. The constant difference between the function machines $+10, \times 3$ and $\times 3, +10$ can be expressed algebraically as follows: $(n+10)\times 3 - (n\times 3+10) = 10\times(3-1)$. Similarly, the difference between the function machines $+20, \times 2$ and $\times 2, +20$ is $20\times(2-1)$. More generally, the difference between $+a, \times b$ and $\times b, +a$ is $ab - a$ or $a(b - 1)$.

Students' mathematical experiences

Students should realise that

- when multiplication and addition are combined, the order matters
- multiplication is distributive over addition.

Students may discover

- ideas can be expressed in different ways: in words, using 'open' expressions, symbolic algebra, diagrams and Cartesian graphs.

Students may express their ideas in different ways, including

- using diagrams and graphs
- using words
- using 'open' expressions.

Assessment and feedback

This lesson has much in common with earlier lessons on 'understanding expressions', including work on comparing expressions, forming expressions and manipulating expressions. In particular, it picks up ideas from Algebra Lessons 4A (*Order matters*) and 4B (*Think of a number*).

You might want to find out what students can remember from these lessons - spontaneously and with prompts as necessary from you.

It is worth remembering that even at this stage, different students will engage with these ideas differently. Some may still want to focus on the results of calculating; others may be making some sense of the structure of the situation, and be able to talk about this in general terms, by means of diagrams and graphs, numbers used generically or algebraic symbolisation. Try to draw out all these approaches, and use each to illuminate the others.

Key questions

How do you know that ...?

Can you use 'open expressions' to show/explain the constant difference rule? Can you use 'algebra'?

Adapting the lesson

You might extend the lesson by choosing a function $\times A, +B$ and asking students to find another function, $+C, \times A$, which always has the *same* output.

For example, the equivalent of $\times 2, +20$ is $+10, \times 2$ [ie the equivalent of $2n + 20$ is $2(n + 10)$].

Similarly, the equivalent of $\times 5, +5$ is $+1, \times 5$ [ie the equivalent of $5n + 5$ is $5(n + 1)$].

The equivalent expressions can be found by trial and improvement, or by manipulating formal expressions, or by something inbetween (eg through insight into the effect of adding then multiplying).

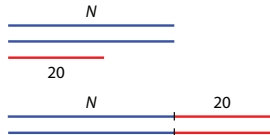
Algebra 10A

Lesson

21A**Outline of the lesson (annotated)**

1. Remind the class of the function machines $\times 3$, $+10$ and $+10$, $\times 3$ from Lesson 4B and the constant difference of 20 in the outputs.
 - Ask students about what they found in Algebra Lesson 4B, but don't go into the reasons again at this stage.

2. Ask the class to find another pair of function machines $\times A$, $+B$ and $+B$, $\times A$ where the difference in the outputs is always 20.
 - Let students work on the task in small groups.
 - List students' results.
 - There are 6 possible whole-number values for A and B :
 2 and 20 (giving $\times 2$, $+20$ and $+20$, $\times 2$)
 3 and 10
 5 and 5
 6 and 4
 11 and 2
 21 and 1.
 There are, of course, infinitely many fractional values (eg 1.5 and 40).
 Students might spot a relationship between A and B [which can be expressed formally as $(A - 1)B = 20$], but don't focus on this here.

3. Choose a correct pair of functions, eg $\times 2$, $+20$ and $+20$, $\times 2$.
 Why is the difference in the outputs always 20?
 - Ask the class for reasons.
 - Record using open expressions or diagrams.
 - Choose a pair where the multiplier is relatively small ($\times 2$ or $\times 5$), so that the gradient of the subsequent graphs is not too steep.
 - For example, consider $\times 2$, $+20$ and $+20$, $\times 2$:
 for an input of 9999, say, we can write $9999 \times 2 + 20$ and $(9999 + 20) \times 2$;
 for an input of N , say, we can write $2N + 20$ and $2(N + 20)$,
 or we can draw a pair of diagrams like these:
 

4. Use a Cartesian graph to represent inputs/outputs for the chosen pair of function machines.
 - Discuss features of the graph.
 “Where is the constant difference?”
 - Discuss what the graphs of other correct pairs of functions from Part 2 would look like.
 - For example,
 - the points lie on straight lines
 - the lines are parallel
 - the lines have a slope of A
 - the vertical distance between the lines is B .
 Discuss reasons for some of these observations.
 What happens when the input is large, or 0, or negative?
 Ask students to *sketch* these, and to compare them with a neighbour's (the sketches may be far from perfect).

5. Explain why order matters.
 - Ask the students in pairs to explain to a younger student why the order of operations matters, using pictures or stories.
 - Discuss their ideas. Which are the best explanations? Why?
 - If students find this difficult, ask “Can you alter one of the explanations on the board?”

Algebra 10A

Lesson

21A

Background

Algebraic structure and the laws of arithmetic

In this lesson we again examine how multiplication and addition interact. Order matters for this pair because multiplication is distributive over addition:

$$(100+10) \times 3 = 100 \times 3 + 10 \times 3 \text{ rather than } 100 \times 3 + 10.$$

Similarly, $(x + b) \times a$ is equivalent to $ax + ab$, not $ax + b$.

Explanation and proof, in words and symbols

In this lesson we again compare outputs of the form $ax + b$ and $(x + b)a$.

Their difference is $b(a - 1)$, which can of course be proved algebraically:

$$(x + b)a - (ax + b) = ax + ab - ax - b = ab - b = b(a - 1).$$

We move tentatively *towards* such a proof in this lesson by encouraging students to make general arguments supported by the use of open expressions or diagrams (see page 260).

However, we should also be receptive to explanations expressed in words, and to explanations that may still be imprecise or incomplete. Our explanations are rarely as good as we would like them to be at a first attempt, and, with regards to this lesson, having some feel or insight into the effect of changing the order of operations is more important than coming up with a formal explanation.

Order, the commutative law and BODMAS

It is likely that some students will have difficulties with order when recording and reading the expressions at the heart of these function machines. The standard way of writing the function machine $+4, \times 6$, say, is $6(n+4)$, rather than $(n+4)6$. Note that a naïve, left to right, reading of $6(n+4)$ might be “multiply by 6 and add 4”, which actually refers to the different function machine $\times 6, +4$, or $6n+4$. Note, too, that we could also write $6n+4$ as $4+6n$, or even as $4+n \times 6$ or $4+n6$.

These alternatives arise from the fact that addition and multiplication are commutative. It is worth periodically taking an open/algebraic expression and considering other legitimate formulations.

Some students may mention BODMAS or BIDMAS. These mnemonics are very particular to the UK but don't always work because they are oversimplifications. Thus while BODMAS might correctly tell us that $2+3 \times 4$ means $2+12$, not 5×4 , it might lead us to interpret $6 \times 8 \div 4$ as 6×2 rather than the conventional left-to-right interpretation of $48 \div 4$. BODMAS in arithmetic does not appear to help students' understanding of algebra because it encourages students to see the ‘open’ expressions commonplace in algebra as incomplete and still needing to be ‘done’.

Variables and parameters

It is worth being aware that we are working with two kinds of variable here, or rather with *variables* and *parameters*.

Thus, when we refer to a function $x \rightarrow Ax + B$, we are thinking of x as a variable and A and B as parameters. It is unlikely that you will want to pursue this distinction with your students, though you might want to touch on it with some of them.

Same and different

In this lesson we build on Algebra Lesson 4B by changing the functions but keeping the difference in output the same (in this case, a difference of 20). By encouraging students to notice what stays the same and what is different, it is hoped that this kind of ‘controlled’ variation in the tasks will help students see structure. A further variation is introduced in that lesson's peer assessment task (Task P), where the pairs of functions produce a difference in output of 21.

Yet another variation occurs in the suggested extension task involving equivalent expressions (see page 259).

Students use and interpret Cartesian graphs

The graphs for these functions are not particularly easy to draw accurately, because the outputs tend to be large and so the scale used for the y axis has to be large. Thus you might want to focus more on *sketching* graphs than drawing them accurately. A virtue of sketching is that rather than simply ‘observing’ a carefully drawn graph, students have to argue about what the graphs *should* look like.

Algebra 10B

Lesson

21B**Solving equations**

For what value of u is this equation true?

Think of different ways of finding the answer.

$$4u + 10 = 38$$

Summary

In this lesson, we look at different methods for solving linear equations: trial and improvement, cover up (or 'inspection'), matching, transforming (eg by 'balancing'), function machines, and number lines (or block diagrams).

We consider how well-suited the various methods are for different equations.

Outline of the lesson

1. Solve the equation in u (above) in different ways.

- Show the equation in u . Ask students to solve it, using any method they like. Discuss two methods that students have used.
- Ask students to find other methods for solving the equation. Discuss their methods. Try to elicit these methods: _____

- trial and improvement
- cover up (or 'inspection')
- matching
- transforming (eg by 'balancing')
- function machines, and
- number lines (or rod diagrams).

2. Modify the equation in u (above). Is it easier or harder?

- Change the equation slightly, to this:
Ask: Does this make any of the methods easier or harder?
Which method does it affect the most?
- Discuss other small changes to the equation.
How do they affect each of the methods?

$$4u + 10 = 39$$

3. Solve this equation in v . Which methods work best?

- Show the equation in v . Ask students to solve it, using any method they like. Discuss the methods they used - which were most popular?
- Discuss all the methods. Which ones work best for this equation?

$$\frac{14}{v + 3} = 2$$

4. Solve this equation in w . Which methods work best?

- Repeat Step 3 for this equation:

$$6w + 10 = 2w + 30$$

5. Compare the equations in u , v and w .

- Show the original three equations in u , v and w .
Which methods work best for which equations? Why?

Algebra 10B

Lesson

21B**Overview****Mathematical ideas**

Here we look at three equations that are markedly different in form and which can be solved in a variety of ways. The first two equations can be solved efficiently using informal methods of some sort; the third equation shows the power of using a formal method.

We look at some of the features that make an equation 'easy' or 'difficult', or suited or not suited to a particular method.

Students' mathematical experiences

Students should discover that

- equations can be solved in different ways
- some methods work better for some equations than for others
- equations have distinct features which may make them more amenable to a particular method of solution.

Key questions

Can you find another way to solve this equation?

Which methods work well for this equation?

Assessment and feedback

Do students tend to favour a particular method?

Are they able to use different methods?

Are they unfamiliar with some methods?

Which methods do they tend to find difficult?

You might want to spend extra time on methods that are unfamiliar to the class.

Ask:

When have we solved equations before?

What methods have we used?

For example, these Algebra lessons:

Lesson 1A: $5a = 10 + a$

Lesson 1B: $3n = n + 3$

Lesson 2A: $4(a+5) = 100$

Lesson 6B: $4n + 20 = 50$

Lesson 7A: $5x - 2 = 2(2x + 7)$

Lesson 7B: $3x + 50 = 5x + 10$

Lesson 8B: $[3(e+11)-7] \div 2 = 25$

Lesson 9A: $60 - 4x = 30$, $2x + 6(10 - x) = 30$

Lesson 9B: $10d + 20 = 100$, $10e - 30 = 100$.

Adapting the lesson

You could ask students to make up their own equations that fit the three forms used in this lesson. They could try to invent 'easy' or 'hard' equations, (or 'easy' or 'hard' for a nominated method of solution). Or they could be asked to invent equations of a given form, with a given solution, eg 100, or -5. They could then give their equations to a partner to solve.

Algebra 10B

Lesson

21B**Outline of the lesson (annotated)**1. Solve the equation in u (above) in different ways.

- Show the equation in u .

Ask students to solve it, using any method they like.

Discuss two methods that students have used.

- Ask students to find other methods for solving the equation.

Discuss their methods. Try to elicit these methods:

[trial and improvement, cover up, matching, transforming, function machines, number lines]

—— Just discuss two methods at this stage - to make clear that there is more than one way.

—— All six methods work quite well for the given equation, though students might not be familiar with the use of *number lines* here. Try to bring out all the methods.

2. Modify the equation in u (above). Is it easier or harder?

- Change the equation slightly, to this: $[4u+10=39]$

Ask: Does this make any of the methods easier or harder?

Which method does it affect the most?

- Discuss other small changes to the equation.

How do they affect each of the methods?

—— *Trial and improvement* is likely to become more difficult as the solution is no longer a whole number. The same applies to *cover up* (here the numbers may induce a shift from “4 times ‘something’ = 39” to “ $39 \div 4 = \text{‘something’}$ ”, which is closer to *transforming* or the use of *function machines*).

3. Solve this equation in v . Which methods work best?

- Show the equation in v . Ask students to solve it, using any method they like. Discuss the methods they used - which were most popular?

- Discuss all the methods. Which ones work best for this equation?

—— This equation looks to be quite complex. However all the methods work quite well here, especially *cover up*. Thus students do not need to know a formal method (*transforming*) to solve the equation.

4. Solve this equation in w . Which methods work best?

- Repeat Step 3 for this equation: $[6w+10=2w+30]$

—— Some students might think that w can have different values in the left hand and right hand expressions. *Cover up* and *function machines* don't really work here, because the unknown, w , appears on both sides. *Trial and improvement* is difficult here for the same reason: as a trial value for w affects both sides of the equation it is difficult to determine whether the value is too small or too big.

5. Compare the equations in u , v and w .

- Show the original three equations in u , v and w . Which methods work best for which equations? Why?

—— You might want to explore what happens when the equations in v and w are modified slightly, as in Step 2.

Algebra 10B

Lesson

21B**Background****Using different methods**

It is empowering for students to realise that equations can often be solved using 'common sense' methods like *cover up*, which are easier to understand than formal methods. By the same token, it is useful for us to realise that students often don't need a formal method to solve an equation - and that they are therefore more likely to see the point of learning to use formal methods when they are given equations that require them!

The equation in u : $4u + 10 = 38$

Students might use one (or a mixture) of these:

trial and improvement

cover up (aka inspection)

matching

transforming (eg by 'balancing')

function machines (or flow diagrams)

number lines (or 'rods', say).

Trial and improvement is **quite efficient here, because the solution is a small whole number ($u = 7$).**

The cover up method is quite straightforward here:

'something' + 10 = 38, so 'something' = 28;

$4 \times$ 'something' = 28, so 'something' = 7.

Matching **could go something like this:**

$$4u + 10 = 28 + 10$$

$$\text{so } 4u = 28$$

$$[\text{so } u + u + u + u = 7 + 7 + 7 + 7]$$

$$\text{so } u = 7.$$

Transforming **might go like this:**

$$\text{Take 10 from both sides: } 4u = 28$$

$$\text{Divide both sides by 4: } u = 7.$$

Some students might first divide by 2 or even by 4:

$$\text{Divide both sides by 4: } u + 2.5 = 9.5$$

$$\text{Subtract 2.5 from both sides: } u = 7.$$

The equation could be represented by a function machine:

input u , $\times 4$, +10, gives output 38;

reverse the process: input 38, -10 , $\div 4$, gives output 7.

One could represent the equation using number lines or rods say, and then 'discern' the value of u :

u	u	u	u	10
38				

The equation in v :

Again, students might use one (or a mixture) of the six methods.

Trial and improvement is **relatively**

efficient here, as the solution is a small whole number ($v = 4$).

However, students might be deterred by the fact that trials are likely to produce fractional values for the left hand side.

The cover up method is quite straightforward here:

$$14 \div \text{'something'} = 2, \text{ so 'something'} = 7;$$

$$\text{'something'} + 3 = 7, \text{ so 'something'} = 4.$$

Matching **is rather obscure here**

but can be done by writing 2 as $14/7$, like this:

$$\text{so } v + 3 = 7, \text{ and so } v = 4.$$

$$\frac{14}{v+3} = 2$$

$$\frac{14}{v+3} = \frac{14}{7}$$

Transforming **is quite challenging here but could go like this:**

$$\text{Multiply both sides by } v+3: 14 = 2v + 6$$

$$\text{Subtract 6 from both sides: } 8 = 2v$$

$$\text{Divide both sides by 2: } 4 = v.$$

It is not so likely that students would use function machines here, as it would involve the *self-inverse* function 'divide into 14':

input v , +3, divide into 14, gives output 2;

input 2, divide into 14, -3 , gives output 4.

The use of rods is also rather obscure here, but it can be done, eg like this:

v	3	v	3
14			

The equation $6w + 10 = 2w + 30$ in w :

Students might use one (or a mixture) of these:

trial and improvement

cover up (aka inspection)

matching

transforming (eg by 'balancing')

function machines (or flow diagrams)

number lines (or 'rods', say).

Trial and improvement **is quite difficult here, as the values of both sides of the equation change so that it is not immediately obvious how to improve the trial.**

The cover up method doesn't work here.

Matching **is fairly straightforward here:**

$$6w + 10 = 2w + 30$$

$$\text{so } 4w + 2w + 10 = 2w + 10 + 20$$

$$\text{so } 4w = 20$$

$$\text{so } w = 5.$$

Transforming **comes into its own here:**

$$\text{Subtract } 2w \text{ from both sides: } 4w + 10 = 30$$

$$\text{Subtract 10 from both sides: } 4w = 20$$

$$\text{Divide both sides by 4: } w = 5.$$

The use of function machines doesn't work here.

(One could, though, treat both sides of the equation as a function and tabulate or graph their values for different values of w , to see where they coincide. It would also be interesting to explore the other two equations graphically.)

The use of rods is straightforward here (and is closely related to matching and to transforming):

w	w	w	w	w	w	10
w	w	30				

Lesson

21B

Notes

Algebra 11AB

Mini-assessment

22AB**Dripping tap**

The bottle is 18 cm tall.

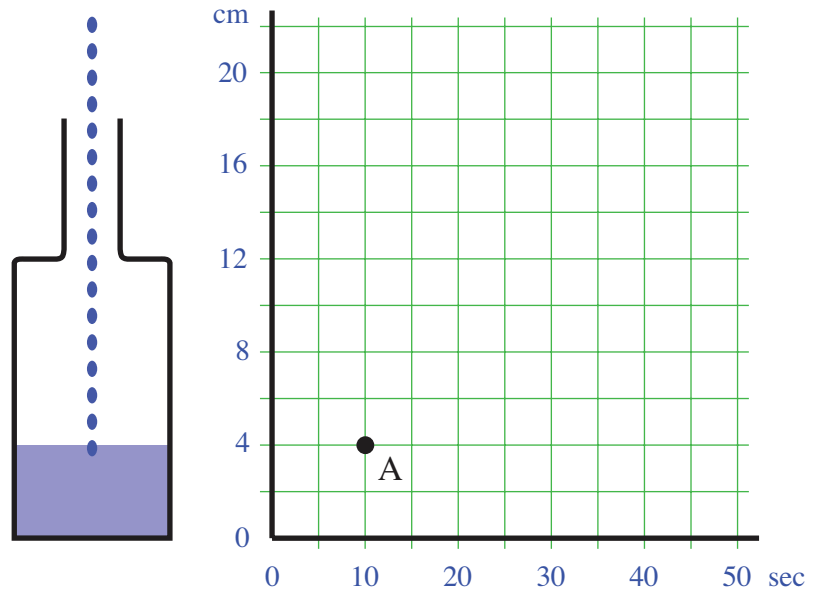
It is being filled from a dripping tap.

Look at Point A on the graph.

How long has it taken for the water to reach a height of 4 cm?

How does the height of the water change with time?

What will the graph look like for the height of the water at different times?

**Commentary**

The aim of this Mini-assessment is to see whether students can visualise a dynamic situation and represent it on a Cartesian graph.

Do students see that the height of the water rises steadily, and then more quickly at the neck of the bottle?

Do students have a qualitative feel for the graph [eg that it is a straight line followed by a steeper straight line, followed by a horizontal line]?

Do students consider the graph analytically [eg by plotting points like (20, 8) and being aware of 'critical' points like (0, 0) and (30, 12)]?

Algebra 11A

Lesson

22A**Drinking trough**

A drinking trough is filled with water from a tap.

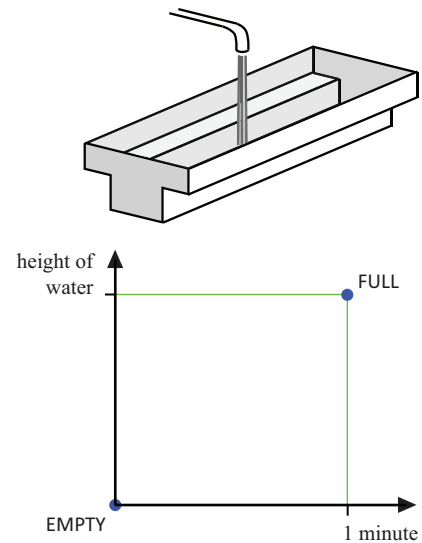
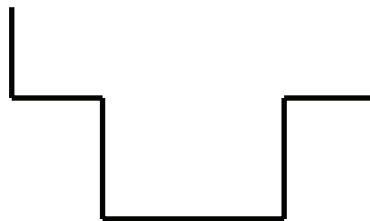
When the tap is full on, it takes 1 minute to fill the trough.

The diagram below shows the cross-section of the trough.

It also shows a graph for the height of water in the trough at different times after the tap has been turned full on.

The two points shown on the graph are for when the tap has just been turned full on (and the trough is still empty) and for when the trough gets full.

Sketch some points in between.

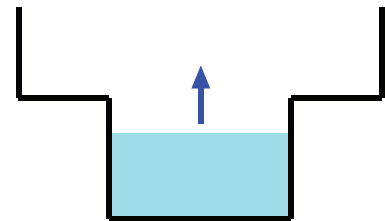
**Summary**

In this lesson we draw a graph to model a situation from 'real life', where the relationship (between time and height) is not of the familiar, linear kind. We start by considering qualitative features of the relationship (eg, "Does the height increase steadily?"). We then use a more analytic approach to test and modify our ideas.

Outline of the lesson

1. Discuss the story and think about the water level.

- Check that students understand the story.
- Focus on the cross-section and ask students to think about the height of the water. *Give them plenty of time.*
"The trough is filling at a constant rate: how is the water level changing?"

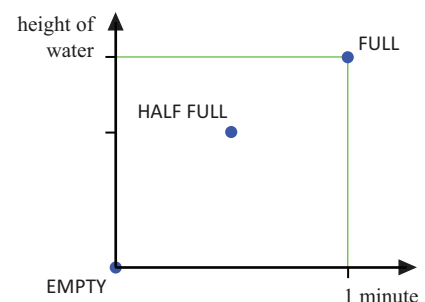


2. Sketch the graph of the changing height of the water level.

- Check that students understand the two given points on the graph.
- Ask students to discuss the shape of the whole graph in groups.
- Put some of the students' sketches on the board. Discuss the sketches, in broad, qualitative terms. *Again, take plenty of time.*

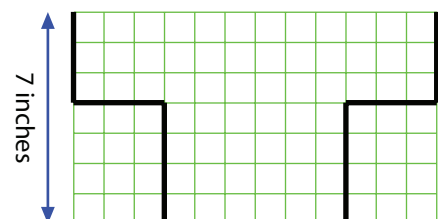
3. When is the trough half full?

- Take a more analytical approach, by considering the moment when the trough is half full: "How long does it take? How high is the water level at this moment? Is it halfway up, or less or more?"
- Plot the approximate position of the corresponding point. Discuss what the graph might look like between this point and the 'full' and 'empty' points.



4. Plot some more specific points.

- Present this scale drawing of the cross-section. Check that the area of the cross-section is 60 sq inches, and remind the class that the trough takes 1 minute (ie 60 seconds!) to fill.
- How long does it take for the water level to be 1 inch high, 2 inches high, etc? Plot points. Draw the graph.



Algebra 11A

Lesson

22A**Overview****Mathematical ideas**

Here we start with a 'real life' situation: filling a trough with water from a constant-flowing tap. We then model this with a Cartesian graph of water height against time.

We have several aims here: to show that we can represent (some aspect of) a real life situation by means of a graph, using both a qualitative and a more analytic approach; that the visual features of a graph can convey a lot of information; but that a graph does not necessarily provide a concrete (or 'photographic') portrayal of the situation it represents and that we need to be aware of this.

Students' mathematical experiences

Students should experience

- that we can represent real life situations on a graph
- that we can convey useful information with a sketch
- that the graph is not a direct portrayal of the situation.

Assessment and feedback

Give students the two height-time graphs about a lift (see page 271).

One is right, one is partly wrong.

Which graph is right?

What does it tell us about the lift?

Look at the textbook activities on real life graphs and ask students to suggest ways of improving them.

Key questions

When does the water level rise more quickly? What does this mean for the graph?

When the trough is half-full, why is the time at the half way mark but not the height?

What would it mean if the graph looked like the (right hand portion of the) cross-section (ie vertical line, then horizontal, then vertical)?

What would the graph look like if the tap was only turned half full-on?

What would the graph look like if the tap was turned to full gradually (in the course of 5 seconds, say)?

Adapting the lesson

You might ask students to sketch graphs for various troughs or bottles. Note that (strictly speaking) the situation with bottles is more complex as the horizontal cross-section is normally circular. [The examples, right, are from the KS3 Framework document, 2001, p173.]

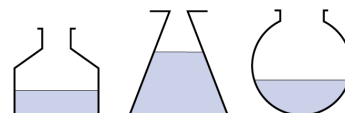
Or you could ask students to consider a trough with the same height and capacity as our original trough but with a rectangular cross section. What would its graph look like? [A line through the original two points.]

How wide is the trough? [$60 \text{ sq inches} \div 7 \text{ inches}$.]

How could we discern the width from the graph? [From the time taken for the height to rise 1 inch!]

There are two related GeoGebra files for the original task: trough-filling.ggb and trough-graph.ggb.

Sketch a graph of the depth of water against time when water drips steadily from a tap into these bottles.



Sketch graphs for other shapes of bottle. Predict the bottle shape from the shape of a graph.

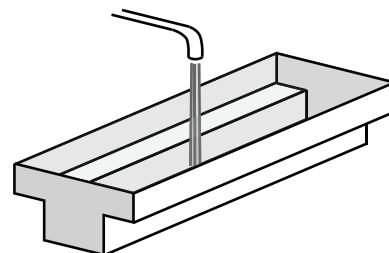
Algebra 11A

Lesson

22A**Outline of the lesson (annotated)**

1. Discuss the story and think about the water level.

- Check that students understand the story.
- Focus on the cross-section and ask students to think about the height of the water.
Give them plenty of time.
“The trough is filling at a constant rate: how is the water level changing?”



2. Sketch the graph of the changing height of the water level.

- Check that students understand the two given points on the graph.
- Ask students to discuss the shape of the whole graph in groups.
- Put some of the students' sketches on the board. Discuss the sketches, in broad, qualitative terms.
Again, take plenty of time.

— Is it a straight line? Is it curved?
Is it several straight line segments?
Does it look like the cross-section?

3. When is the trough half full?

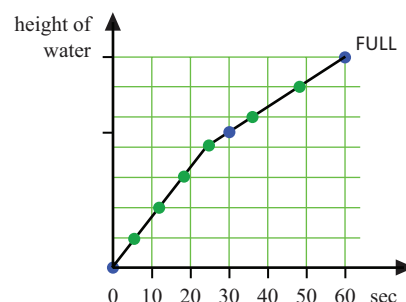
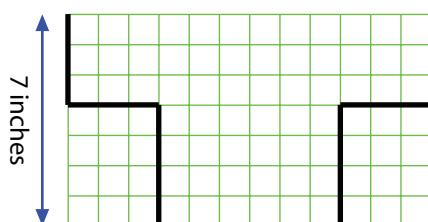
- Take a more analytical approach, by considering the moment when the trough is half full: “How long does it take? How high is the water level at this moment? Is it halfway up, or less or more?”
- Plot the approximate position of the corresponding point.
Discuss what the graph might look like between this point and the ‘full’ and ‘empty’ points.

— Do students realise that it is half full after 30 seconds?
When the level is halfway up, why is the trough less than half full?
Why is the water level more than half way up at this moment?
Some students might surmise that the trough is half full at the place where it widens. Say that later in the lesson we will use a scale drawing to see whether this is the case (it is not).

4. Plot some more specific points.

- Present this scale drawing of the cross-section. Check that the area of the cross-section is 60 sq inches, and remind the class that the trough takes 1 minute (ie 60 seconds!) to fill.
- How long does it take for the water level to be 1 inch high, 2 inches high, etc?
Plot points. Draw the graph.

— At what height is it half full? [4.5 inches]
Where/when is there a change in the speed at which the water level rises? [4 inches, 24 sec.]



Algebra 11A

Lesson

22A**Background****Using 'real life' situations**

This lesson is about graphing, but we start with a 'real life' situation concerning a drinking trough being filled with water. The virtue of such a situation is that it should be perfectly accessible to the students: even if they have never seen a drinking trough, they should be able roughly to visualise what is going on (eg, that the water level rises relatively quickly at first and then more slowly), based on experiences like filling a bath, a bucket or a bottle with water. Also, the situation is engaging, made more so by the different widths of the trough.

However, our prime interest is obviously not in water troughs. Nor are we claiming that in real life we would specifically want to graph the water level of a trough. Rather, our interest is in understanding graphs and, in particular, in showing the power of graphs to represent aspects of a 'real' situation - while acknowledging that this, and probably any other classroom-based graphing activity, is rather contrived.

Graphing 'real life' situations

In this graphing activity, there are clear correspondences between the shape of the trough and the resulting graph. However, the graph is more abstract (despite the fact that we are representing height vertically, as in real life): the graph is not a direct representation of the shape of the trough.

Thus it is important for students to learn that a graph does not provide a 'photographic' representation of a situation.

Pioneering work on students' understanding of 'real life' graphs was undertaken by Daphne Kerslake, and by Claude Janvier with Alan Bell. This led to the development of some rich classroom materials, especially by Malcolm Swan and colleagues at the Shell Centre, Nottingham.

Kerslake, D. 1978. The understanding of graphs. *Mathematics in School*, Vol. 6, No. 2, pp. 22-25.

Bell, A and Janvier, C. 1981. The interpretation of graphs representing situations. *For the Learning of Mathematics*. 2(1), 34-42.

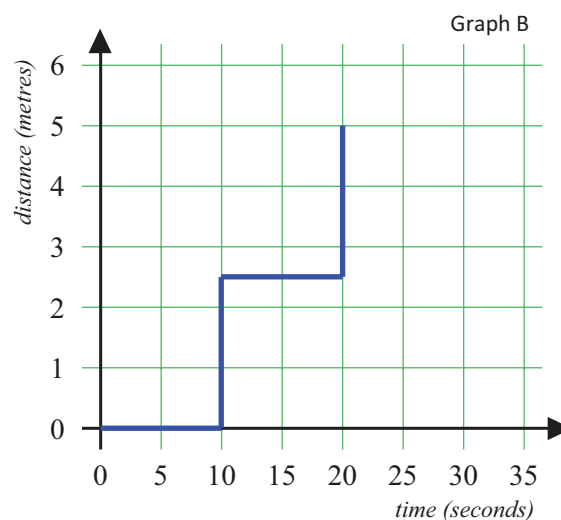
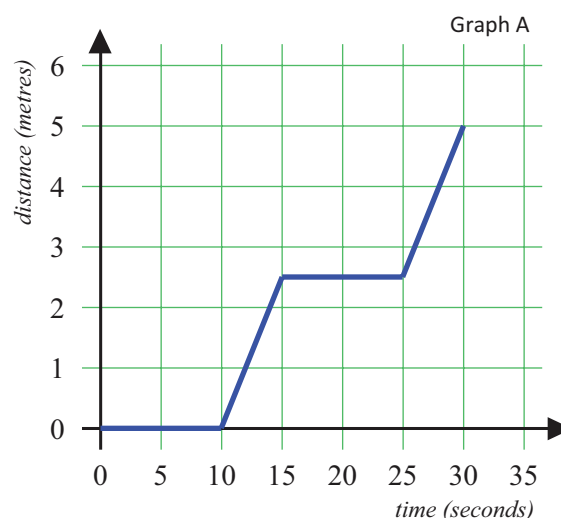
Swan, M. (and the Shell Centre team) 1985. *The language of functions and graphs*. Shell Centre & Joint Matriculation Board.

Interpreting 'real life' graphs

The graphs below are for the distance travelled by a lift, from the time it reaches the ground floor to the time it reaches the second floor of a building. Both graphs are oversimplifications, but the second graph is clearly not correct.

You might want to ask students

- to identify the incorrect graph [Graph B]
- to describe the information about the lift's journey conveyed by the other graph [eg, each floor is about 2.5 m high; the lift waits for 10 seconds; it travels at about 0.5 m per second or 50 cm per sec, or about 1.1 mph]
- to consider how the representation might be oversimplified [eg, the start and stop are both shown as instantaneous rather than gradual].



Algebra 11B

Lesson

22B**Swimming club membership**

The Minnows swimming club and the Dolphins swimming club are mainly for school children.

Their membership fees depend on your age.

The number of pounds (£) you pay to join is given by these expressions, where a is your age (in years).

Which club is trying to attract younger children and which is trying to attract older children?

$$\text{MINNOWS } 6a - 30$$

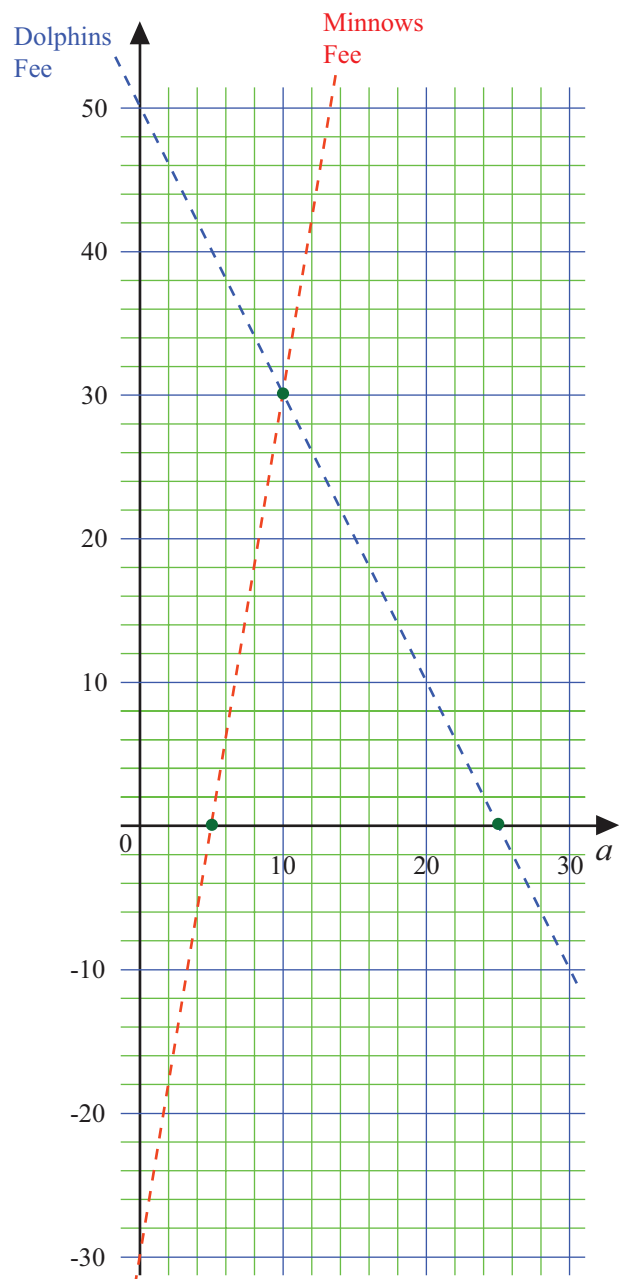
$$\text{DOLPHINS } 50 - 2a$$

Summary

Here we go full circle, by having a lesson like *Boat hire*, where we compare two linear relationships, albeit involving more complex expressions this time. As with *Boat hire*, tables of values are used initially, but the main focus is on representing the relationships as points on a Cartesian graph and on interpreting the result.

Outline of the lesson

1. Display the Membership problem and ask students for their immediate responses.
 - Try to draw out some qualitative responses.
 - Ask students to explore this further in small groups.
 - Collect data on the membership fees for various ages.
 - Represent the data
 - 'randomly' on the board
 - in (randomly ordered) tables
 - in ordered tables.
 - Discuss anomalies: eg, the value of the expressions is zero or negative for some ages.
 - Ask, "Are the fees for the two clubs ever the same?"
2. Ask students to represent the data as points on a Cartesian graph.
3. Discuss the sets of points, and key individual points on the graph.
 - Can we draw a line through each set of points? Should it be solid or dotted, or should it be stepped?
 - Relate points to the tables of values, to the original algebraic expressions and to the original story.
 - Discuss how we might modify the rules for the membership fees (eg, what do we do with people of 5 years and younger, and 25 years and older?).



Algebra 11B

Lesson

22B**Overview****Mathematical ideas**

As with *Boat hire*, and other lessons, this lesson gives students the opportunity to compare algebraic expressions as the value of the variable changes. The expressions are more complex than in *Boat hire*, and one of them decreases as the variable increases.

The focus is on drawing and interpreting a Cartesian graph, which is a particularly powerful way of representing expressions, since it allows us to represent a range of values simultaneously. The graph has several salient features which can be related back to the story, the given algebraic expressions and to students' tables of values. These features include the point of intersection of the lines, the points where the lines cut the x -axis and the gradients of the lines.

Students' mathematical experiences

Students might discover some of the following

- for some values of a , the membership fee for *Minnows* is less than for *Dolphins*, but for others it is greater
- for some values of a , the membership fee for *Minnows* or for *Dolphins* can be zero or negative
- when $a = 10$ the fees are the same (and this value can be found from a table of values, from a graph, or by solving an equation)
- as a gets further from 10, the difference between the two expressions gets greater
- if a increases by 1, then the *Minnows* fee ($6a-30$) increases by 6, but the *Dolphins* fee ($50-2a$) decreases by 2
- each set of points on the graph forms a pattern: each lies on a straight line
- the lines have different slopes and these relate to the change in fee as members get a year older.

Students might discuss

- continuity, ie whether some or all points on the line fit the relationship.

Assessment and feedback

Be flexible over the organisation and timing of the lesson - in particular, allow sufficient time for students, collectively and in small groups, to explore the situation qualitatively.

For some (positive) values of a , the expressions are negative. Check whether students perform such calculations correctly.

Assess how well students make links between the story, the given algebraic expressions, their tables of values and features of the graph; encourage them to make such links.

Key questions

When is *Minnows* (or *Dolphins*) cheaper?

How could we record these values more systematically?

How does the fee change as a member's age changes?

Should the clubs have age limits?

Adapting the lesson

Decide how much support to give for Stage 2 of the lesson (drawing the graph): eg, you might let students come up with scales for the axes on their own, or you might discuss the scales, or provide ready-made scales, or even provide the complete graph.

You might want to explore the effect on the graph of changing the expressions (by, for example, adding 10 to each to give $6a-20$ and $60-2a$); or the effect on the expressions of changing the graph (eg by drawing lines which intersect or cut the x -axis at a different place).

Algebra 11B

Lesson

22B**Outline of the lesson (annotated)**

1. Display the Membership problem and ask students for their immediate responses.
 - Try to draw out some qualitative responses. ——— For example:
 “The older you are, the cheaper Dolphins membership becomes”
 “Minnows costs more as you get older, but it costs very little if you’re very young ...”
 “Dolphins costs a lot if you are very young”.
 - Ask students to explore this further in small groups.
 - Collect data on the membership fees for various ages. ——— Listen to students’ arguments and conclusions - but don’t pursue them at this stage.
 - Represent the data
 - ‘randomly’ on the board
 - in (randomly ordered) tables
 - in ordered tables.
 ——— Try to prompt the need for an ordered table, rather than simply produce such tables.
 - Discuss anomalies: eg, the value of the expressions is zero or negative for some ages. ——— We might therefore want to modify the membership rules - which we’ll discuss later in the lesson.
 - Ask, “Are the fees for the two clubs ever the same?” ——— We could solve this by trial and improvement, by drawing graphs, or by solving the equation $6a - 30 = 50 - 2a$.

2. Ask students to represent the data as points on a Cartesian graph. ——— You might want to discuss the scales that one might use, and the possibility of leaving space for negative values for the fees. Or you could hand out graph paper with pre-prepared axes.

3. Discuss the sets of points, and key individual points on the graph.
 - Can we draw a line through each set of points? Should it be solid or dotted, or should it be stepped? ——— If we treat age as discrete, for example by only counting whole numbers of years, the relationship between age and fee will be a step function.
 - Relate points to the tables of values, to the original algebraic expressions and to the original story. ——— In an ordered table, we can relate changes in age to changes in fees and relate this to the gradient of the lines on the graph.
 Particular points of interest are the point of intersection (10, 30) and the ‘zero fee’ points (5, 0) and (25, 0).
 If we factorising the expressions, we reveal the ages for which the fees are zero, ie 5 and 25:
 $6a - 30 = 6(a - 5)$
 $50 - 2a = 2(25 - a)$.
 - Discuss how we might modify the rules for the membership fees (eg, what do we do with people of 5 years and younger, and 25 years and older?). ——— Extension: Find ways of modifying the expressions for the fees, so that the clubs become attractive to other age groups.

Algebra 11B

Lesson

22B**Background****Students interpret algebraic expressions**

Students are given two expressions in a . Their values change as a changes, as do their relative values: sometimes one expression is larger than the other, sometimes it is smaller, sometimes their values are the same. In particular, as a increases, one expression increases quite rapidly, while the other expression decreases, but less rapidly.

Students develop their understanding of 'variable'

The kind of variable we are learning about (be it called ' a ' or 'your age in years')

- is a number
- can take on lots of different values;
- as it changes in a systematic way, so the 'dependent variable' (in this case 'the club membership fee, in £') may also change in a systematic way.

Students might also discover that, in terms of the *context* that the expressions represent, their values don't make sense for some (perfectly sensible) values of the variable. For example, when $a = 4$, the expression $6a - 30$ has a value of -6 , ie for an age of 4 years, the membership fee is $-\text{£}6$.

Students engage with the notion of continuity

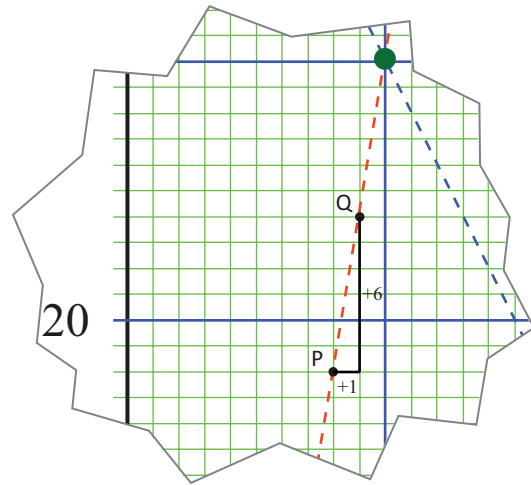
The graph prompts the question of 'intermediate' values: given that each set of points lies on a line, what about some other points on that line, eg where $a = 10.5$? Do they fit the algebraic relationship, and do they fit the membership fee story? And do 'all' points on the line fit the relationship? For the current story, what about values of a outside the range $5 \leq a \leq 25$?

Students develop their ability to 'read' a Cartesian graph

Each of the properties of variable mentioned above can be seen, to a greater or lesser extent, in these three representations. For example, if a value of a in a table goes up by 1, then the corresponding values of $6a - 30$ have a difference of 6 (see the second table, below); similarly, if two points representing $(a, 6a - 30)$ are 1 horizontal unit apart on a graph, they will be 6 vertical units apart (see points P and Q on the graph, below). Also, because the two sets of points on the graph form a pattern (in this case, they lie on two straight lines), they help students get a general sense of the

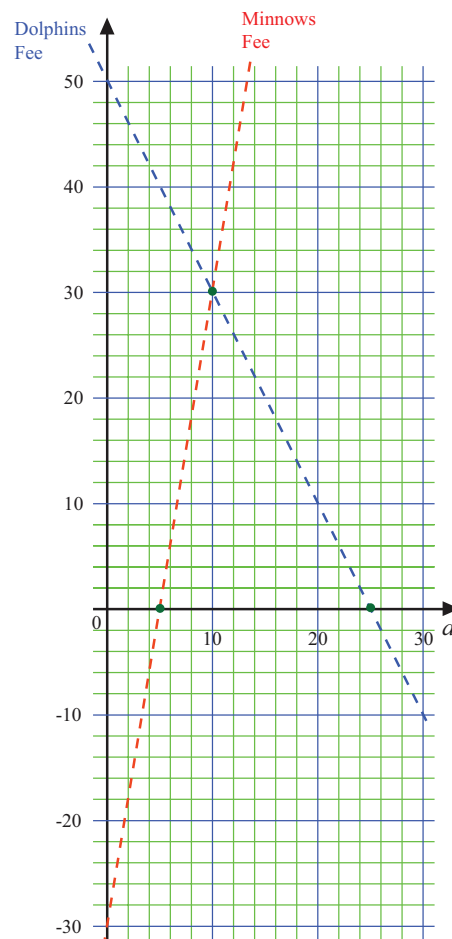
not ordered		ordered	
a	$6a - 30$	a	$6a - 30$
6	6	6	6
9	24	8	18
8	18	9	24

+1 +6



relationships, ie a sense of what happens for a range of values rather than just isolated numerical cases.

As already mentioned, some points on the graph may not make sense in terms of the story. However, they may still have meaning algebraically. So for example, the two straight lines cut the scale on the y axis at -30 and 50 , which are the constant terms in the expressions $6a - 30$ and $50 - 2a$. This makes sense algebraically, since this is where $a = 0$, even though it would be peculiar to think of a membership fee for a 0 year old.



Algebra 11B

Lesson

22B

Algebra 12AB

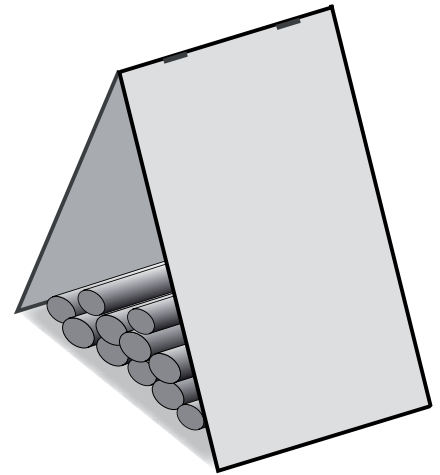
Mini-assessment

23AB**Garden shelter**

Linda has hinged two doors together to make a shelter for some logs that she wants to keep dry.

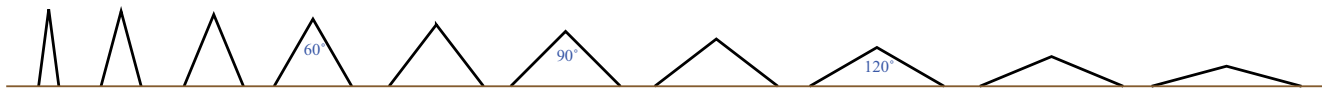
She wants the shelter to be as roomy as possible.

What is the best size for the angle between the doors?



Here are some possible front-views of the shelter.

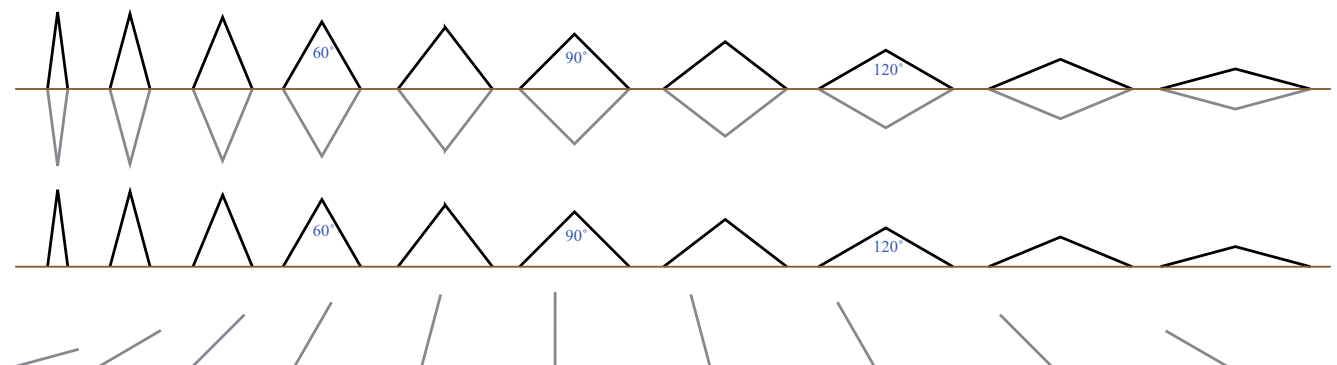
Describe the shelter that you think would have the largest capacity.

**Commentary**

This is potentially a very rich task - it is reminiscent of the 'rectangular sheep pen' task (involving a wall and some fencing), which many people solve using calculus. However, here it should be treated with a very light touch: our main purpose is to give students some help in thinking dynamically, by showing the series of front views, and to assess what they make of this help. To start with, though, it is worth seeing how students respond to the task without the front-views diagram.

A dynamic view is useful for Lesson 23A, Garden plot, in which students are asked to imagine a garden that is to be partitioned into a square and three rectangular regions, and to consider what happens as their sizes are varied.

It is not important that students solve this assessment task - though you might want to try for that on another occasion. Some students might see that the task boils down to comparing the areas of the triangles shown in the front-views diagram. Can this be done through geometric reasoning, or only empirically by measuring bases and heights? One or other diagram below might help.



Algebra 12A

Lesson

23A**Garden plot**

Piet has a 14 m by 19 m rectangular garden.

He also has lots of 1 m square paving tiles, coloured red or yellow or blue.

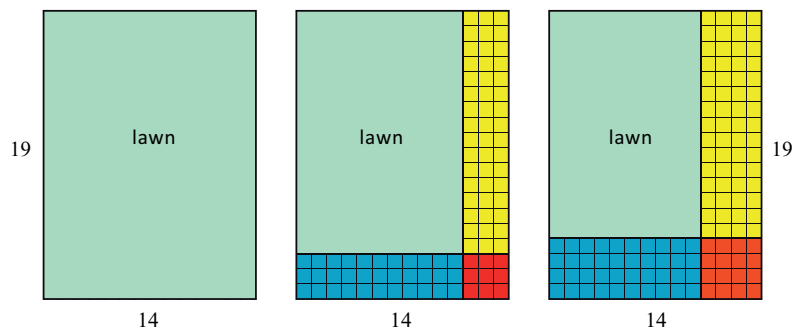
He wants to divide the garden into four rectangular regions and to tile three of them.

The red region is to be a square but he is not sure how big to make it.

The diagram shows how the garden looks with a 3m and a 4m red square.

Imagine bigger and bigger red squares.

What happens to the number of blue tiles?

**Summary**

In this lesson we present students with a task involving a quadratic relationship, to provide a contrast with the more familiar linear relationship. Encourage students to visualise and to make predictions before gathering lots of data. It may come as a surprise to students that the area of the blue rectangle does not increase in step with the length of side of the red square, and that it doesn't keep getting bigger.

Outline of the lesson

1. Visualise changes in the red square and the blue rectangle.

- Carefully present the above task to the class.
- Ask students to describe what happens to the blue rectangle as the red square gets bigger.
- "As the side of the red square gets steadily bigger, what happens to the number of blue tiles?"
"Does the number increase? Does it increase steadily? Will it always increase?"
Leave the questions unresolved.

2. Count the blue tiles.

- Count the number of blue tiles when the length of side of the red square is 3m, then 4m. Record the information in a table.
- "How many blue tiles do you think there will be when the red square's sides are 5m long?"
- Find the actual number of blue tiles. Discuss the result. Add the information to the table.
- "How many blue tiles do you think there will be when the red square's sides are 6m long?"
- Find the actual number. Add the information to the table. Find other values to put in the table.
- Discuss the values in the table.
- Extension: **Find a rule** for working out the number of blue tiles when you know the length of side of the red square.

Length of side of RED square	Number of BLUE tiles
3	33
4	40

3. Compare with a linear relation.

- Add a third column to the table, and enter the initial values 33 and 40 again (for 3m and 4m). Discuss how the values would continue if they 'changed regularly'. Complete the column accordingly and compare it to column 2.

Length of side of RED square	Number of BLUE tiles	Number of REGULAR tiles
3	33	33
4	40	40
5	45	47
6	48	54

4. Graph the information.

- Sketch Cartesian axes showing 'length of side of red square' and 'number of blue tiles'. Plot the initial values (3, 33) and (4, 40). Consider what other points would look like on the graph.
- Extension: Ask students to imagine plotting similar points for the yellow tiles. "How would the two sets of points compare?"

Algebra 12A

Lesson

23A**Overview****Mathematical ideas**

This optional extension lesson builds on Lesson 13B (Algebra 5B) to explore quadratic expressions and relationships. In the lesson, we address how quadratic and linear relationships differ and how we can discern this from tables and Cartesian graphs.

Students' mathematical experiences

Students should have the opportunity to

- visualize the dynamic nature of the relationships
- represent expressions in different ways
- observe that a quadratic expression does not increase (or decrease) steadily.

Key questions

What can you do well when multiplying numbers?

What do you need to do better?

What are the advantages and disadvantages of the various methods?

Assessment and feedback

Consider how different students were able to think dynamically in the Mini-assessment 23AB. You may want to allow students time in pairs to consider each other's ideas in order to "re-tell", comment and improve on them.

Some students may find it helpful to sketch the layout for different sizes of the red square. In the initial stage of the lesson, encourage them to sketch rather than draw accurately.

Some students may have difficulty with deciding upon appropriate scales for the Cartesian graph. Address this quickly and directly by suggesting scales.

Adapting the lesson

This is an optional lesson that is likely to be more appropriate for middle and high attaining students and classes. If students have difficulty with the task, you may find it helpful to repeat Lesson 13B.

Algebra 12A

Lesson

23A**Outline of the lesson (annotated)**

1. Visualise changes in the red square and the blue rectangle.

- Carefully present the above task to the class.
- Ask students to describe what happens to the blue rectangle as the red square gets bigger.
- “As the side of the red square gets steadily bigger, what happens to the number of blue tiles?”
“Does the number increase? Does it increase steadily? Will it always increase?”
Leave the questions unresolved.

— Stress that students should *not* count (or calculate) actual values at this stage.

2. Count the blue tiles.

- Count the number of blue tiles when the length of side of the red square is 3m, then 4m. Record the information in a table.
- “How many blue tiles do you think there will be when the red square's sides are 5m long?”
- Find the actual number of blue tiles.
Discuss the result. Add the information to the table.
- “How many blue tiles do you think there will be when the red square's sides are 6m long?”
- Find the actual number. Add the information to the table.
Find other values to put in the table.
- Discuss the values in the table.

— Use the given diagram. (We have deliberately shown every tile on the diagram so that students who are unsure about ‘area’ can count individual squares.)

It is likely that some students will use the previously found values (33 and 40 blue tiles) to conclude that there will now be 7 more, which is 47, blue tiles. The actual value is 45.

Some students might correctly surmise that there will now be 48 blue tiles, based on the correct hunch that the difference between successive values decreases by 2.

- Extension: **Find a rule** for working out the number of blue tiles when you know the length of side of the red square.

— It is useful for students to realise that there *is* a rule that directly connects the variables (ie a *function* rule), rather than just a *scalar* term-to-term rule. However, you might not want to pursue this too deeply here. And you might be content with a verbal rule or an informal symbolic rule. Getting the syntax right [eg, $N = L(14 - L)$] is quite demanding.

3. Compare with a linear relation.

- Add a third column to the table, and enter the initial values 33 and 40 again (for 3m and 4m). Discuss how the values would continue if they ‘changed regularly’.
Complete the column accordingly and compare it to column 2.

Length of side of RED square	Number of BLUE tiles	Number of REGULAR tiles
3	33	33
4	40	40
5	45	47
6	48	54

Extension: What sort of arrangement of tiles could produce column 3?

4. Graph the information.

- Sketch Cartesian axes showing ‘length of side of red square’ and ‘number of blue tiles’. Plot the initial values (3, 33) and (4, 40). Consider what other points would look like on the graph.
- Extension: Ask students to imagine plotting similar points for the yellow tiles. “How would the two sets of points compare?”

— Because some of the x and y values are so different you need to think carefully about the scales for the axes. Using the same scale produces a very ‘thin’ graph. Using different scales might make the graph harder to interpret.

Algebra 12A

Lesson

23A**Background**
Linear and quadratic relations

The purpose of this lesson is to present students with a relationship that is not linear, to help them better to appreciate the nature of a linear relationship. In the latter, as one variable repeatedly changes by a fixed amount, so does the other variable. Thus students might be surprised that, in *Garden plot*, as the length of side of the red square repeatedly increases by 1, the number of blue tiles increases by changing amounts. Similarly, they might be surprised by the fact that the area of the blue rectangle does not simply get bigger and bigger as the red square gets bigger.

It turns out that the relationship between the length (call it L , say) of side of the red square and the number of blue tiles (N , say) is quadratic. (The same applies to the number of yellow tiles, M , say.)

We can write

$$N = L(14 - L) \quad \text{and} \quad M = L(19 - L).$$

However, you may not want to go as far as representing the relationship algebraically in this lesson, though we offer it as an Extension activity.

We start by asking students to *visualise* what happens to the number of tiles in the blue rectangle (or to the area of the blue rectangle) as the length of side of the red square (composed of red tiles) increases. Here we don't want students to count (or calculate) but to try to think *qualitatively*. It may not be easy to see that the area of the blue rectangle grows at a varying rate. However, some students might notice that as the red square gets bigger, the room available for the blue tiles starts to diminish at some stage and eventually disappears.

We then switch to counting the number of blue tiles in the given diagram: 33 for the 3m red square and 40 for the 4m red square. Rather than ask students to rush headlong into generating more data, we pause to ask students to *predict* the number of blue tiles for a 5m square. The hope is, that students who predict that there will be 47 (ie another 7 tiles) will be jolted/intrigued into thinking again.

We repeat the process for the 6m red square.

Our aim here is not to reveal all the intricacies of a quadratic relation, but to heighten students' awareness that not all relations are linear and that, in turn, linear relations behave in ways that other relations do not.

To take this further, we suggest plotting values on a Cartesian graph. This is not straightforward as the x and y values we are dealing with are sometimes very different and thus not easy to show clearly on a graph. However, it should be possible to convey key features - namely that the points don't lie on a straight line, and they don't keep going 'up'. [In the adjacent graph, the axes are drawn to the same scale. Would it be helpful to stretch the horizontal axis, eg by a factor of 2?]

It would be interesting to see students' attempts at drawing a graph, but you might decide to restrict the approach in

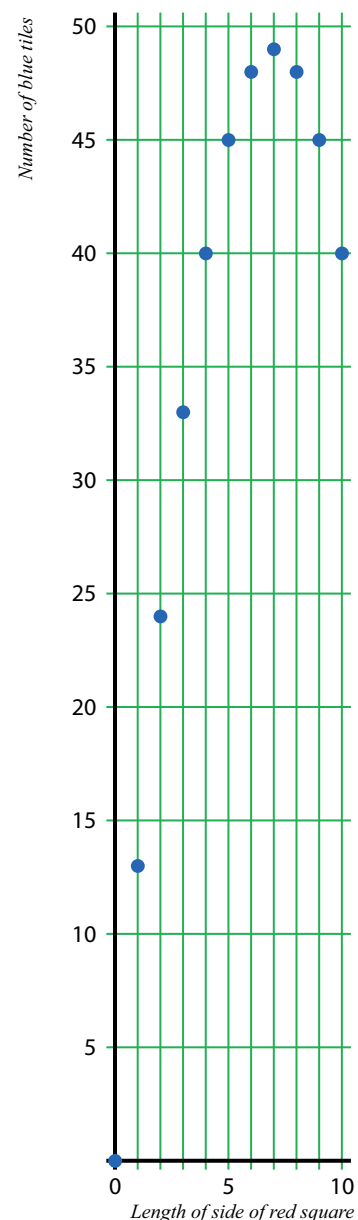
this lesson to sketching a graph on the board.

There are various other issues that you could pursue, either here or in subsequent lessons. For example, how do we join the points - with a curve or with line segments? And should we use a solid or a dotted line - in other words, what can we say about intermediate points? Would things change if we used smaller tiles, or if we used painted concrete, say, and measured the blue area?

You might also want to look at the yellow rectangle.

It behaves like the blue rectangle, but as one of its dimensions can range from 0 to 19, rather than 0 to 14, it has a maximum value for a different value of L .

It is also interesting to consider the perimeter of the blue (or yellow) rectangle. Having found that the area of the blue rectangle varies, it may come as a surprise that its perimeter stays the same. This can be looked at geometrically and algebraically.



Algebra 12B

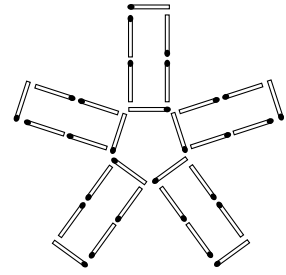
Lesson

23B**Growing flowers**

Tim has a numerical rule for making flower-shapes out of matchsticks.

This is his 5-petals flower.

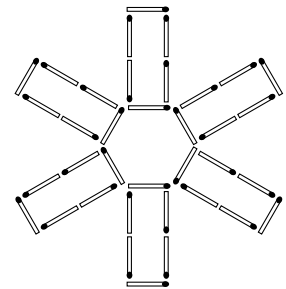
What might his rule be? Write it down. Sketch a 6-petals flower using this rule.

**Summary**

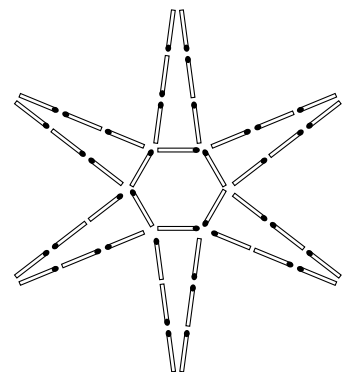
In this lesson we look at various rules, linear and quadratic, for generating a figurative pattern (flowers). We compare how the number of elements (matchsticks) used to make the patterns vary for the different rules and consider how this can be represented on a graph.

Outline of the lesson

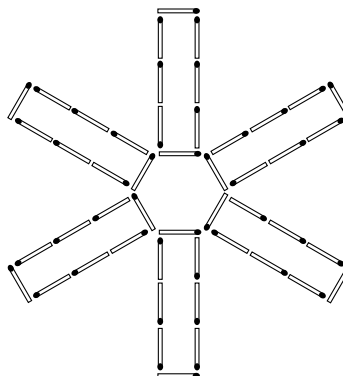
1. Invent and apply a rule for the 5-petals flower pattern.
 - Present the above task. Ask students to work on it, individually or in pairs.
 - Discuss some of the students' rules and 6-petal flowers.
2. Find a rule (A).
 - Show this 6-petals flower pattern.
Say, "Imagine that this is Tim's 6-petals flower. Find a rule that fits both of his flowers. Now use the rule to sketch a 7-petals flower".
 - Show this rule:
Rule A: Make a regular polygon using a matchstick for each side. Fix a petal made of five matchsticks to each side.
Say, "Tim might have used this rule. Check that it works for 5 and 6 petals".
 - Use the rule to find the matchsticks needed for a 7-petal and 8-petal flower. Put all four pairs of numbers in a table.
 - Look at the rule again. Are there alternative ways of expressing it?



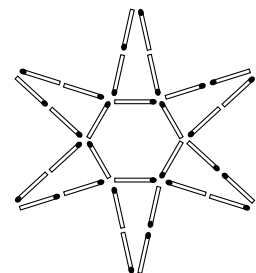
3. Find another rule (B).
 - **Repeat Stage 2** using this 6-petal flower and this rule:
Rule B: Make a polygon using a matchstick for each side. Fix a petal to each side, made of the same number of matchsticks as there are sides.



4. Find yet another rule (C) of (D). [Optional]
 - **Repeat Stage 2** for one or both of these patterns and rules.



C: Make a polygon... Fix a petal to each side, made of two lines of sticks plus one stick. Have 3 fewer sticks in each line than there are sides.



D: Make a polygon... Subtract the number of sides from 10. Use that number of matchsticks for each petal. Fix to each side of the polygon.

Algebra 12B

Lesson

23B

Overview

Mathematical ideas

In this optional extension lesson, we again consider how quadratic and linear relationships differ. There is also the option, for students for whom you think this is suitable, to express rules algebraically and to see how this sometimes allows us to re-write rules in a simpler form and to re-interpret them accordingly.

Students' mathematical experiences

Students should have the opportunity to realise that

- in some growing flower-patterns, as we increase the number of petals, the number of matches increases steadily (as in the initial linear pattern described by Rule A)
- in other growing flower-patterns this is not so, because as the number of petals increases, the number of sticks within the petals also changes (as in the initial quadratic pattern described by Rule B).

Some students may realise that

- a single example of a flower-pattern can belong to different families of flower-patterns (unless the pattern is already tightly defined as in the case of 'Row of tiles' in Algebra Lesson 6A).

Assessment and feedback

The students are likely to have had some experience of creating simple linear patterns with matchsticks. The patterns in this lesson are more complex. You may find it valuable to use a *visualiser* to construct the initial flower patterns as a class.

Some students may have difficulty sketching the patterns. Encourage them to draw quick, schematic sketches, but neat enough for them to 'see' regularities in the structure.

Encourage students to compare and discuss each other's sketches: Does the sketch represent the pattern sufficiently well?

Some classes may have difficulty finding the rules. If you show students the rule, encourage them to check whether the rule 'works'.

Key questions

Can you make sense of the rule in terms of the matchstick pattern?

Why does the number of petals not increase steadily?

Can you find a simpler expression for the rule? How can you be sure that the two expressions are equivalent?

Adapting the lesson

This is an optional lesson that is likely to be more appropriate for middle and high attaining students and classes. You will need to make a judgment about whether you concentrate on the verbal expression of the rules or whether you also use algebraic symbolisation.

You can extend the task by considering rules C and D.

Algebra 12B

Lesson

23B**Outline of the lesson (annotated)**

1. Invent and apply a rule for the 5-petals flower pattern.
 - Present the above task. Ask students to work on it, individually or in pairs. — If the class seems to be stuck, show Rule A (Stage 2) and say, “Use this rule if you like, or invent another rule that works”.
 - Discuss some of their rules and 6-petal flowers. If students think they have a rule, encourage them to draw the 6-petals flower even if they can't put their rule into words. But overall, don't spend too long on this stage.
2. Find a rule (A).
 - Show this 6-petals flower pattern.
Say, “Imagine that this is Tim's 6-petals flower. Find a rule that fits both of his flowers. Now use the rule to sketch a 7-petals flower”. — If finding a rule proves difficult, skip to the next step and show the class Rule A.
 - Show this rule:
Rule A: Make a regular polygon using a matchstick for each side. Fix a petal made of five matchsticks to each side.
Say, “Tim might have used this rule. Check that it works for 5 and 6 petals”.
 - Use the rule to find the matchsticks needed for a 7-petal and 8-petal flower. Put all four pairs of numbers in a table. — The rule is linear: as the number of petals increases by 1, the number of matchsticks increases by 6.
 - Look at the rule again. Are there alternative ways of expressing it? — You will need to decide how far you want to take this in this lesson, and whether you want to stick with the verbal format or also use algebraic symbolisation.
Some students may come up with this alternative structuring:
Each petal is a rectangle made with 6 matchsticks. Draw the petals touching each other in a 'circle'.
This can be expressed symbolically as:
For n petals, the number of matchsticks is $6n$.
Similarly, the original version of the rule can be expressed as:
For n petals, the number of matchsticks is $n + 5n$.
Students might see that we can simplify this to $6n$.
3. Find another rule (B).
 - **Repeat Stage 2** using this 6-petal flower and this rule:
Rule B: Make a polygon using a matchstick for each side. Fix a petal to each side, made of the same number of matchsticks as there are sides. — This rule is *not* linear: as the number of petals increases by 1, the number of matchsticks increases by more and more. “Why?”
It turns out that the rule is quadratic. It can be expressed thus: $n + n^2$, which factorises to $n(1 + n)$ [though you might decide not to use this algebraic form with the students].
4. Find yet another rule (C) of (D). [Optional]
 - **Repeat Stage 2** for one or both of these patterns and rules. — Rule C can be expressed thus: $n + [2(n - 3) + 1]n$.
This simplifies to $2n^2 - 4n$ or $2n(n - 2)$ or $n(2n - 4)$.
The latter can be expressed as:
Decide on the number of petals. Make the petals using 4 less than twice that number of matchsticks. Arrange the petals in a 'circle'.
Rule D can be expressed as $n + (10 - n)n$, or $11n - n^2$.
“What does the 10-petal flower look like?”
“What about the 11-petal flower?!”
5. Compare rules.
 - Compare the table of values for Rule A and Rule B (and/or C and/or D).
 - Discuss how the values would look on a Cartesian graph. Draw (or sketch) the graph. — For Rules B and C, the points lie on a curve that gets more and more steep. What about A and D? How/why are they different? The rules all have one point in common - what are its coordinates?

Algebra 12B

Lesson

23B**Background****More linear and quadratic relations**

The main purpose of this lesson is again to present students with relationships that are not linear, and to contrast them with a relationship that is.

Students are asked to invent, or use, rules that fit two examples of a pattern, starting with a 5-petals flower made of 30 matchsticks, followed by a 6-petals flower made, in the first instance, of 36 matchsticks.

If we were working purely numerically, with just the ordered pairs (5, 30) and (6, 36), say, then there would be an infinite number of rules that map the first number on to the second within both pairs. Indeed, this would normally be true for any number of such ordered pairs - think of the pairs as points on a graph and think of all the curves we could draw through them. On the other hand, there is only one *linear* rule that fits two pairs such as (5, 30) and (6, 36).

However, in this lesson we are working with figurative patterns rather than just with numbers. This may still allow for a variety of rules if the pattern is not tightly defined, although it is likely that one rule will be more obvious than any other. Also, for most of the lesson, students are asked to use a given rule rather than invent rules of their own, so here the issue doesn't arise of there perhaps being a multiplicity of valid rules.

Some of the rules that students are shown in this lesson might appear to be rather opaque, as here:

C: Make a polygon using a matchstick for each side. Fix a petal to each side, made of the same number of matchsticks as there are sides.

You might thus need to take extra care to ensure that students understand what the rule is trying to say.

The lesson provides the option of expressing such rules algebraically, and you will need to decide whether that is appropriate for the class and at this time - you might want to return to this aspect in a later lesson.

However, this does offer the potential of showing some of the affordances of algebra. It can be much neater to express the rule algebraically. Also, it allows for the possibility of simplifying the algebra and thereby coming up with a simpler rendition of the rule. For example, for the above rule, if we use n , say, for the number of petals in a flower, then we can express the number of matches as $n + n^2$. This can be rewritten as $n(1 + n)$ which in turn can be interpreted as something like this:

Decide on the number of petals. Make the petals, each with one more matchstick than that number. Arrange the petals in a 'circle'.

We present four rules, A, B, C and D, though you may not have time to use all of them in a single lesson. Rule A is linear, the others turn out to be quadratic.

The table below shows the number of matchsticks arising from the rules, for flower-patterns with 5, 6, 7 and 8 petals.

There are many interesting features to the table, some of which you might be able to tease out in the lesson. For example, the Rule A values increase steadily, while those for B and C seem to 'accelerate'. On the other hand, the Rule D values seem to be falling - is that always the case?

It is interesting to relate these observations to the figural patterns. For example, in Rule A, the number of matches increases steadily because we keep adding petals of the same 'size'. On the other hand, in Rule B, as the petals get more numerous they also get 'bigger'.

It is also interesting to consider the range of values for which these rules make sense. For example, for Rule D, the 10-petals pattern seems to have *no* petals, while the 11 petals-pattern seems not to exist!

	Rule A	Rule B	Rule C	Rule D
	$n + 5n$	$n + n^2$	$n+n[2(n-3)+1]$	$n + n(10 - n)$
Number of petals, n	Number of matchsticks			
5	30	30	30	30
6	36	42	48	30
7	42	56	70	28
8	48	72	96	24

Algebra 12B

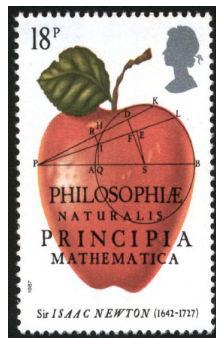
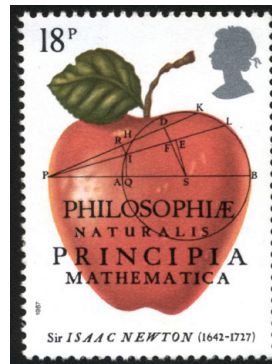
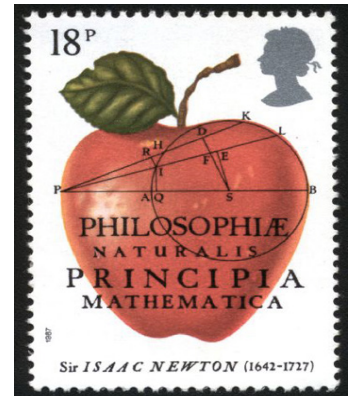
Lesson

23B

Notes

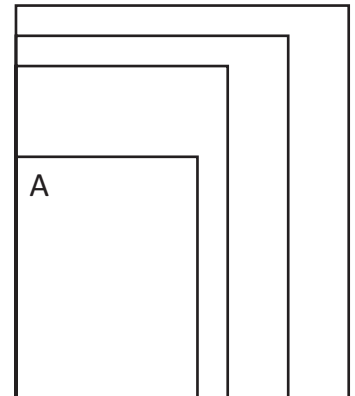
Multiplicative Reasoning 12AB

Mini-assessment

24AB**Stamps****A****B****C****D**

1. Which stamp has the same shape as stamp A ?
2. The stamps have been placed on top of each other.

How could we use this to help us decide which stamp has the same shape as stamp A ?

**Commentary**

Use the file MR-12AB-SLIDES.pdf for this Mini-assessment [pages 3-4 show a version with a heavier circle].

The aim again is to probe students' awareness of *similarity* (expressed as 'the same shape'). However, you might also want to use the Mini-assessment to *prompt* students into looking for visual cues - something they will need to do in the subsequent lessons.

The second part gives students the opportunity to use notions of dragging and of enlarging about a centre.

Help students sharpen the quality of their answers by giving them *plenty of time to talk* to their neighbours before fielding their responses.

Task 1: It is fairly easy to see that B and D are *not* similar to stamp A, but what *criteria* and what *language* do students justify the choice of stamp C?

Do students ask for numerical information, or can they, in its absence, give qualitative arguments?

What features of the stamps and what geometric properties do they use?

Do they notice what happens to the circle?

What mathematical terms are used, and by which students?

Task 2: Can students make use of their experience of dragging?

Can students make use of their knowledge of enlarging about a centre?

Multiplicative Reasoning 12A

Lesson

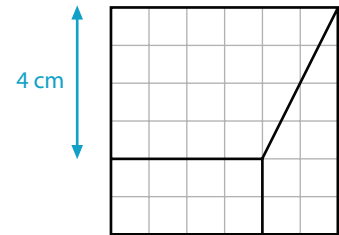
24A**Tangram**

This tangram consists of three pieces. We want a larger version of the tangram where the 4 cm length becomes a 7 cm length.

Work in a group of three. Choose one piece each.

Draw the larger version of your piece on 1 cm squared paper.

Carefully cut it out. Check that the 3 new pieces again fit together.

**Summary**

In this lesson students enlarge a 3-piece tangram puzzle. Initially they enlarge just *one piece each* - the key to the lesson is the considerable surprise that occurs in cases where the resulting pieces don't fit together.

Students are asked to map 4 cm onto 7 cm, which involves the quite demanding scale factor $\times 1.75$, and which is likely to lead some students to adopt the inappropriate addition strategy $+3$. They are then asked to enlarge the tangram again, using the scale factor $\times 1.5$. This is a less demanding task and should allow students to draw on (and develop) some of their geometric knowledge about enlargement.

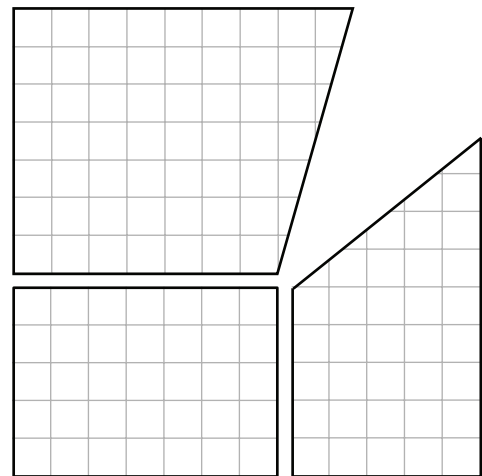
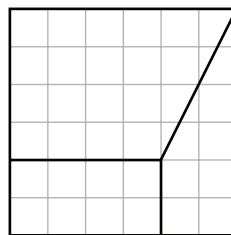
Outline of the lesson

1. Ask students to solve the task in groups of three.

- Explain the above task and ask students to solve it in groups of three. Provide students with 1 cm squared paper and scissors; make sure **each** student enlarges just **one** piece and cuts out the result.

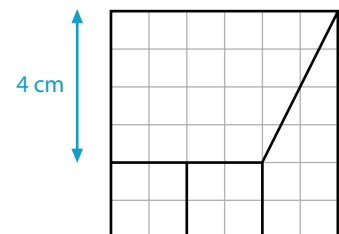
2. Discuss students' solutions.

- Check / briefly discuss students' solutions.
[If students have used the addition strategy, by adding 3 cm to various lengths, the new pieces won't fit.] $\rightarrow$



3. Simplify the task.

- Display this shape. $\rightarrow$
"This is a 6 cm square tangram again, but now with 4 pieces (2 of which are squares). Draw a larger version that forms a 9 cm square. Use 1 cm squared paper but *don't* cut out the pieces."
- Discuss students' drawings. "What clues can we use to decide whether the result is right or wrong?"



4. Revisit the original task.

- Ask students to repeat the original $4 \rightarrow 7$ enlargement.
"Draw a larger version but *don't* cut out the pieces."
"What size square does the enlarged tangram form?"
- Evaluate the drawings. "What clues can we use...?"
- Why does the $+3$ addition strategy not work?
- Discuss the scale factor. "If I know a length on the tangram, how can I work out the enlarged length?"

Multiplicative Reasoning 12A

Lesson

24A

Overview

Mathematical ideas

As with *Expanded house* (Lesson MR-5A) and *Stamps* (this lesson's Mini-assessment), this lesson involves geometric enlargement. We start with a fairly demanding scale factor, $\times 1.75$, of the sort we have met before, and test students' understanding with a task which dramatically highlights the error of using the addition strategy. We go on to use geometric properties of shapes to further bring out the 'uniform', multiplicative nature of enlargement.

Files are also provided (see pages 291 and 292) for investigating enlargements by dragging, either in this lesson or later, and for considering the direction and magnitude of the drag (which readily links to the use of centres of enlargement for constructing images). There is also an opportunity to use Cartesian coordinates to represent and analyse enlargements.

Students' mathematical experiences

Students may discover some of the following

- the addition strategy does not preserve an object's shape
- an enlargement is 'uniform'
- all squares within a shape remain as squares after an enlargement
- slopes (and angles) stay the same
- all distances are enlarged by the same factor.

Assessment and feedback

The Mini-assessment activity should give you an indication of what use students make of geometric properties for judging 'shape' and of what language they use.

Thus you might want to focus attention on features like the slope of the slanting line in the tangram, or the fact that the rectangular shape consists of two squares (and that therefore the enlarged version will also consist of two squares).

And you might want to use an activity to improve students' mathematical language - for example, asking students in pairs to craft an explanation as to why two shapes are or are not similar. A *visualiser* can be useful here for displaying the shapes under discussion.

Key questions

Why does adding a constant amount (3 cm) not work?

How can you tell whether the drawing is an enlargement?

Adapting the lesson

The *Stamps* Mini-assessment activity gives students the opportunity to think about similarity and enlargement *qualitatively* (eg, "Stamp A is too tall for its width"; "In stamp D the circle has become egg-shaped"). Such thinking is important in this lesson, though students' initial focus is likely to be numerical. If students struggled with the activity, you might want to start the lesson with a $4 \rightarrow 6$ enlargement rather than $4 \rightarrow 7$. Stages 2 and 3 could then proceed as outlined while it might be worth considering a $\times 2$ or $\times 1.25$ enlargement rather than $\times 1.75$ in Stage 4.

For other students it is worth going through the four given stages of the lesson, since it is useful to be able to explain why the addition strategy does not work, and to be able to elucidate geometric properties of enlargement, even when students can perform challenging enlargements successfully.

For such students there might be time to explore enlargement in terms of dragging and/or by using coordinates and a centre of enlargement, as discussed on pages 291 and 292. Or this could be pursued at a later date.

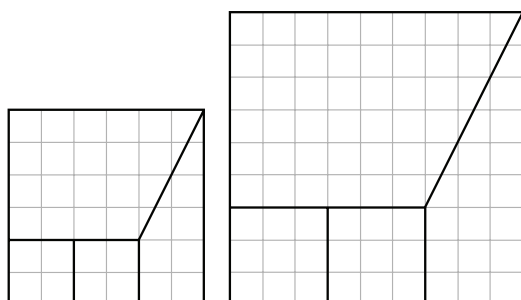
Multiplicative Reasoning 12A

Lesson

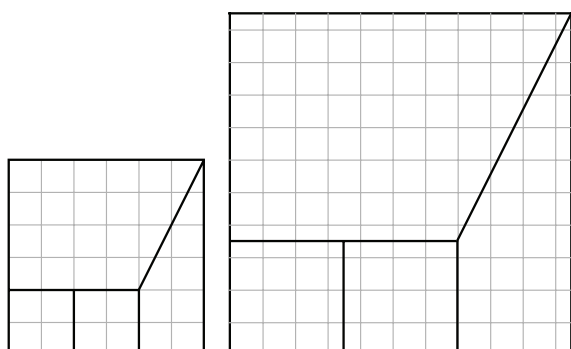
24A

Outline of the lesson (annotated)

1. Ask students to solve the task in groups of three.
 - Explain the above task and ask students to solve it in groups of three. Provide students with 1 cm squared paper and scissors; make sure **each** student enlarges just **one** piece and cuts out the result.
2. Discuss students' solutions.
 - Check / briefly discuss students' solutions.
[If students have used the addition strategy, by adding 3 cm to various lengths, the new pieces won't fit.] →
3. Simplify the task.
 - Display this shape. →
"This is a 6 cm square tangram again, but now with 4 pieces (2 of which are squares). Draw a larger version that forms a 9 cm square. Use 1 cm squared paper but *don't* cut out the pieces."
 - Discuss students' drawings. "What clues can we use to decide whether the result is right or wrong?"



4. Revisit the original task.
 - Ask students to repeat the original 4 → 7 enlargement.
"Draw a larger version but *don't* cut out the pieces."
"What size square does the enlarged tangram form?"
 - Evaluate the drawings. "What clues can we use...?"



- Why does the +3 addition strategy not work?
- Discuss the scale factor. "If I know a length on the tangram, how can I work out the enlarged length?"

— It is essential that each student works on their single piece, and cuts out the result. Errors will be far less obvious if students simply enlarge the tangram as a whole (as will be done in Stages 3 and 4 of the lesson).

— The two square-shapes will remain square-shaped.

The right-hand shape has the same-size base as the square-shapes and the enlarged shape will have the same-size base as the enlarged square-shapes.

The slanting line has a slope of '1 across, 2 up' (ie a gradient of 2), and will continue to have this slope.

Any 1 cm line becomes a 1.5 cm line.

The original tangram can be split into nine 2 cm squares (or thirty six 1 cm squares); the enlarged tangram can be split into nine 3 cm squares (or thirty six 1.5 cm squares).

— The addition strategy increases all lengths by the same amount, regardless of how long they are. Rectangles become more like squares.

Adding 3 cm to the heights of the original 3 shapes would increase the left hand edge of the tangram by 6 cm while the right-hand edge would only increase by 3 cm.

Multiplicative Reasoning 12A

Lesson

24A

Background

The tangram task

This lesson is based on a tangram task developed by Guy and Nadine Brousseau in France in the 1970s. Our tangram is simpler than theirs, but in Stage 1 of the lesson we have adopted their idea of asking students to enlarge pieces of a tangram separately, and then to fit the resulting pieces together. This means that students' errors, in particular those resulting from the addition strategy, will show up very dramatically, through the pieces not fitting together. Of course, such a demonstration does not of itself offer a way of resolving such errors.

Stage 1 of the lesson also involves a quite demanding scale factor ($\times 1.75$) and thus provides quite a stringent as well as dramatic test of how well students have come to understand that the addition strategy does not preserve proportions in geometric shapes. Later stages of the lesson encourage students to use their geometric knowledge to consolidate or enhance their understanding of the 'uniform', ie multiplicative, nature of enlargement.

Enlarging from a centre

The related file tangram-results.pdf (below) shows a drawing of the original tangram (in green) and enlargements produced by dragging, using the same scale factors as in the lesson, namely $\times 1.5$ and $\times 1.75$. The file can be displayed on a screen or printed out as a worksheet.

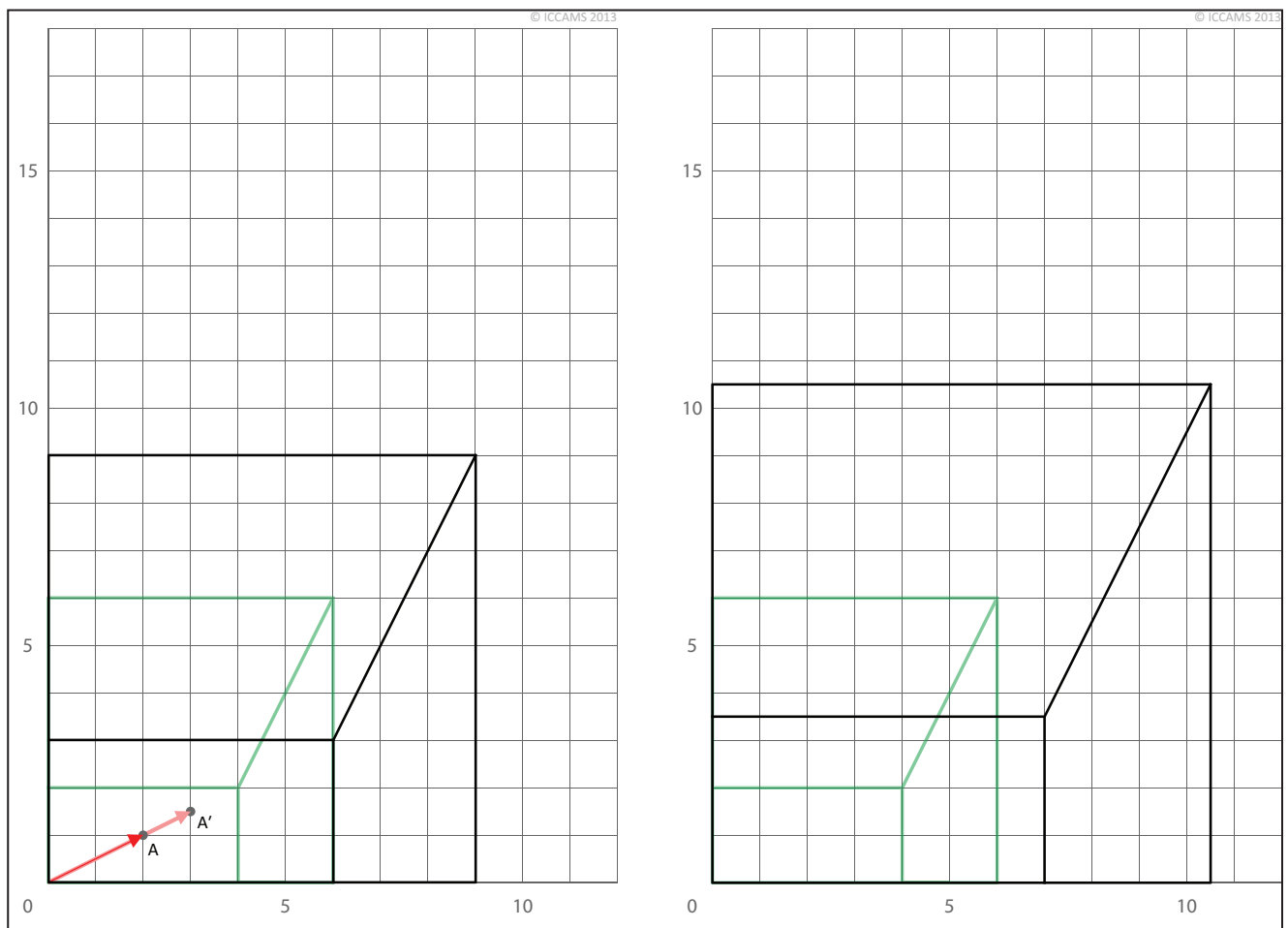
For the $\times 1.5$ enlargement we have selected a point A (at the centre of the rectangular tangram piece) and drawn arrows to show how its image, A', is the result of dragging. Students could investigate the images of other points in this way:

What can be said about the position, direction and length of the drag-arrows?

The file also shows numbered axes (chosen, for the sake of simplicity, with the fixed corner, or *invariant point*, of the tangram at the origin). We can thus write down the coordinates of points and their images, eg $A(2, 1) \rightarrow A'(3, 1.5)$.

So students could investigate this:

What is the relation between the coordinates of a point and its image?



tangram-results.pdf

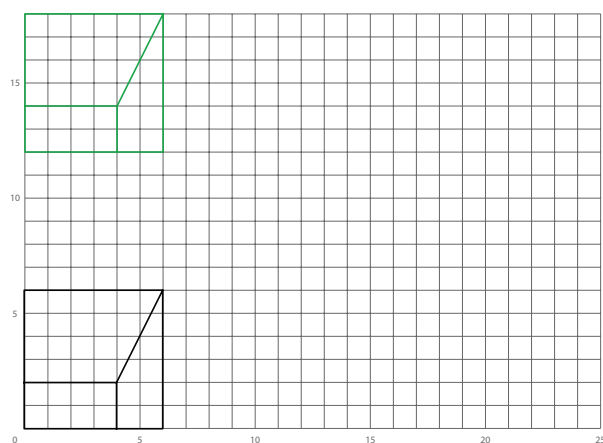
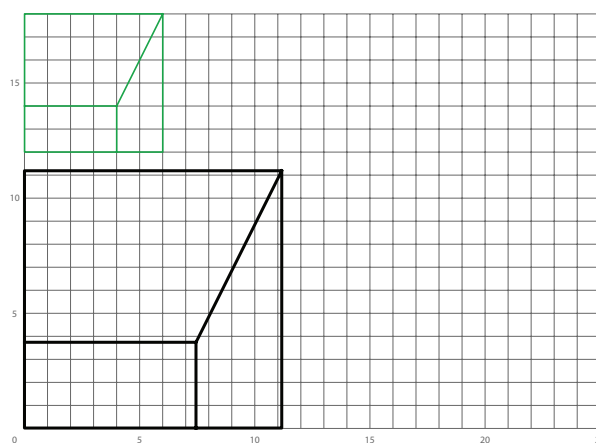
Multiplicative Reasoning 12A

Lesson

24A**Using the accompanying Word files****Enlarging by dragging**

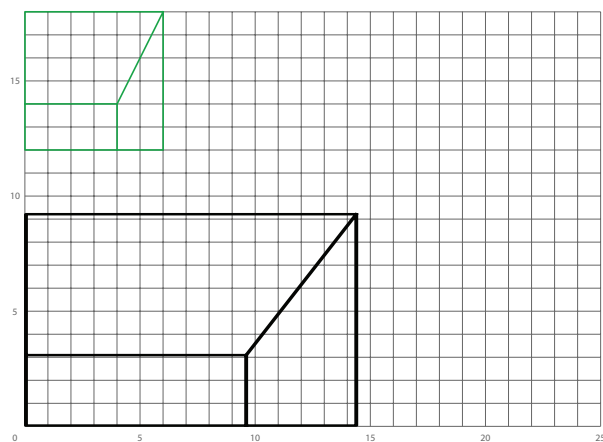
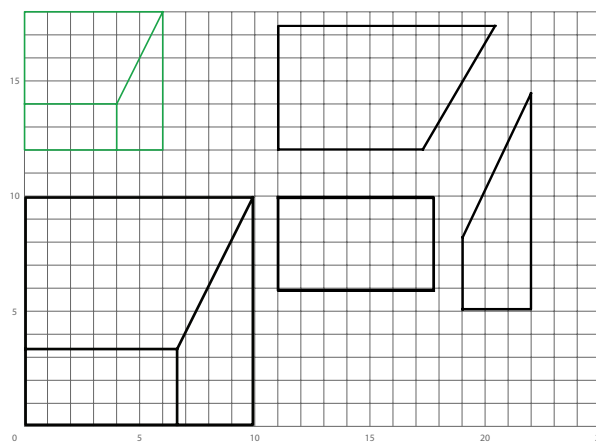
We have provided three Word files that can be projected on a screen and which allow the tangram to be enlarged by dragging. It is worth trying these for yourself, to get a sense of what they do and to help you decide whether to use any of them in class.

In the file *tangram-drag-constrain.docx* (below left and right) the aspect ratio of the tangram is preserved regardless of the direction of drag, and so this file can be used to provide a simple demonstration of what various enlargements of the tangram will look like.

*tangram-drag-constrain.docx* (initial state)*tangram-drag-constrain.docx* (after 'random' dragging)

In contrast to this, the aspect ratio is not constrained in the file *tangram-drag-free.docx* (below, left), so the shape of the 'expanded' tangram depends on the direction of the drag.

A third file, *tangram-drag+freepieces.docx* (below, right), is like the first in that the aspect ratio of the tangram is constrained so that one can easily produce enlarged versions; however, it also shows the three pieces of the tangram separated out and able to be 'expanded' freely. Here it can be both challenging and illuminating to try to reproduce an enlarged version of the tangram by dragging the individual shapes separately.

*tangram-drag-free.docx* (after 'random' dragging)*tangram-drag+freepieces.docx* (after extensive dragging)

Lesson

24A

Notes

Multiplicative Reasoning 12B

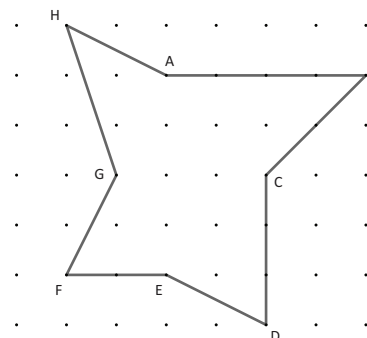
Lesson

24B**Stars**

This star has been drawn on
1 cm dotted paper.

Use the dots on Worksheet A.

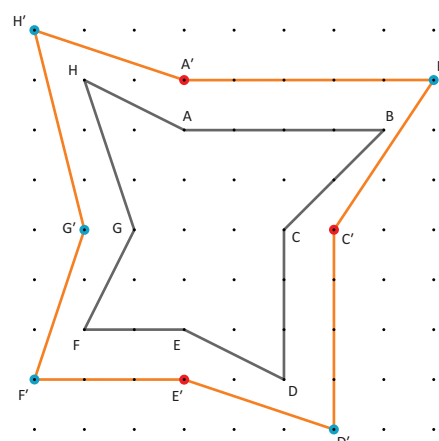
Draw a larger version of the star,
but exactly the same shape.

**Summary**

In this lesson we draw similar shapes and consider the scale factor of enlargement. As with the *Stamps* Mini-assessment, we use visual criteria to check the enlargement. We also measure lengths to find the scale factor and to develop the notion that *all* lengths in the plane are enlarged by this factor. Finally, we look critically at a way of expanding the star that does not produce an enlargement.

Outline of the lesson

1. Enlarge the star on an enlarged grid.
 - Give students a copy of Worksheet A.
“Use the dots to draw the enlarged version of the star.”
 - Discuss ways of checking the enlargement.
2. Measure corresponding distances on the object and image.
 - “How much bigger is the enlarged shape?” Discuss briefly.
 - Measure some corresponding distances; put them in a table.
 - Discuss the measurements. Can we find a rule to map object-lengths to image-lengths? [The scale factor is $\times 2.2$]
3. Enlarge the star on a 1 cm square grid.
 - Give students a copy of Worksheet B.
“Use the dots to draw the enlarged version of the star.”
 - What should the scale factor be? [$\times 1.5$]
4. Evaluate an attempt to enlarge the star.
 - Show this drawing (right). “The red points are correct. How can we tell that the blue points are wrong?”
 - Select one of the blue points. “Where should it go?”
[Use the file **STAR-expanding.ggb** to drag the point.]
5. Predict what a continued expansion would look like.
 - Each point (right) has been moved like this:
1 unit left or right and/or 1 unit up or down.
“Imagine that we move each point the same amount again. And again ... What happens to the star’s shape?”
[Use the file **STAR-expanding2.ggb** to move all the points.]



Multiplicative Reasoning 12B

Lesson

24B

Overview

Mathematical ideas

Here we consider two enlargements of a star shape, and examine why a third transformation is *not* an enlargement.

In the first task, using an enlarged grid, students might discover that all lengths on (and inside) the star have been transformed in the same way, namely *multiplicatively*, by a scale factor of $\times 2.2$. They might also notice that the star's angles are unchanged - ie corresponding object/image lines are parallel.

In the second task, students might realise that they can produce an enlargement by *applying* a scale factor (of $\times 1.5$).

In these and the third task, geometric properties are used to assess whether the resulting shapes are enlargements, or why they are not.

Students' mathematical experiences

Students may discover some of the following

- in an enlargement all corresponding lengths are multiplied by the same amount
- in an enlargement angles remain the same
- we can use a 'centre' to enlarge a shape.

Key questions

Are they the same shape? How can you tell?

How have the lengths changed?

Assessment and feedback

The Mini-assessment activity should give you an idea of how readily students might use visual cues and how much you might need to encourage them to do so.

How readily do students spot visual inconsistencies (eg spot that lines are not parallel when they should be)?

On a more practical level,

- how well can students sketch lines *without* a ruler?
- And how well can they measure the lengths of lines with a ruler? Do they use a sensible degree of accuracy?

Adapting the lesson

You might want to simplify the tasks by enlarging just part of the star shape in Stage 1 and/or Stage 3 - eg, just the triangle ABH - this would also reduce the time spent measuring lengths in Stage 2.

In Stage 4, you might want to give more emphasis to the idea of a *centre of enlargement* (especially if it arose during the exploration of *dragging* in the *Tangram* Lesson MR-12A). Where is the centre? What properties does it have? How can it be used to check whether a point (eg H') is in the right place?

Stage 5 provides an opportunity to think about limits. Do students see that as the star-shape increases in size, its shape gets more like a rectangle? But that the rectangle-like shape remains convex (rather than puffing out like a balloon)?

A task on 'nested' shapes is given in the Revisits section.

Multiplicative Reasoning 12B

Lesson

24B**Outline of the lesson (annotated)**

1. Enlarge the star on an enlarged grid.
 - Give students a copy of Worksheet A.
“Use the dots to draw the enlarged version of the star.”
 - Encourage students to draw freehand (though they will need a ruler in Stage 2 to measure distances).
 - Discuss ways of checking the enlargement.
 - The dots can be used as a coordinate system. Angles and slopes are preserved.

2. Measure corresponding distances on the object and image.
 - “How much bigger is the enlarged shape?”
Discuss briefly.
 - Measure some corresponding distances; put them in a table.
 - You might want to start with the integer lengths AB, CD and EF (assuming your printer/photocopier has not enlarged or shrunk the worksheet).
It is also interesting to consider ‘internal’ lengths, such as the 5 cm length from D to the dot above G - are all lengths enlarged in the same way?
 - Discuss the measurements. Can we find a rule to map object-lengths to image-lengths?
[The scale factor is $\times 2.2$]
 - The 1 cm grid has become a 2.2 cm grid.

3. Enlarge the star on a 1 cm square grid.
 - Give students a copy of Worksheet B.
“Use the dots to draw the enlarged version of the star.”
 - What should the scale factor be? [$\times 1.5$]
 - You might want to consider (or return to) this after Stage 4. Check: Are all lengths multiplied by the same amount? Have angles been preserved?

4. Evaluate an attempt to enlarge the star.
 - Show this drawing (right). “The red points are correct. How can we tell that the blue points are wrong?”
 - Image-lines should be parallel to object-lines, and 1.5 times as long.
You might want to tease out the idea of a *centre* (1 cm to the right of G) from which the red-point mappings $A \rightarrow A'$, $C \rightarrow C'$, $E \rightarrow E'$ radiate.
 - Select one of the blue points. “Where should it go?”
[Use the file **STAR-expanding.ggb** to drag the point.]
 - Or use Worksheet C here and/or for Stage 5.

5. Predict what a continued expansion would look like.
 - Each point (right) has been moved like this:
1 unit left or right and/or 1 unit up or down.
“Imagine that we move each point the same amount again. And again ... What happens to the star’s shape?”
[Use **STAR-expanding2.ggb** to move all the points.]
 - Use the *slider* to expand the whole shape - but sparingly to start with: give students a chance to *imagine* what happens and to justify their predictions.
“What happens if we ‘shrink’ the star?”
The result can be very surprising!

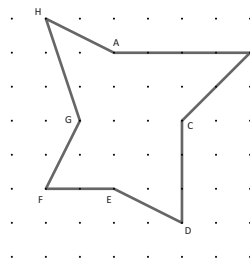
Multiplicative Reasoning 12B

Lesson

24B**Background****Worksheets**

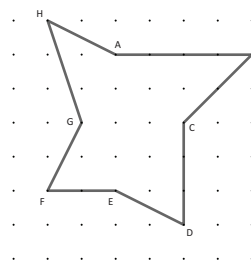
Worksheet A is for Stage 1 of the lesson. Here the grid itself has been enlarged, by a scale factor $\times 2.2$. Students should thus discover that their drawings are enlargements of (roughly) this amount.

Stars Worksheet A



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Stars Worksheet B

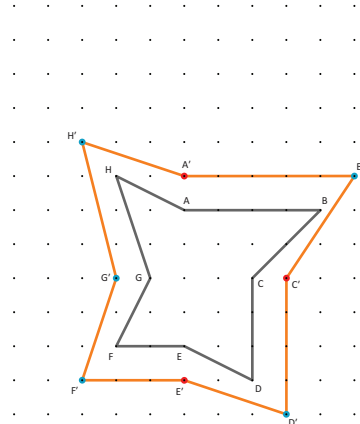


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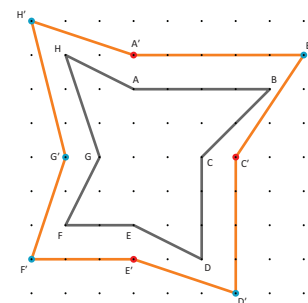
Worksheet B is for Stage 3 of the lesson. The length of AB is 4 cm, while A'B' is 6 cm, so students' drawings should be enlargements with scale factor $\times 1.5$.

Students who think additively (eg who try to add 2 cm to various lengths) are likely to get a very distorted shape or one that doesn't join up.

Stars Worksheet C



Adjust the blue vertices (B', D', F', G' and H')



Add the 'next' star

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Worksheet C can be used for Stages 4 and 5, instead of or to support the use of the GeoGebra files **STAR-expanding.ggb** and **STAR-expanding2.ggb**.

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