11 Geometry

Mastery Professional Development

11.3 Trigonometry

Guidance document | Key Stage 4

Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The fifth of the Key Stage 4 themes (the eleventh of the themes in the suite of Secondary Mastery Materials) is Geometry, which covers the following interconnected core concepts:

11.1 Transformations and relative position

11.2 Reasoning with the properties of a circle

11.3 Trigonometry

11.4 3D shapes

This guidance document breaks down core concept 11.3 Geometry into two statements of **knowledge, skills and understanding:**

11.3 Trigonometry

- 11.3.1 Reason with relationships in right-angled triangles
- 11.3.2 Reason with trigonometric relationships in non-right-angled triangles

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

- 11.3.1 Reason with relationships in right-angled triangles
  - 11.3.1.1 Use reasoning to derive exact trigonometric values
  - 11.3.1.2 Use Pythagoras' theorem and trigonometric ratios to solve increasingly complex 2D problems
  - 11.3.1.3 Use Pythagoras' theorem and trigonometric ratios to solve problems in 3D
- 11.3.2 Reason with trigonometric relationships in non-right-angled triangles
  - 11.3.2.1 Use chains of reasoning to derive the general formula for the area of a triangle
  - 11.3.2.2 Use the general formula for the area of a triangle to solve a range of problems
  - 11.3.2.3 Use chains of reasoning to derive the sine rule
  - 11.3.2.4 Use the sine rule to solve a range of problems
  - 11.3.2.5 Use chains of reasoning to derive the cosine rule
  - 11.3.2.6 Use the cosine rule to solve a range of problems

Overview

This core concept draws together and builds upon foundational work involving triangles, with a particular focus on extending the application of Pythagoras' theorem and the trigonometric ratios to three-dimensional (3D) space. The trigonometric relationships known to students are also extended to include non-right-angled triangles. Throughout, emphasis is placed on using geometric, numerical and algebraic reasoning to derive formulae.

The introduction of the sine and cosine rules at Key Stage 4 means that students' work on trigonometry is further deepened to include problems involving non-right-angled triangles. It is important that students are supported in developing their understanding of when a particular rule can be applied, so that they are able to problem solve authentically in contexts where they do not know in advance what mathematics might be required. This is an essential pre-requisite for future study of mathematics, particularly if students go on to apply their geometrical knowledge in fields such as architecture, aerospace, engineering or healthcare. Regardless of the direction students take, the ability to identify connections and select the appropriate resource or knowledge when faced with an unknown problem is a useful cross-curricular skill that will serve them well.

The ability to derive the rules is fundamental to deepening students' grasp of how the sine and cosine rules connect to their existing knowledge. Emphasising using chains of reasoning to explore ways to derive the rules, rather than providing new formulae to be memorised, provides an opportunity for students to work on algebraic manipulation in a meaningful context and a basis for developing their understanding of proof. It is important that students can recognise when they have proved or shown something to be true, skills which are essential for further study.

At Key Stage 3, students' understanding of using trigonometry in right-angled triangles in two-dimensional (2D) space was developed. This is extended at Key Stage 4 to include the application of the trigonometric ratios, to determine unknown angles and lengths in 3D. The ability to visualise right-angled triangles in 3D space is an essential part of solving 3D problems. The use of concrete objects can support students to visualise a 3D situation and extract the information needed to be able to work with 2D figures.

When working with relationships in both right-angled and non-right-angled triangles at Key Stage 4, students need to recall and apply existing knowledge when solving problems. It is important that they develop the ability to view geometrical problems through both a numerical and algebraic lens. Supporting students in making connections between the spatial/geometric and numerical/algebraic worlds is fundamental to their development of mathematical structure and fluency in solving problems in a variety of geometrical contexts.

The mathematics in this core concept is challenging and rewarding in its own right, but it also provides a set of knowledge, skills and understanding that can be applied in a diverse range of situations. Some of these are explored at Key Stage 4 – such as bearings, geometrical properties of polygons, volume of 3D shapes, and plans/elevations. It is therefore important to consider carefully the Key Stage 4 curriculum, to ensure that there is sufficient time to embed and then consolidate, so that students feel confident in applying this learning to solve problems in other contexts.

Prior learning

Trigonometry is an area of mathematics that straddles several areas of knowledge, skills and understanding, including functional thinking, graphical representations, algebraic manipulation and geometrical reasoning. It also draws on prior knowledge and experience that is scattered across the primary and early secondary curriculum. It is critical not only that students are supported to make connections between different mathematical themes, but also that teachers are alert to potential gaps and misconceptions and how they might be compounded.

At Key Stage 2, students built an appreciation of the multiplicative relationship between similar triangles, using a given scale factor to identify missing lengths. This was developed further at Key Stage 3, with students exploring similarity in more depth. Also in Key Stage 3, students extended their knowledge of right-angled triangles to include applying Pythagoras' theorem to find missing lengths. Work on similarity provided a way in to understanding the trigonometric ratios as a consistent, multiplicative relationship shared by all right-angled triangles, which they can then apply to find missing lengths and angles.

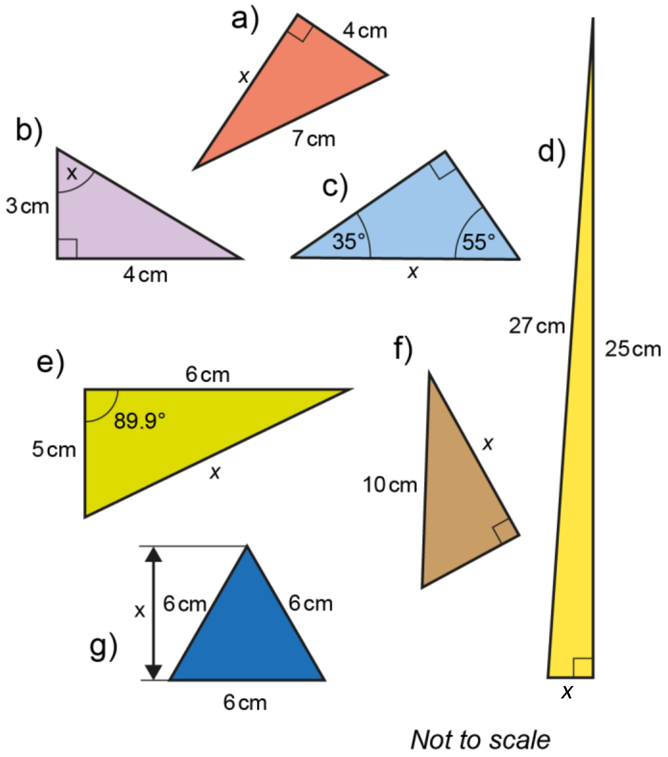
Trigonometric reasoning involves making connections between and within multiple mathematical ideas across number and geometry. It is important to check students understand that multiplicative relationships can be expressed as ratios and fractions; know how to manipulate them; and recognise that similar triangles have sides that are proportional, but angle sizes are preserved. Students should also bring extensive experience of substituting values into equations and rearranging formulae. It is important to check their ability to form and interpret algebraic expressions and equations. This is learning that has its beginnings in Key Stage 2 and will have been consolidated over Key Stage 3.

At Key Stage 4, students should encounter increasingly complex 2D problems, and problems in 3D. Students will have named and categorised 3D shapes from their earliest experiences in primary school, and that knowledge should have gradually developed to include representing 3D shapes using 2D diagrams and finding the volume and surface area. Confidence with all these aspects of working with 3D shape is important, so that they do not present a barrier to students' application of Pythagoras' theorem or trigonometric ratios in 3D contexts.

The following core concept documents from the Key Stage 3 PD materials all explore the prior knowledge required for this core concept in more depth: '*1.4 Simplifying and manipulating expressions, equations and formulae'*, '*3.1 Understanding multiplicative relationships'*, '*3.2 Trigonometry'*, '*4.2 Graphical representations'* and '*6.1 Geometrical properties'*.

Checking prior learning

The following activities from the NCETM secondary assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

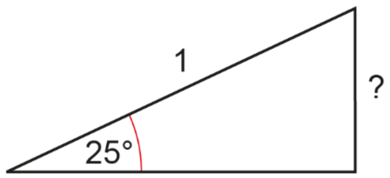
Reference	Activity
Key Stage 3 PD materials document '*6.1 Geometrical properties*', Key idea 6.1.3.2, Example 1	In which of these diagrams could Pythagoras' theorem be used to calculate x? 

Key Stage 3 PD materials document *'3.2 Trigonometry'*, Key idea 3.2.1.3, *Example 3*

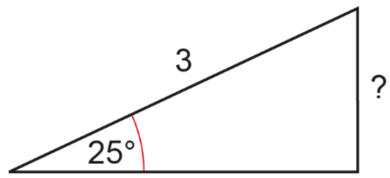
Using the relevant values in the table below (or using your calculator), find the missing side lengths.

Angle	Sine	Cosine	Tangent
10	0.1736	0.9848	0.1763
15	0.2588	0.9659	0.2679
20	0.3420	0.9397	0.3640
25	0.4226	0.9063	0.4663
30	0.5000	0.8660	0.5774
35	0.5736	0.8192	0.7002
40	0.6428	0.7660	0.8391
45	0.7071	0.7071	1
50	0.7660	0.6428	1.1918
55	0.8192	0.5736	1.4281
60	0.8660	0.5000	1.7321
65	0.9063	0.4226	2.1445
70	0.9397	0.3420	2.7475
75	0.9659	0.2588	3.7321
80	0.9848	0.1736	5.6713
85	0.9962	0.0872	11.4301
90	1	0	∞

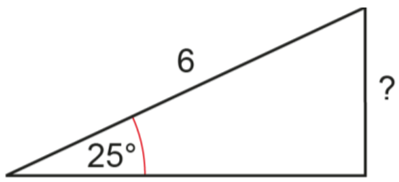
a)



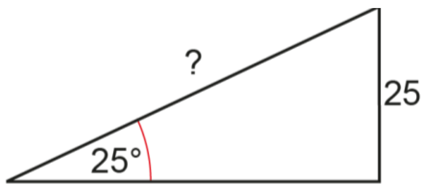
b)



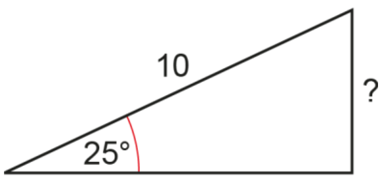
c)



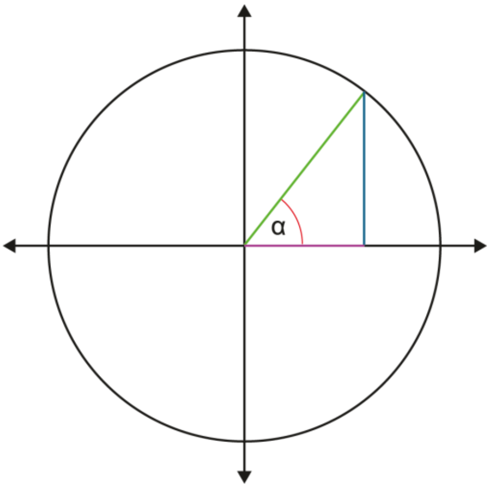
d)



e)



Not to scale

Key Stage 3 PD materials document *'3.2 Trigonometry'*, Key idea 3.2.1.3, Example 6	 What are the values of α if the length of the vertical (blue) line in this unit circle diagram is: a) 0.5? b) 0.38? c) 0.9? d) 1.2? What are the values of α if the length of the horizontal (purple) line is: d) $\frac{3}{4}$? e) 0.7777? f) −0.4?

Key vocabulary

Key terms used in Key Stage 3 materials

- adjacent
- hypotenuse
- opposite
- Pythagoras' theorem
- trigonometric functions (sine, cosine, tangent)

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
cosine rule	The cosine rule, or law of cosines, is a geometric formula that links all three sides of a triangle with one of the angles: $$a^2 = b^2 + c^2 - 2bcCosA$$ $$b^2 = a^2 + c^2 - 2acCosB$$ $$c^2 = a^2 + b^2 - 2abCosC$$ If three side lengths are known, it can be used to calculate one angle. Similarly, if two sides and an angle are known, it can be used to calculate the missing side. It is a generalisation of Pythagoras' theorem.
sine rule	The sine rule, or law of sines, is a geometric formula that links angles and sides within any triangle. It states that the ratio between each angle and its opposite side length is the same for all three sides of the triangle: $$\frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC}$$ If two angles and the lengths of one of the opposites sides are known, it can be used to calculate the length of the other opposite side. Similarly, a missing angle can be calculated if its opposite side and one other angle/side pair are known.

Key ideas

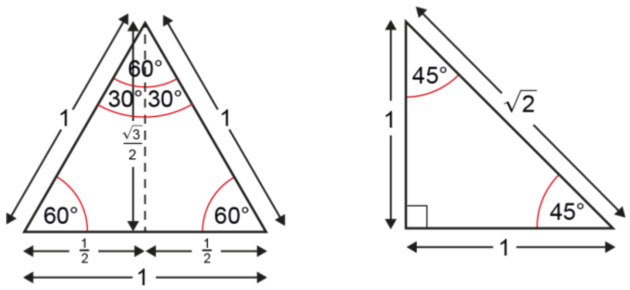
In the following list of the key ideas for this core concept, selected key ideas are marked with a 🔍. These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

11.3.1 Reason with relationships in right-angled triangles

At Key Stage 3, students are introduced to the three trigonometric ratios and develop an understanding of how to use them to solve problems involving right-angled triangles. These are explored deeply in '*3.2 Trigonometry'* from the Key Stage 3 PD materials, and so this new core concept at Key Stage 4 works from the position that, for students, the following ratios are firmly established.

$$\sin\theta = \frac{opposite}{hypotenuse},\ \cos\theta = \frac{adjacent}{hypotenuse}\ \text{and}\ \tan\theta = \frac{opposite}{adjacent}$$

At Key Stage 4, students' familiarity with the ratios is deepened further, and ways of deriving the exact values for sin, cos and tan of 30°, 60° and 45° explored. While the unit circle can be used to deduce exact trigonometric values, including for angles greater than 90°, the unit equilateral and isosceles right-angled triangles below also offer an accessible way for students to derive the necessary values:



The values for the height of the equilateral triangle, and hypotenuse of the isosceles triangle, can be derived using Pythagoras' theorem. There are then two different right-angled triangles – one with angles of 30° and 60°, and one with two angles of 45° – with which the trigonometric ratios can be explored. Substituting the relevant lengths and angles into the trigonometric ratios results in the exact values for each angle being obtained. For example, $\cos(60) = \frac{\frac{1}{2}}{1} = \frac{1}{2}$.

Providing opportunities for students to use reasoning to derive exact trigonometric values promotes a deeper understanding of the underlying mathematical structures, and develops students' fluency when recalling and applying trigonometric values. It also helps to prepare them for the geometric reasoning and algebraic manipulation that they will later need when using the perpendicular height of non-right-angled triangles to derive the sine and cosine rules.

At Key Stage 4, students' confidence with Pythagoras' theorem and the three trigonometric ratios is developed. They should encounter increasingly complex problems in 2D, involving the combination of Pythagoras' theorem and trigonometry ratios, or multiple applications of one or the other, to determine missing angles and lengths. Working on solving problems develops their understanding of how to relate diagrams of triangles to numerical relationships, and then manipulate the symbols involved.

The introduction of 3D problems requires students to visualise different 2D planes within 3D shapes and recognise what is needed to be able to determine a solution. Students may, initially, adopt a multi-stage approach: annotation of diagrams and clear notation should be encouraged. As students' work becomes more sophisticated, their understanding of the mathematical structure should extend to three dimensions. For example, they should identify that a cuboid with edges a, b and c has a diagonal of length $\sqrt{a^2 + b^2 + c^2}$, by applying Pythagoras' theorem twice.

Students should be given the opportunity to solve a variety of problems involving right-angled triangles, to develop their ability to interpret the problem and recognise the need to use Pythagoras' theorem, or one of the three trigonometric ratios, in either two or three dimensions.

11.3.1.1 Use reasoning to derive exact trigonometric values

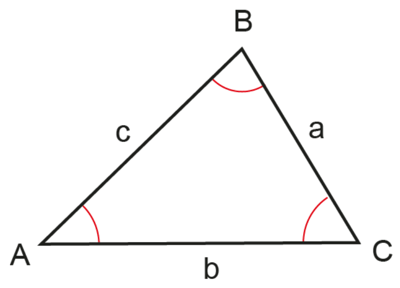
11.3.1.2 Use Pythagoras' theorem and trigonometric ratios to solve increasingly complex 2D problems

11.3.1.3 Use Pythagoras' theorem and trigonometric ratios to solve problems in 3D

11.3.2 Reason with trigonometric relationships in non-right-angled triangles

The relationship between the areas of rectangles and triangles is first introduced at Key Stage 2, with students recognising that the area of a triangle is half the area of a rectangle with the same length and width. This is formalised into the formula $area = \frac{1}{2} \times base \times perpendicular\ height$, used throughout Key Stage 3. At Key Stage 4, there is the opportunity to extend students' knowledge to include finding the areas of triangles, where the height is unknown. Students need to appreciate that any triangle can be divided into two right-angled triangles, and the trigonometric ratios applied to these triangles to find the height of the original triangle.

Application of the trigonometric ratio $\sin\theta = \frac{opposite}{hypotenuse}$ to any triangle ABC, where the 'base' is AC and the perpendicular height formed at B, leads to the height being defined as either $a \sin C$ or $c \sin A$. Time should be spent exploring this generalisation, and identifying which features are arbitrary and which are necessary. Students should understand that a perpendicular height can be drawn from any of the angles, so it does not matter what label is chosen for the angle in the expression for height; what matters is the relationship between the angle and side length used.



Application of the area formula $\frac{1}{2} \times base \times perpendicular\ height$ and the trigonometric ratio $a \sin C$ enables the general formula for the area of a triangle $\frac{1}{2}ab \sin C$ to be derived, where a and b are side lengths with enclosed angle C. Knowing how to derive the formula $\frac{1}{2}ab \sin C$ is fundamental to students making connections and recognising how it relates to, and can be derived from, their existing knowledge of the area of a triangle.

It is important also that students not only know the standard conventions for labelling the sides and angles of triangle ABC, but are supported to understand the relationships inherent in the formulae. If students are too reliant on memorising the formulae without understanding meaning, then they are vulnerable to instances where the triangles are labelled differently.

The introduction of the sine rule ($\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$) and cosine rule ($a^2 = b^2 + c^2 - 2bc \cos A$) at Key Stage 4 provides an opportunity to further deepen students' understanding of the relationships between the sides and angles in triangles and the trigonometric functions. It is important that students recognise when the two rules can be applied, as well as when rearrangement is required:

- The sine rule can be used to work out a missing side when information about two angles and a side, or two sides and a non-included angle is known. Students can rearrange the formula so that $\sin A$, $\sin B$, $\sin C$ are the numerators when finding a missing angle.

- The cosine rule can be used to determine a side of a triangle when two side lengths and the included angle are already known. It can also be applied to find the three angles in a triangle when all side lengths are known, using the rearranged formula $\cos A = \frac{b^2+c^2-a^2}{2bc}$.

The above points are fundamental to developing students' grasp of the mathematical structures of each rule, and identifying when they can be used. It is important that students recognise that, in contrast to the three trigonometric ratios (which are used to calculate angles and lengths *only* in right-angled triangles), the sine and cosine rules can be used to find the missing sides or missing angles in *any* triangle.

11.3.2.1 Use chains of reasoning to derive the general formula for the area of a triangle

11.3.2.2 Use the general formula for the area of a triangle to solve a range of problems

11.3.2.3 Use chains of reasoning to derive the sine rule

11.3.2.4 Use the sine rule to solve a range of problems

11.3.2.5 Use chains of reasoning to derive the cosine rule

11.3.2.6 Use the cosine rule to solve a range of problems

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

11.3.1.3 Use Pythagoras' theorem and trigonometric ratios to solve problems in 3D

Common difficulties and misconceptions
Students often struggle to visualise problems when working in 3D, especially when identifying right angles in a 2D representation of a 3D shape. The ability to add lines to a 3D representation to reveal a right-angled triangle is an essential skill when solving problems in 3D. It is important that students have opportunities to make these additions themselves, rather than always being presented with problems where the structure is given. It is also important to be able to visualise planes within 3D objects, as the right-angled triangles in problems involving Pythagoras' theorem are contained within a plane. Cutting solid cuboids in various ways can help with this. Concrete objects, such as empty cardboard boxes – or even the classroom – can help students to visualise planes and diagonals. This will in turn support the process of extracting the information needed to be able to apply existing knowledge of Pythagoras' theorem and trigonometric ratios in 2D, to solving problems in 3D. Recognising that 3D problems do not require any additional formulae, but instead can be solved through the application of the trigonometric skills developed for 2D triangles, is an important part of developing students' understanding of solving problems in 3D.

Students need to	Guidance, discussion points and prompts
Know how to identify right-angled triangles in 3D shapes *Example 1:* *Simon is buying a cuboid-shaped tank for his two fish, Flip and Flop. To stretch their fins, he wants them to be able to swim a distance of five feet in a straight line.* *Simon says, 'I need to buy a tank that has at least one side of length of five feet for this to be possible.'* *a) Is Simon correct?* *Simon buys a tank where the longest side is five feet. Once his fish are in the tank, he notices Flip swimming diagonally from the bottom vertex, along the glass, to the top vertex of the same rectangular face. He realises that Flip must have swum a distance greater than five feet.* *b) How does Simon know this?* *Flop swims in a straight line and covers a distance even greater than Flip did.* *c) Where in the tank might Flop have swum from and to?*	Before students can apply their learning around Pythagoras' theorem or trigonometry in a 3D context, they need to build confidence in identifying 2D shapes within 3D situations. *Example 1* provides an accessible context but does not include a visual **representation**, so that teachers can explore how readily students are able to visualise and imagine the situation. Teachers should consider how they will support those who find this challenging, perhaps by having an image or physical object ready if required. It can be a huge leap for students to understand that a 2D plane can be created using cross-sections of a 3D shape, and so this question is structured to support them with **deepening** their thinking. Students who say that Simon is correct for part a do not necessarily need to be immediately corrected, as parts b and c naturally offer a way to challenge this incorrect thinking. Teachers might like to revisit students' answers to part a after they have explored the whole of the example, to give them an opportunity to change their minds and explain why.
Example 2: *Shigeru is designing a cube for their computer game. They describe the cube using coordinates. One face has vertices: (0, 0, 0), (3, 0, 0), (0, 3, 0), (3, 3, 0)* *a) What does the third value in each coordinate mean?* *b) What is the side length of the cube?* *c) What are the coordinates of the other four vertices of the cube?* *d) Give the coordinates of the vertex that is furthest from (0, 0, 0).*	3D coordinates are not explicitly part of the Key Stage 4 curriculum, but they can be a helpful **representation** to support students in describing 2D figures within 3D shapes. Students may be unfamiliar with the idea of a third coordinate and teachers will need to use their judgment about whether to continue with this example based upon students' responses to part a. The intention is that the provision of coordinates will encourage students to bridge from thinking two dimensionally to working in three dimensions. Encourage them to sketch the shapes to support them in this. Pay attention to the **language** that students feel comfortable with and encourage them to use mathematical terms such as 'hypotenuse' and 'perpendicular' accurately.
Example 3: *The net of a cuboid is shown below this question.* *a) How many vertices, edges and faces does a cuboid have?* *b) How might you describe a particular vertex, edge or face using the letters shown in the diagram?*	In 3D space, right angles occur between lines that are perpendicular in the same 2D plane. In *Example 3*, students identify right-angled triangles within cuboids, starting with the diagonals of each face. The net **representation** is a useful starting point, as it places all six faces on the same plane. Imagining the net made up into a cuboid is the point at which students may find it more challenging to visualise the diagonals. The structure of part

c) *Complete the table to identify all the side lengths of all twelve edges. Three sides have been done for you.*

15 cm	***8 cm***	***6 cm***
DE	*DA*	*DC*

The net is folded to form the cuboid.

d) *How many different lengths of diagonal will there be?*

e) *Complete the table to identify the diagonals that will be found by each calculation. Three diagonals have been done for you.*

$\sqrt{15^2+8^2}$	$\sqrt{15^2+6^2}$	$\sqrt{6^2+8^2}$
DH	*DF*	*DB*

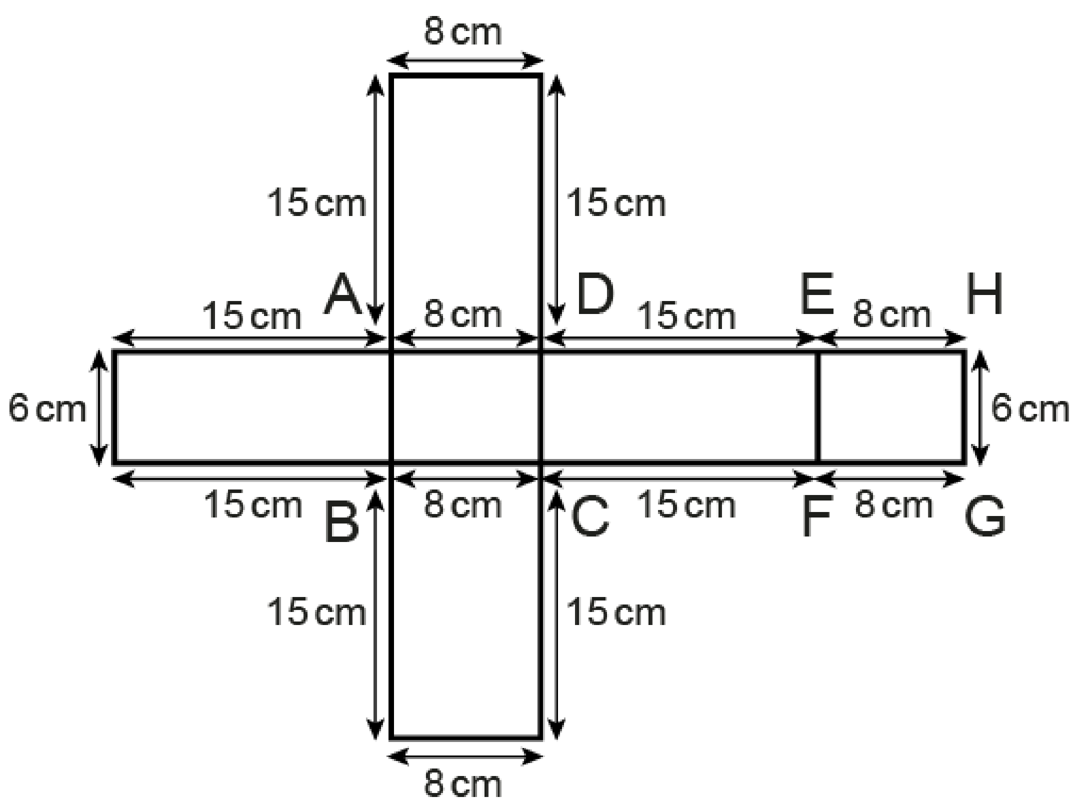
f) *Which diagonals are missing from your table? How might you calculate these?*

c is designed to draw attention to this, but having a physical net to work with may also be helpful.

At this stage, there is no expectation on students to actually calculate the lengths of the diagonals, but parts c and d explore valid approaches for doing so. Depending on the order of topics in the curriculum, this could also be an opportunity for **deepening** students' understanding about what mathematical structures are available to them based upon the information available. For example, teachers could ask:

- *'What information do you have? Why does this mean that Pythagoras' theorem is the most appropriate method?'*
- *'How would this question change if you did not know one of the lengths? What other information might you need in this case?'*

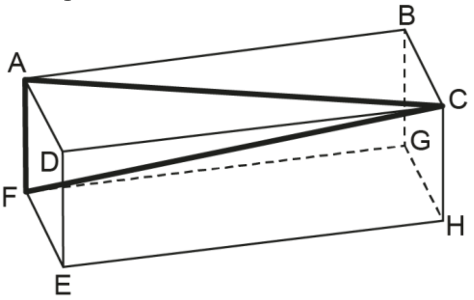
Examples 1 to *3* are presented without diagrams, so that students can start to build an understanding of the 2D triangles that can possibly be created within a 3D shape, without needing to refer to potentially confusing diagrams. It could be a useful PD exercise to ask your department to sketch the diagrams that could accompany these examples. Compare the sketches you draw: what is the same and what is different? What was the most challenging feature to capture in 2D? How might students find these diagrams misleading, and what support might they need to interpret them?



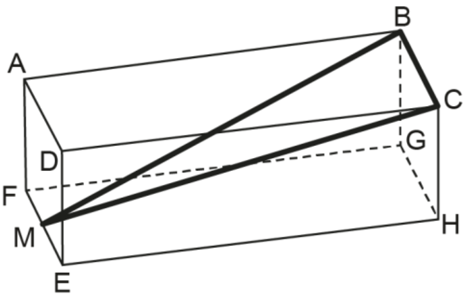
Example 4:

Three different triangles have been drawn inside a cuboid.

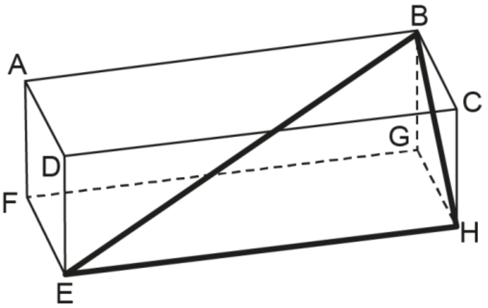
Triangle ACF:



Triangle BCM (where the point M is the midpoint of edge EF):



Triangle BEH:



Identify which of the three triangles are right-angled triangles, explaining how you know.

Students' exploration of the 2D triangles that can be found in 3D shapes culminates in an exploration of the different types of triangles that can be constructed using the edges and vertices of a cuboid. Students may assume that all such triangles are right angled, which can be accidentally reinforced by the distorted appearance of 2D shapes in 3D diagrams. This is the first time in this sequence of examples that students encounter a non-right-angled triangle. The **variation** of position of the base of the triangle is intended to support students in thinking about structure and help them to recognise the necessary constraints for a right angle to be present.

Students often struggle to visualise problems in 3D, and can find 2D **representations** of 3D shapes difficult to work with. Students who do not appreciate the plane ACFH that the triangle ACF sits on, for example, may just see an angle that looks acute at FAC. Provide students with a shoe box or something similar, where the interior is accessible; or a net, so that they can construct an open cuboid for themselves. This can be invaluable in supporting the development of proficiency when solving problems in 3D.

Students may assume that the angle between edge BC of the cuboid and CM is a right angle, due to its appearance in the 2D representation. It may be helpful to use a box, or constructed open cuboid, with pieces of string to replicate triangle BCM, to demonstrate why angle BCM is not a right angle. Encourage students to use reasoning in their explanations. As M is the midpoint of edge EF of the cuboid, if angle BCM is assumed to be a right angle, then angle CBM would also be a right angle. Recognising this, and identifying that it is not possible, is an important step in **deepening** students' ability to reason geometrically.

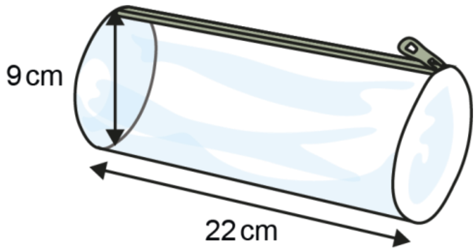
The correct use of **language** when describing the triangles and cuboids should be emphasised; it is important that terms such as vertex, edge, face and diagonal are used appropriately.

Discuss students' issues when working within 3D space. Identify available resources that can enable them to work with physical objects, rather than having to rely solely on visualisation.

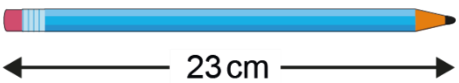
Apply Pythagoras' theorem to determine lengths in three-dimensional contexts

Example 5:

A cylindrical pencil case has length 22 cm and diameter 9 cm.



Wim has a pencil that is 23 cm long.



Show that the pencil will fit in the pencil case. Include diagrams in your answer.

In *Example 5*, students need to recognise that the base of a cylinder and the curved surface are perpendicular to one another, so Pythagoras' theorem can be applied to a right-angled triangle with length 22 cm and height 9 cm to determine the hypotenuse. They may perform the correct calculation, based on an assumption that a right-angled triangle exists, without demonstrating a good grasp of the properties of a right cylinder (i.e. a cylinder where the axis through the centre of the circular faces meets the base at right angles). Discuss how they can be sure a right-angled triangle is identifiable; focus on **deepening** understanding of the way 2D planes intersect 3D shapes.

Encourage the correct use of **language** when referring to parts of the cylinder, such as axis, base etc., to support students in identifying the structure of 3D shapes.

While the **representation** of the pencil case in this example helps to demonstrate the presence of right-angled triangles within cylinders, it is important for students to explore a physical cylinder when discussing its properties. There may be a student with a cylindrical pencil case in the class, for example, or teachers could bring in a cylinder (such as a food container) to model this.

How might students respond differently if the question was, '*Will the pencil fit in the pencil case?*' Are there other ways the question could be asked to support them in recognising the need to identify a right-angled triangle and apply Pythagoras' theorem?

Example 6:

Shigeru is designing a cube for their computer game. They describe the cube using the coordinates (0, 0, 0), (3, 0, 0), (0, 3, 0), (3, 3, 0), (0, 0, 3), (3, 0, 3), (0, 3, 3) and (3, 3, 3)

a) What is the diagonal length from:

(i) (0, 0, 0) to (3, 3, 0)?

(ii) (3, 0, 0) to (0, 0, 3)?

(iii) (0, 0, 0) to (3, 0, 3)?

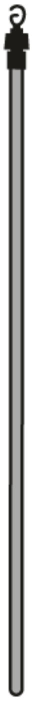
(iv) (0, 0, 0) to (3, 3, 3)?

b) For each of your answers in part a, give a different pair of coordinates that creates a right-angled triangle with the same length of hypotenuse.

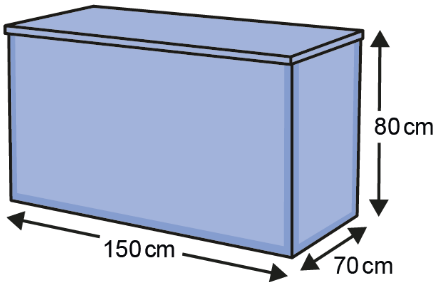
Example 6 revisits the cube explored in *Example 2,* but this time asks students to actually apply Pythagoras' theorem to find the diagonals. The **variation** is such that students should come recognise that, in a cube, there are only two possible lengths of diagonal – a property reinforced in part b, where students are asked to work in reverse and identify coordinates that form triangles with the same lengths. Students' thinking is then extended to include cuboids, but only one dimension is changed so that only some of the possible triangles change. Teachers can exploit this variation by drawing students' attention to what is the same and what is different about the two shapes explored in this example.

Teachers may find that a 3D axis **representation** helps students to visualise the coordinates. Alternatively, they could create a $3 \times 3 \times 3$ cube from multi-link blocks and demonstrate how each vertex corresponds to one of the coordinates given. Working between a physical 3D shape and 2D sketches of triangles, labelling the lengths with measurements obtained from the actual cube, can provide students with a means to understand how 2D geometry can be applied in 3D contexts.

Example 7:



Max and Nell have bought an outdoor storage container and want to store their extendable washing line prop in it.



The line prop extends to 2.4 m but only measures 175 cm when not extended.

Max says that the line prop cannot fit in the storage container, even when not extended. Nell manages make it fit.

Explain, with calculations to support your answer:

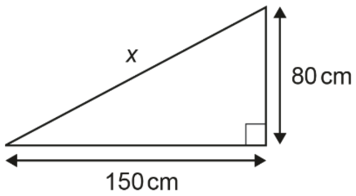
a) Why Max thought the prop wouldn't fit in the storage container.

b) How Nell manages to make it fit.

In *Example 7,* the diagonal of a cuboid is once again recognised as the longest line that can be drawn inside a cuboid. Students may need support in identifying the diagonal of the cuboid and calculating its length. It is important that students recognise which 'face diagonals' can be used to calculate the base length needed, to be able to apply Pythagoras' theorem and determine the diagonal of the cuboid. Precise **language** is key, to prevent students becoming muddled between different diagonals. Encourage them to label vertices and use two-letter notation to refer to specific lengths, even though the diagram is presented without

While the diagonal of the cuboid can be calculated by applying Pythagoras' theorem twice, it can also be expressed as $\sqrt{l^2 + w^2 + h^2}$, where l is the length, w is the width and h is the height. It is important that students understand where this **representation** of the length of the diagonal comes from and do not merely memorise it as an additional formula.

Discuss with your team the value of including a calculation as a basis for Max's conclusion in the question rubric. For example:



$$x^2 = 150^2 + 80^2$$
$$x^2 = 28900$$
$$x = 170$$

How could including this benefit students? What are the drawbacks of including a calculation or not?

Example 8:

Leo and Sabrina are improving the storage in the garden shed. They buy a cupboard in the standard shape: a cuboid.

The website says it has dimensions $100 \times 58 \times 201$ cm.

Leo plans to build the cupboard on the floor then rotate it so that it stands up properly.

He says, 'I know the internal height of my shed is 205 cm, so I will definitely be able to do that.'

Sabrina is not so sure.

a) Why might Sabrina have doubts?

b) How might Sabrina convince Leo to change his plans?

Sabrina wants to store a fishing rod in the cupboard. The fishing rod is 226 cm long. She is confident that it will fit, but Leo

Example 8 brings together students' learning on the lengths of the diagonals of cuboids. Unlike the previous example, where students had to imagine the various possible orientations of something inside the shape, this time it is the shape itself that is being hypothetically rotated. It might therefore be difficult for students to appreciate that the significant length is the diagonal of the cuboid. Students may need some help in visualising a cuboid being rotated within a limited space to appreciate why it is not as straightforward as the height of the wardrobe being less than the height of the room. Dynamic **representations**, sourced online or made from classroom materials, can be key. For example, modelling a rotation from horizontal to vertical first with a flat piece of card (i.e., a negligible depth), and then with cuboids of increasing depth, can help students to see how the length of the diagonal increases with the depth of the cuboid.

Discuss with your team the range of 'real-life' contexts that students experience when exploring geometry problems such as this. It is easy for situations to become contrived when the mathematical modelling needs to be relatively straightforward, and it is *remembers the consequences of his hubris (having to rebuild the cupboard) and has doubts.*

c) Is it possible to fit the fishing rod in the cupboard?

important to get a balance between realism and accessibility. Are there any particular contexts that will be relevant to your students, that could be adapted to reinforce learning on applying Pythagoras' theorem in 3D? For example, local landmarks, recent building work or school events.

Estimate angles in the right-angled triangles that can be created within 3D shapes

Example 9:

In the diagrams below this example, three different right-angled triangles are shown in the cube ABCDE.

a) Organise the angles DBC, CDB, HED, EDH, GEC and ECG into the table below:

< 45°	*= 45°*	*> 45°*

The lengths AB, CD, EF and GH are extended so that the cube becomes a cuboid.

b) How will this change the position of the angles in your table from part a?

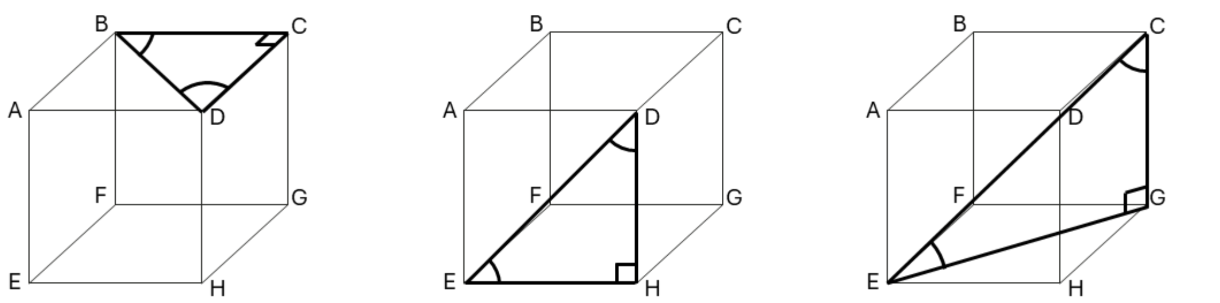
c) For each of the angles in your table, identify other angles within the cube that are the same size.

Before launching into work on 3D trigonometry, it is important to take stock and consider what additional challenge there might be for students. It is tempting to assume that the preliminary work on identifying right-angled triangles for applying Pythagoras' theorem in 3D will be sufficient, but identifying and working with angles provides another layer of difficulty. The mathematical demand of *Example 9* is therefore kept relatively low – estimating angles in relation to half of a right angle – so that the focus can be on **deepening** their capability with interpreting and visualising 2D shapes in 3D space.

2D **representations** of 3D shapes are not always easy to construct or decipher. Ensure students are alert to any distortions in the perspective of the diagram, and that they draw conclusions from mathematically sound reasoning, rather than the appearance of the shape on the page.

The **variation** of the triangles is such that, in part b, only two of the triangles will change as the lengths increase. This should support students in recognising which plane the triangles exist on. Students should be encouraged to notice that the angles of each triangle come in pairs: either both in the centre column, or one in each of the outer columns.

Asking students to reason and categorise, rather than find precise answers, can help to develop their reasoning and sense of structure. It can be easy to get out of the habit of providing students with experience such as this – particularly at Key Stage 4, where there is a competing need to prepare them to answer examination questions. With colleagues, reflect on the curriculum materials you plan to use over the next few weeks. Are there opportunities to adapt any exercises to develop reasoning, using an approach like this?



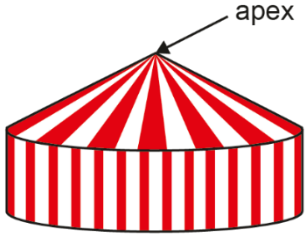
Use trigonometric ratios to determine lengths and angles in three-dimensional contexts

Example 10:

One of the acts in a circus is performed by a trapeze artist. The starting position for the act is the apex of the circus tent.

The trapeze artist climbs a 5 m ladder that is attached to the side wall of the tent, and then uses a rope attached from the top of the side wall to the apex of the tent

The circus tent has a height of 10 m and a diameter of 24 m.



a) *Use the information above to sketch and label a diagram of side view of the circus tent.*

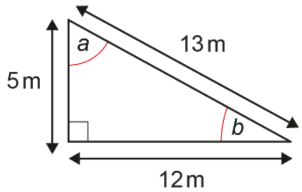
b) *Calculate the length of rope needed for the trapeze artist to be able to climb to the apex.*

c) *What angle does the trapeze artist make with the axis of the tent when climbing the rope?*

In *Example 10*, students need to appreciate that a right angle exists between the axis of the cone that forms the roof of the tent and the radii of its base (also the cylinder that forms the walls of the tent). This may need unpicking. Students should be encouraged to annotate the illustration, or their own **representation** of the situation.

This context is an opportunity for **deepening** students' understanding of how to work with the information in the question rubric and diagram; there are some preliminary steps before the relevant information can be extracted. For part a, subtracting the length of the ladder (equivalent to the height of the wall of the tent) from the total height of the tent gives the height of the roof. A right-angled triangle can then be extracted and Pythagoras' theorem applied to determine the length of rope.

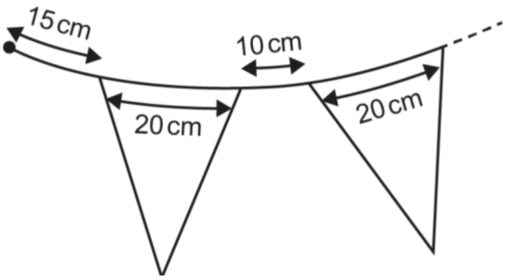
When determining the angle for part c, the **language** of 'axis' may need clarification. Drawing a diagram of the relevant triangle (right) may help. Some students may calculate angle a, others, the size of angle b in this right-angled triangle. Rather than saying that the latter is wrong, work with them to identify what further step they can take to find angle a using their information that $a = 90 - b$.



Students often struggle to recognise what is an appropriate degree of accuracy, especially when it is not specified in the question. Discuss with your colleagues what degree of accuracy students might use when giving the angle in part b, and what degree of accuracy they would expect in their classroom. Is there cohesion among your team about this?

Example 11:

Laurie is making bunting to be used to decorate the inside of the baking competition marquee for the school fete.

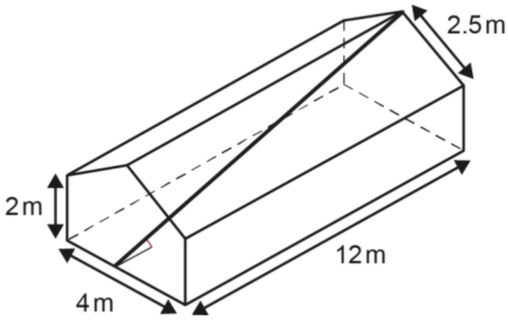


The fabric bunting triangles are 20 cm wide and are attached to the bunting string with a 10 cm gap between each fabric triangle and a 15 cm length of string at each end.

In *Example 11,* students apply their learning to a real-life scenario, with an emphasis on making connections between different mathematical ideas when solving a problem. Students' experience with this context may be variable, and so teachers may first need to check that students know that a marquee is a type of tent, and bunting a decoration made up of flags on a string. The need to apply Pythagoras' theorem is also not made explicit. Students must recognise that, to be able to determine the number of fabric triangles required when making the bunting, they need to first determine the distance between the top vertex at the back of the marquee and the midpoint of the floor at the front of the marquee. Providing the slant length (hypotenuse) and width of the roof requires students to carry out a rearrangement of Pythagoras' theorem, to determine the roof height. Emphasise the careful use of **language**, to distinguish slant length from perpendicular height.

When attempting a complex problem of this type, it is important that students can explain both what they are

The bunting is to be attached to the top vertex at the back of the marquee to the midpoint of the floor at the front.



The marquee is 12 m long and 4 m wide and the height of the side wall is 2 m. The roof of the marquee has a slant length of 2.5 m.

a) How many fabric triangles will Laurie need, to be able to make the bunting?

Laurie wants to check that when the bunting is in place in the marquee, it won't cause a trip hazard.

b) What angle will the bunting make with the floor of the marquee? Give your answer rounded to the nearest degree.

doing and why they are doing it. Encouraging them to give reasons for each of their calculations helps with **deepening** their understanding of the problem-solving process. Once the height of the marquee has been established, the distance between the top vertex at the back of the marquee and the midpoint of the floor at the front of the marquee can be calculated.

When identifying the number of triangles Laurie needs for part a, it is important that students recognise there will be one less 10 cm gap than the number of triangles. They may form an algebraic equation such as:

$$20n + 10(n-1) + 30 = 1250$$

(where n is the number of fabric triangles)

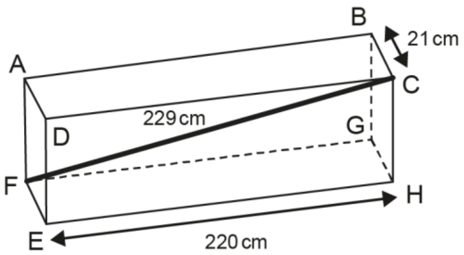
Encouraging the use of algebra as a **representation** of the unknowns in the problem is key to developing students' conceptual understanding. It also emphasises that, while the application of Pythagoras' theorem is an important part of the solution process here, there is often a need to recall and apply other ideas and methods to solve a problem.

When calculating the angle that the bunting makes with the floor of the marquee for part b, students can choose any of the three trigonometric ratios to use, as all three side lengths of the right-angled triangle are known. Discuss with teachers how they handle situations where there are multiple routes to the right answer. How do they achieve a balance between helping students work efficiently, and understanding the equivalence of the different methods?

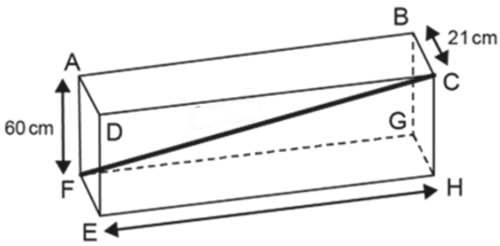
Example 12:

CF is the diagonal of a cuboid.

a) Find the angle between CF and the plane EFGH.

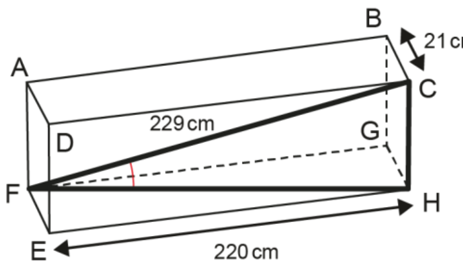


b) Find the angle between CF and the plane ABCD.



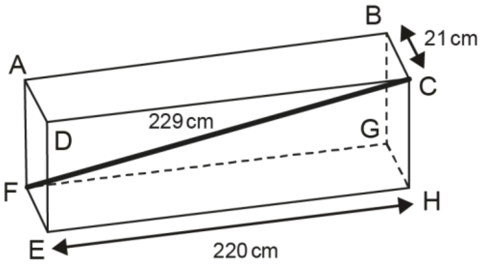
The focus of *Example 12* is on **deepening** students' understanding of when a particular trigonometric ratio is applicable. There is a particular emphasis on the way in which, unlike the fixed hypotenuse, the opposite and adjacent sides are dependent on the specified angle.

Annotating the **representation** so that all three sides of the triangle, as well the relevant angle, are clear each time will help students in identifying relevant information. For example, for part a:



The **variation** between parts a and b draws students' attention to the relationship between the angle between CF and the plane EFGH, and the angle between CF and the plane ABCD. Students should recognise that, regardless of the height of the cuboid, the angle between the diagonal of the plane ABCD and the diagonal of the cuboid CF, and

c) *Find the angle between CF and the plane BCGH.*



the angle between the diagonal of the plane EFGH and diagonal CF, are equal and alternate. This plays an important part in developing an understanding of structure and relationships between angles. In part c, the cuboid is presented in the same way as for part a, with the length of the cuboid's diagonal given, to draw students' attention to the angle that needs to be found.

Ensure that students are precise with the **language** they use and are differentiating between the diagonal of a face of a cuboid and the diagonal of the cuboid itself. Students may recalculate the diagonal FH, without recognising that it is the same length as diagonal AC found in part a. Students may also not recognise that the same cuboid has been explored in all three parts of this example. Ask them to comment on what they notice about the angles calculated, supporting them to reason about why the three angles considered are connected in the way that they are.

Unpick the design of this question with your team. How has this example been constructed to ensure that students have an opportunity to apply all three of the trigonometric ratios? How does supplying the length of the cuboid's diagonal affect the focus of the example? What difference would it make to change the order of parts a to c? What support could be given to students to help them to identify the salient information in the diagram?

11.3.2.5 Use chains of reasoning to derive the cosine rule

Common difficulties and misconceptions

The cosine rule is a generalisation of Pythagoras' theorem. It relies on an understanding that all triangles can be divided into two right-angled triangles by selecting a vertex and drawing a perpendicular line to the opposite side. This recognition provides an opportunity for students to make connections between problems involving non-right-angled and right-angled triangles. However, it can sometimes result in students incorrectly applying Pythagoras' theorem in triangles that do not contain a right angle.

When deriving the cosine rule, it is important that students understand and can use the standard conventions for labelling a triangle ABC, and take care to ensure that the correct sides and angles are being used. This can cause confusion where students use different letters to label the same angles, and so they need to be clear that what matters is the relationship between corresponding sides and angles.

Being confident in their knowledge of what constitutes a proof is fundamental to developing students' understanding, and it is important that they recognise whether or not a proof has been achieved. Refer back to earlier work on algebraic proof, and on proving angle rules, to check that students are aware of what information is required to demonstrate that something is true in the general case.

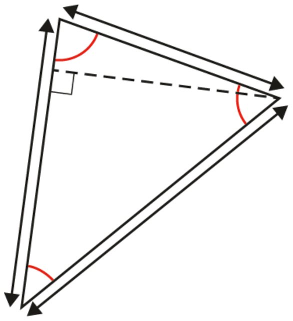
Students need to	Guidance, discussion points and prompts

Students need to

Identify relevant features of triangles

Example 1:

Mr Cooper's class is learning about trigonometry. He asks the students to label this triangle so that they can find the perpendicular height shown:



When he looks along the front row, he sees four different responses. These are shown below the questions.

a) *What is the same and what is different about the way each student has labelled the triangle?*

b) *Some of the differences are because of a mistake. What should Mr Cooper say to help these students?*

c) *Are any of the differences unimportant? Why?*

d) *Which of the students would be able to find the perpendicular height shown if they were given values for the three things they have labelled so far?*

e) *Which of the students has enough information to also calculate the area of the triangle?*

Guidance, discussion points and prompts

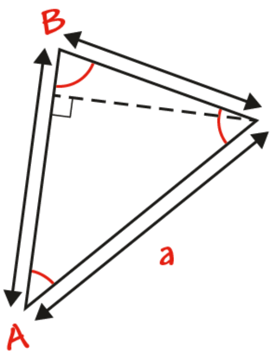
Some of the difficulties that students have with complex equations such as the cosine rule can be addressed by ensuring that teachers are aware of common errors and misconceptions. For example, before students can work with the rule to solve problems, teachers need to be confident that they can use the labelling conventions that it refers to. The **variation** in this example is designed to draw out some key preparatory learning points – such as ensuring students know that, conventionally, angles are labelled with capital letters and sides in lower case, and that opposite sides/angles have the same letter.

The questioning that teachers employ alongside this example can be used for **deepening** students' understanding about the information that they need when working with triangles. For example:

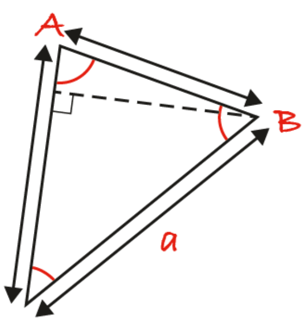
- *'Could Daisy find a different perpendicular height with the information she has? If so, which one?'*
- *'Beth hasn't labelled the angle she needs, but why **doesn't** she need more information to find the height?'*
- *'Anwar has the right information to find the height, but has not used the normal labelling convention. What would the equation look like using his labels? Why do we use a labelling convention?'*
- *'Why can't Daisy use trigonometric ratios to find the side that she has labelled? Why can't Beth use the angle that she has labelled?'*

This example has been stepped in such a way as to support teachers to understand the intended learning points. However, a valuable professional development exercise could be achieved by offering teachers just parts d and e of the example to enable them to discuss their own understanding of the conventions for labelling sides and angles, and the difference between arbitrary and necessary information. For example, whilst Anwar is 'wrong' in identifying side a as adjacent to angle A, rather than opposite it, he still nonetheless could find the perpendicular height using the calculation $a \sin A$.

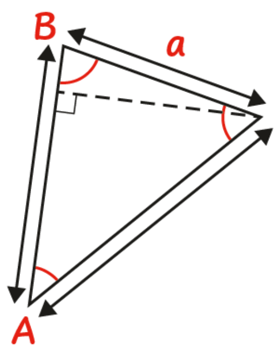
Anwar:



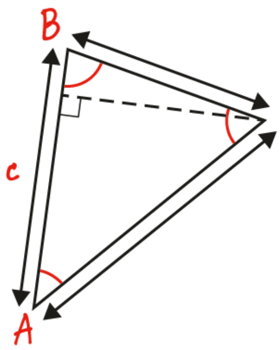
Beth:



Callum:



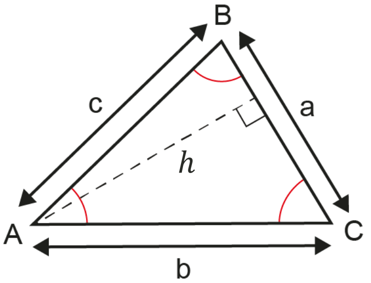
Daisy:



Use the fact that all triangles can be divided into two right-angled triangles

Example 2:

Ms Hawkins offers her class three pieces of information about the triangle below. She asks students to choose which three pieces of information they would like.



Shane chooses A, B and C.

Annabelle chooses a, b and c.

Chinraj chooses A, b and c.

Nic chooses a, b and C.

a) *Which of the students would be able to find h?*

b) *For those that could not find h, how might you change their chosen information so they can?*

c) *Which of the students would be able to find the area of the triangle?*

d) *For those that could not find the area of the triangle, how might you change their chosen information so they can?*

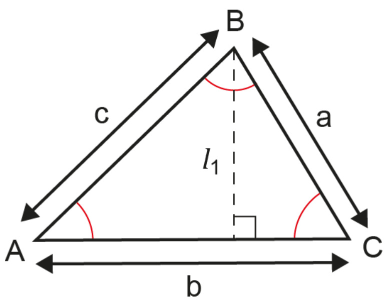
Example 2 continues to explore the measurements needed to find missing information in triangles. The **variation** draws students' attention to the relationships between sides and angles, so that they realise which combinations can prove fruitful when trying to calculate the perpendicular height. By comparing Chinraj and Nic's choices, students should realise that only two pieces of information are needed: the hypotenuse of the right-angled triangle formed by h, and the angle this length forms with the base. Nic does not need to know a to find h; Chinraj cannot find h because the angle she has chosen is split, by h, into two angles of unknown size. Since the cosine rule is derived from Pythagoras' theorem, it is possible to for Annabelle to find h with some algebraic manipulation (as explored in *Example 9*). Teachers may therefore like to revisit this example once the cosine rule, and its relationship to Pythagoras' theorem, has been established.

Repeated **language** structures, such as 'always, sometimes or never' statements, can help students notice patterns and draw conclusions about how the values in the triangles interrelate. For example, are the bulleted statements below always, sometimes or never true?

- *'If you know two angles in a triangle, you always know the third.'*
- *'If you know two lengths in a triangle, you always know the third.'*
- *'Knowing the base and one of the sides is enough to find the perpendicular height of any triangle.'*
- *'The height of any triangle can be found with $x \sin y$, where x is one of the sides that meets the apex, and y is the angle this side makes with the base.'*
- *'It is not possible to use $\tan$ or $\cos$ to find the height of a triangle.'*

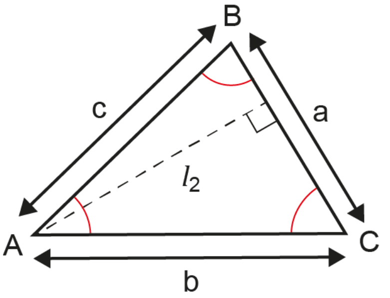
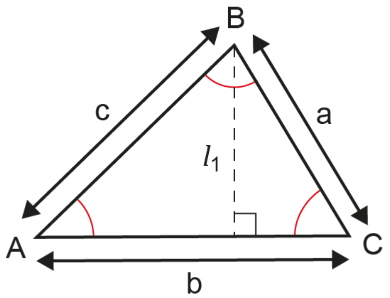
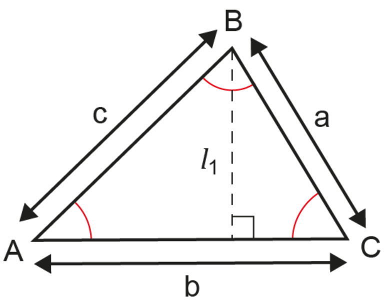
Example 3:

a) *What is the minimum information that you need to find the perpendicular height of this triangle, labelled l_1?*

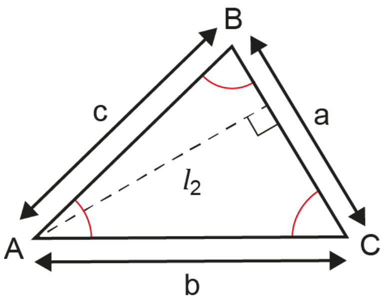


Example 3 builds on the learning of *Example 2,* asking students to consider what information they need to find the perpendicular height. Students should be encouraged to share and compare answers, as they could choose to use either of the complete angles to form a right-angled triangle. Ensuring that students are clear about the relevant relationships, as opposed to just the relevant labels, can help with **deepening** students' understanding of the structures behind the cosine rule.

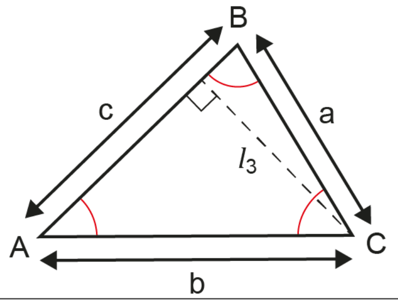
It may be helpful to use the **language** of 'arbitrary' and 'necessary' with students in defining which information is essential. For example, when using $\cos\theta$ to find the height in this way, it is crucial that the angle being used is complete (i.e., not subdivided by the perpendicular height), and that the length being used is the hypotenuse formed between that angle and the base. It is unimportant what labels are used, as long as those relationships are

b) *What is the same and what is different about the information you need to find the perpendicular height of this triangle, labelled l_2?*  	maintained – although it is helpful to use labelling conventions consistently, to prevent confusion. It is important that students experience different orientations of triangles, so that they do not associate the cosine rule solely with a **representation** where the base is horizontal and the perpendicular height is vertical. This can help them to go on to identify situations where the cosine rule is needed in different contexts.
Example 4:  *Write an equation to find l_1 if:* a) *A = 42° and c = 8.5 cm* b) *A = A° and c = c cm* c) *A = 38° and b = 15 cm* d) *A = A° and b = b cm.*	In *Example 4*, students continue to write equations using the trigonometric ratios but begin to move from using values to using expressions to represent lengths. The **variation** in this example is such that the questions come in pairs: part b is simply a repeat of part a, but with letters instead of numbers. Parts c and d are similarly related, but students need to recognise that they do not have sufficient information here to write an equation. They should, instead, be encouraged to generate their own unknown to represent the missing length. Students could designate the length from A to the point where l_1 meets b as x, so that part c is $l_1 = x \sin 38$ and part d is $l_1 = x \sin A$. Alternatively, they could designate the length from C to the same point as x, so that the answers to c and d are $l_1 = (15 - x) \sin 38$ and $l_1 = (b - x) \sin A$ respectively. Recognising the equivalence of these two different ways to find l_1 is an important step in understanding the derivation of the cosine rule.
Example 5: *Triangle ABC is divided into two right-angled triangles in three different ways, as shown below.* *Express the lengths of the three dividing lines in terms of lengths a, b, c, and angles A, B and C.* *(i)*  	*Example 5* explores the ways in which a non-right-angled triangle can be divided into two right-angled triangles, by selecting a vertex and drawing a perpendicular line to the opposite side. It is important that students understand why this is helpful. Ask them to describe the lengths of the three perpendicular divides in terms of lengths a, b, c, and angles A, B and C. This will focus their attention on selecting which angle to use and correctly identifying the sides as opposite and hypotenuse. Confidence with these **representations** is essential, as students need to be happy that all triangles can be split into two right-angled triangles in order to understand how the cosine rule can be derived. Students should be encouraged to label the perpendicular divides as 'opposite' and identify the hypotenuse they wish to use in each case, to create equations for the sine of the angle. Rearranging the relationships gives two equations for the height each time: $l_1 = a \sin C \;\; or \; c \sin A$, $l_2 = c \sin B \;\; or \; b \sin C$ and $l_3 = b \sin A \;\; or \; a \sin B$. Students should understand that it is not necessary to identify both relationships in each case, but recognising both exist is key to **deepening** their understanding of mathematical

(ii)



(iii)

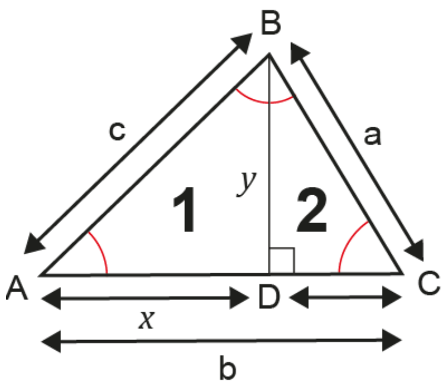


structure. Emphasising that one may be preferable to another, depending on the context, is particularly helpful when exploring the derivation of the cosine rule.

While this example considers the division of a non-right-angled triangle into two right-angled triangles, any triangle, including right-angled triangles can be divided in the same way. When might it be helpful to consider dividing a right-angled triangle into two right-angled triangles?

Example 6:

A triangle is split into two right-angled triangles, labelled 1 and 2.



a) *Write an expression for the base of triangle 2.*

b) *Write an equation for y^2 using first triangle 1 and then triangle 2.*

c) *What is the same and what is different about your two answers in part b?*

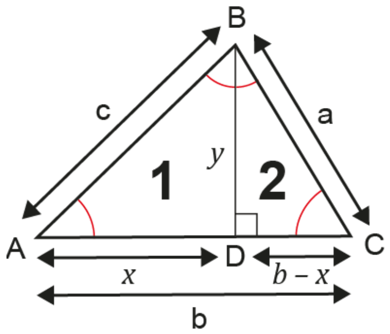
d) *Rewrite your answers to part b to eliminate y^2.*

In *Examples 6* to *8*, the steps for proving the cosine rule are broken down into smaller stages. None of these examples is sufficient alone to understand the proof. Teachers could choose to work through all of them, or to use a more complete example such as *Example 10* below, and then refer back to the parts in this example if any specific details need further explanation. In *Example 6*, the focus is on **deepening** understanding of the two different ways in which it is possible to define the perpendicular height of the triangle.

The **language** in this example refers back to students' work on simultaneous equations, to help them recognise that they are employing similar skills of algebraic manipulation. It is in recognising the equivalence of the two different equations for y^2, and then eliminating this variable, that students are able to create an equation in terms of x. Proof of the cosine rule requires students to draw on learning from across the mathematics curriculum, and it is important to draw their attention to these links.

Example 7:

A triangle is split into two right-angled triangles, labelled 1 and 2.



Which of the equations below are correct?

$\cos A = \frac{b}{c}$	$\cos A = \frac{y}{c}$	$\cos A = \frac{x}{c}$
$\cos B = \frac{x}{c}$	$\cos B = \frac{b-x}{a}$	$\cos B = \frac{y}{c}$
$\cos C = \frac{b-x}{a}$	$\cos C = \frac{b}{a}$	$\cos C = \frac{a}{b}$

The skill isolated in *Example 7* is that of identifying the relevant values when generating equations involving the cosine ratio. When working with right-angled triangles within a non-right-angled triangle, it is common for students to struggle to identify which sides and angles they can use. The **variation** in this example supports teachers to facilitate discussions around which information will never be valid. For example, students should reject the entirety of the middle row, as, in this case, angle B has been subdivided into two angles of unknown size. Similarly, any equation involving the length *b* is not valid, as this length is not part of either of the two right-angled triangles.

From this example, students could be supported to generalise that the height of any non-right-angled triangle is $c \cos A$, where A is one of the complete angles and c the hypotenuse that is used to form that angle with the base of the triangle. Achieving this level of fluency in generating expressions for unknown lengths in this way is a significant step towards **deepening** students' understanding of trigonometry in preparation for further study.

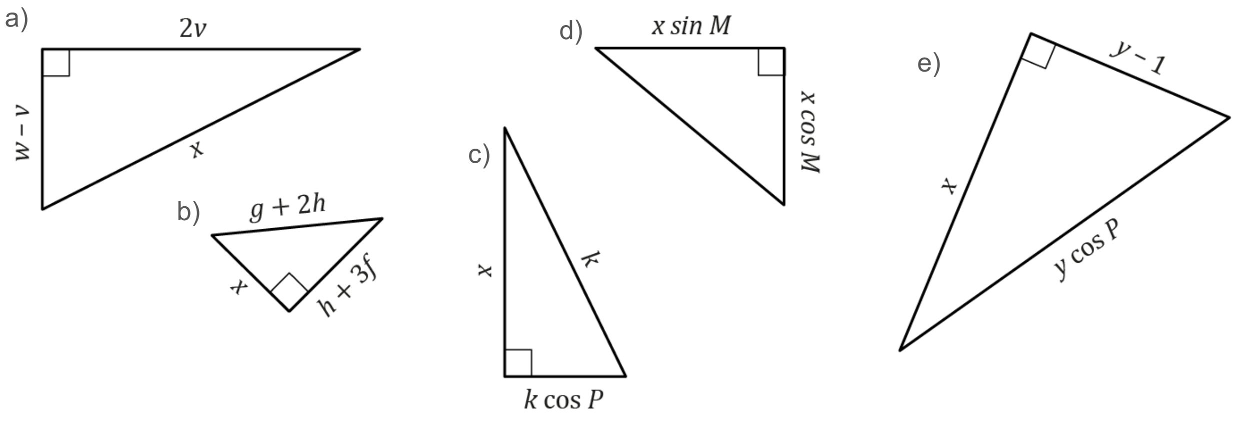
Example 8:

Below this example are five right-angled triangles.

Use Pythagoras' theorem to write equations to find the missing length x each time.

In *Example 8*, students are asked to create equations using lengths that are not single values or unknowns, but expressions involving a combination of both. The intention is to explore whether or not students are aware that the entirety of the expression needs to be squared (or square rooted), and to identify any misconceptions with algebraic manipulation. The **language** used when reading out such expressions can play a large part in both creating and addressing such misconceptions. For example, if a student hears '*w – v squared plus two v squared*' for part a, then they may not be clear that the whole of $(w - v)$ is squared.

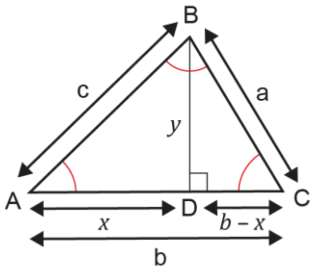
Discuss with both students and teachers the need for precision with language in such cases. As a team, do you have a consistent approach to 'reading' expressions and equations?



Follow the chains of reasoning involved in deriving the cosine rule

Example 9:

Quince is using the diagram below to prove the cosine rule.



a) *Using the table below, place cards A to D in the correct positions to complete the steps of the derivation.*

b) *Complete the explanation boxes for steps 4 to 7.*

	Derivation	Explanation
1	$c^2 = x^2 + y^2$ $\Rightarrow y^2 = c^2 - x^2$	Applying Pythagoras' theorem to triangle ABD.
2		Applying Pythagoras' theorem to triangle BCD.
3		Equating equations from steps 1 and 2.
4		
5	$\cos A = \frac{x}{c}$ $\Rightarrow x = c \cos A$	

In *Example 10,* students place the steps to proving the cosine rule (provided) in the correct order and complete the explanations for each step. The subtle shift involved in asking students to explain the steps in a proof, rather than completing the steps for themselves, helps with **deepening** understanding of the reasoning involved in a convincing argument. When the correct order has been established (D, B, A and C), it is important to check that students are convinced that Quince's derivation constitutes a proof, identifying that it involves the application of theorems and rules that can be/have been proved in order to achieve the cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$. Discuss the different rearrangements of the cosine rule; $b^2 = a^2 + c^2 - 2ac\cos B$ and $c^2 = a^2 + b^2 - 2ab\cos C$ and how they can be derived in a similar way. Recognising that $\cos A = \frac{b^2+c^2-a^2}{2bc}$ is a rearrangement of $a^2 = b^2 + c^2 - 2bc\cos A$ and does not require an additional proof, is an important step in the development of students' understanding of mathematical structure.

Students may manipulate the algebra slightly differently to the way that Quince has, and it is important that this is discussed and validated as an alternative method. When discussing the steps of the proof, check students' use of **language** when explaining the ways in which the algebra has been manipulated. They may refer to the expansion of brackets, collection of like terms and substitution, for example, and it is important that they are using the correct language to describe the algebraic manipulation that has taken place.

There are a number of ways to approach deriving the cosine rule. For example, the cards containing the missing steps could be omitted and students asked to complete the missing steps themselves, or cards containing the reasoning explanations could be added. Discuss the ways that the derivation could be explored and what each one helps to focus on. What, if any, are the benefits of looking at more than one proof of the cosine rule? How might consideration of a proof that includes mathematics at a higher-than-expected level affect students' engagement with it? What key ideas do students need to understand to be able to access a derivation of the cosine rule?

6	$c^2 = a^2 - b^2 + 2bc\cos A$	
7		

A: $c^2 = a^2 - b^2 + 2bx$

B: $c^2 - x^2 = a^2 - (b^2 - 2bx + x^2)$

C: $a^2 = b^2 + c^2 - 2bc\cos A$

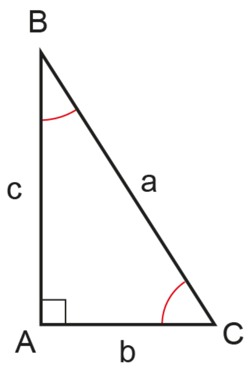
D: $a^2 = (b-x)^2 + y^2 \Rightarrow y^2 = a^2 - (b-x)^2$

Appreciate Pythagoras' theorem as a special case of the cosine rule

Example 10:

Kit and Leigh are talking about the cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$.

Kit tells Leigh that the cosine rule is just Pythagoras' theorem, because in triangle ABC angle A is a right angle.



Explain whether or not what Kit says is true.

Example 10 focuses on the fact that the cosine rule is a generalisation of Pythagoras' theorem. While the cosine rule can be used in any triangle where we are trying to relate all three sides to one angle, its most useful application is in non-right-angled triangles. However, applying the cosine rule to a right-angled triangle helps with **deepening** students' understanding of how it relates to Pythagoras' theorem and supports them in making connections between what they already know and this new formula.

It is likely that students will recognise that $a^2 = b^2 + c^2 - 2bc\cos A$ resembles the Pythagorean theorem, but with an additional third term on the right-hand side. To make the third term equal to zero, $\cos A$ needs to be equal to zero, which is indeed the case when $A = 90°$.

$$\boldsymbol{a^2} = b^2 + c^2 - 2bc\cos 90 = b^2 + c^2 - 2bc \times 0 = \boldsymbol{b^2 + c^2}$$

When exploring the above relationship, students need to understand the importance of identifying the hypotenuse. The correct use of **language** to describe the sides of the triangle in relation to the angles should be encouraged.

Students often struggle to remember which formula they need to apply, and it is important to develop their understanding to reduce the reliance on memory recall. How can the exploration of the application of the cosine rule to a right-angled triangle help to emphasise that the cosine rule works for any triangle and not just right-angled triangles as is the case with Pythagoras' theorem?

Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 11 Geometry* can be found here:

https://www.ncetm.org.uk/media/3xcpkpft/ncetm_ks4_cc_11_solutions.pdf