MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Multiplication and Division

2.12 Division with remainders

Teacher guide | Year 4

Teaching point 1:

Objects can be divided into equal groups, sometimes with a remainder; objects can be shared equally, sometimes with a remainder; a remainder can be represented as part of a division equation.

Teaching point 2:

If the dividend *is* a multiple of the divisor, there is *no* remainder; if the dividend *is not* a multiple of the divisor, there *is* a remainder. The remainder is always less than the divisor.

Teaching point 3:

When solving contextual problems involving remainders, the answer to a division calculation must be interpreted carefully to determine how to make sense of the remainder.

2019 pilot

Overview of learning

In this segment children will:

- build on what they have learnt, in segment *2.6 Quotitive and partitive division, Teaching point 1*, about dividing quantities into equal groups with some left over
- use the language of remainders to describe the quantity left over
- represent situations with remainders using both mixed-operation multiplication equations (e.g. $14 = 3 \times 4 + 2$) and division equations (e.g. $14 \div 4 = 3\ r\ 2$)
- explore division with remainders in both quotitive (grouping) and partitive (sharing) contexts, using both skip counting and known multiplication facts to solve problems
- explore how the remainder *must* be less than the divisor, using manipulatives and systematically increasing the remainder to support their understanding that when the 'remainder' is equal to the divisor, another whole group can be made
- learn to predict whether a division calculation will involve a remainder, based on whether the dividend is a multiple of the divisor or not.

Because this segment has such strong links to segment *2.6*, it is recommended that teachers ensure that they are familiar with that segment, and are confident with the quotitive and partitive structures of division, and the language used to describe them.

This segment begins by reviewing the preparatory learning about remainders from segment *2.6, Teaching point 1*, where children learnt to describe the action of dividing quantities into 'equal groups with some left over' using the language of remainders (for example, *'nine is divided into four groups of two with a remainder of one'*), and mixed-operation multiplication equations (e.g. $9 = 4 \times 2 + 1$). The learning then builds directly from both this and children's experience of solving quotitive and partitive division problems (without a remainder). Time is spent working through both quotitive and partitive division contexts, in each case beginning with no remainder and then increasing the dividend in order to introduce a remainder, which is included in the resulting division equation:

- Quotitive division context:
  - *'A baker has **twelve** cakes. He sells cakes in boxes of **four**. How can he box the cakes?'*
    $12 = 4 \times 3$ $12 = 3 \times 4$ $12 \div 4 = 3$
  - *'A baker has **thirteen** cakes. He sells cakes in boxes of **four**. How can he box the cakes?'*
    $13 = 4 \times 3 + 1$ $13 = 3 \times 4 + 1$ $13 \div 4 = 3\ r\ 1$
- Partitive division context:
  - *'**Eighteen** toy cars are shared equally between **three** children. How many cars does each child get?'*
    $18 = 3 \times 6$ $18 = 6 \times 3$ $18 \div 3 = 6$
  - *'**Nineteen** toy cars are shared equally between **three** children. How many cars does each child get?'*
    $19 = 3 \times 6 + 1$ $19 = 6 \times 3 + 1$ $19 \div 3 = 6\ r\ 1$

Note that the two multiplication equations in each example represent the same context, a reminder that the factors can be written in either order.

As in segment *2.6*, a number-line representation is used to support understanding, with backward 'jumps' linked to repeated subtraction of multiples of the divisor (i.e. for the cakes example above, starting at 13 and making backward jumps of 4); in this way, the remainder is highlighted and connected to the left over quantity.

Teaching point 2 involves examining the relative value of the divisor and the remainder, using pattern spotting and manipulatives, so that children come to understand that we can never have a remainder equal to or larger than the divisor (e.g. *17 ÷ 5 = 2 r 7*) , because in that case another group could be made (*17 ÷ 5 = 3 r 2*). The same pattern spotting and reasoning also highlight why there will only be remainder if the dividend is *not* a multiple of the divisor.

The final teaching point introduces children to a variety of contexts where they need to make sense of the remainder by 'rounding' the quotient up or down to find the solution to the problem:

- 'Rounding' up:
  'At scout-camp, four scouts can fit in each tent. How many tents will be needed for thirty scouts?'
  30 ÷ 4 = 7 r 2 (correct calculation; incomplete solution)
  8 tents are needed. (correct, complete solution)

- 'Rounding' down:
  'Stephanie is having a party. She has 34 biscuits and wants to put them onto plates of six. How many full plates of six can she make?'
  34 ÷ 6 = 5 r 4 (correct calculation; incomplete solution)
  She can make 5 full plates. (correct, complete solution)

An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast*. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.*

Teaching point 1:

Objects can be divided into equal groups, sometimes with a remainder; objects can be shared equally, sometimes with a remainder; a remainder can be represented as part of a division equation.

Steps in learning

<table>
<tr><th></th><th>Guidance</th><th>Representations</th></tr>
<tr><td>1:1</td><td>
This teaching point reviews and builds on the ideas and language used in segment 2.6 Quotitive and partitive division, Teaching point 1, working towards formally representing division with remainders.
Begin by briefly reviewing the learning from segment 2.6, Teaching point 1. Present a quantity of counters (e.g. nine), and group them equally with a remainder (for example four groups of two with a remainder of one), as shown opposite. As before:

use a multiplication equation to represent the counters
use the following stem sentence to describe the action of dividing the counters into groups:
'___ is divided into groups of ___. There are ___ groups and a remainder of ___.'

Note that the order of the numbers in the stem sentence does not match the order of the numbers in the mixed-operation equation. However, it is used here to prepare children for representing remainders in division equations (step 1:3).
Repeat for a range of total quantities, group sizes and remainder sizes (for example, two groups of five with a remainder of four, four groups of six with a remainder of two, etc.).
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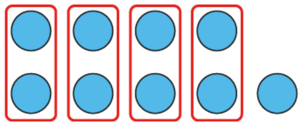
$9 = 4 \times 2 + 1$

'Nine is divided into groups of two. There are four groups and a remainder of one.'

'What does the "9" represent?'<br>'The "9" represents the total number of counters.'
'What does the "4 × 2" represent?'<br>'The "4 × 2" represents the four groups of two.'
'What does the "1" represent?'<br>'The "1" represents the remaining one counter.'

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1:2

Now explore a quotitive division context. Begin with a situation with no remainder, for example: *'A baker has twelve cakes. He sells cakes in boxes of four. How can he box the cakes?'*

Go through the process, introduced in segment *2.6 Structures: quotitive and partitive division*, of:

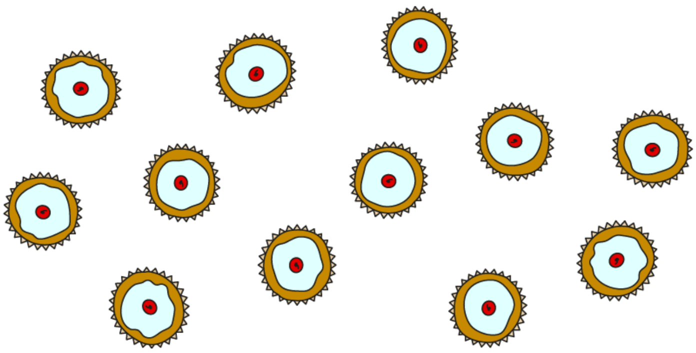
- writing multiplication and division equations (with missing factor/quotient) to represent the problem
- skip counting according to the divisor to find the quotient (work pictorially to move the cakes, or use manipulatives, making groups of four as you count); you can support the skip counting using forward jumps on a number line
- describing the solution in terms of the context.

Note that, although children already know that they can use their multiplication facts (here *3 × 4 = 12*) to find the quotient, skip counting is used in order to support understanding when a remainder is introduced into the problem in step *1:3*.

Once you have found the quotient, represent backward jumps of four on a number line, alongside a repeated subtraction equation and the division equation (for more detail, see segment *2.6*, step *2:7*).

Quotitive division *without* remainder – presenting the problem:

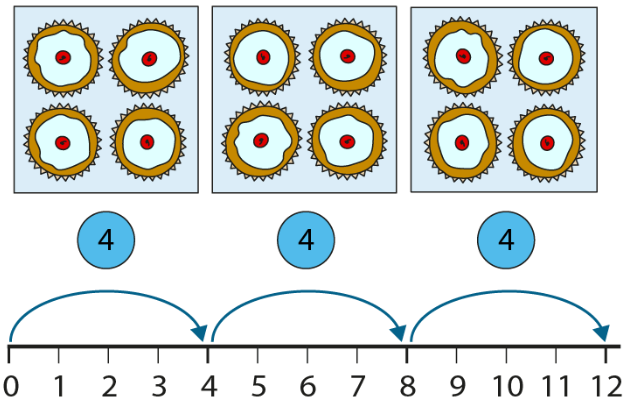
'A baker has twelve cakes. He sells cakes in boxes of four. How can he box the cakes?'



$12 = \boxed{} \times 4$ $\qquad$ $12 = 4 \times \boxed{}$

$12 \div 4 = \boxed{}$

Quotitive division *without* remainder – skip counting to find the solution:



- *'One box of four is four.'*
- *'Two boxes of four are eight.'*
- *'<u>Three</u> boxes of four are twelve.'*

Quotitive division *without* remainder – describing the solution:

$12 = 3 \times 4$ $\qquad$ $12 = 4 \times 3$

$12 \div 4 = 3$

- *'Twelve divided into groups of four is equal to <u>three</u>.'*
- *'So, the baker can make three boxes of cakes.'*

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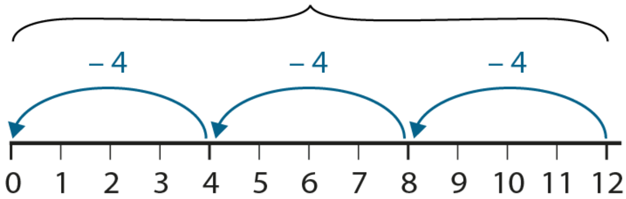
Number line – removing groups of four:



$12 - 4 - 4 - 4 = 0$

$12 \div 4 = 3$
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<tr>
<td>**1:3**</td>
<td>
Present a related problem, but now with a remainder; for example: *'A baker has fourteen cakes. He sells cakes in boxes of four. How can he box the cakes?'*

Work through the same process as in step *1:2*, but now:

- when working pictorially (or with manipulatives) draw attention to the left-over cakes that can't be made into a group of four
- as you skip count forwards, supported by the number line, emphasise the stopping point (i.e. we don't skip count to the next multiple of four (sixteen) since there aren't enough cakes to make another group of four)
- include the remainder in the multiplication equations
- use the following stem sentence to describe the solution: ***'___ divided into groups of ___ is equal to ___, with a remainder of ___.'***
- demonstrate how the stem sentence is mirrored by the recording of the remainder as part of the division equation
- ask children to describe what each part of the division equation represents, using full sentences, as exemplified on the next page
- highlight the remainder on the number line with backward jumps, and in the associated repeated subtraction and division equations.
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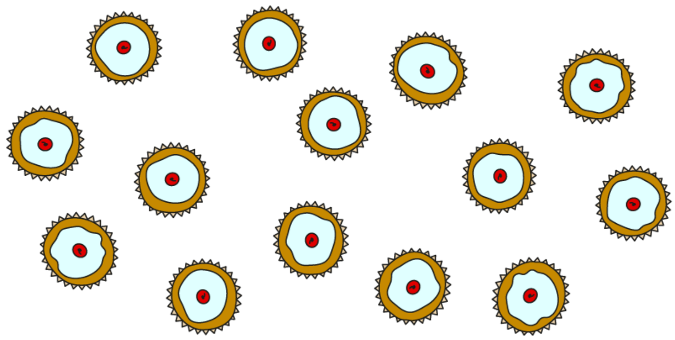
Quotitive division *with* remainder – presenting the problem:

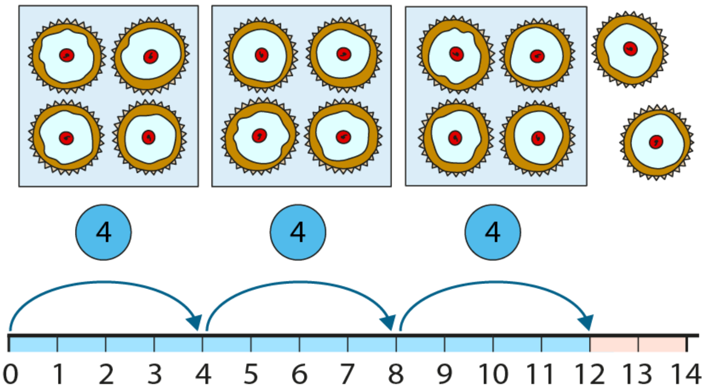
'A baker has fourteen cakes. He sells cakes in boxes of four. How can he box the cakes?'



$14 = \boxed{} \times 4$ $14 = 4 \times \boxed{}$

$14 \div 4 = \boxed{} \times 4$

Quotitive division *with* remainder – skip counting to find the solution:



- *'One box of four is four.'*
- *'Two boxes of four are eight.'*
- *'Three boxes of four are twelve.'*
- *'There are two cakes left over.'*
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Quotitive division with remainder – describing the solution:

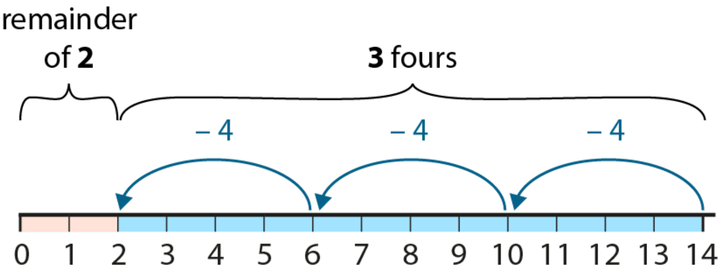
$14 = 3 \times 4 + 2$ $\quad$ $14 = 4 \times 3 + 2$

$14 \div 4 = 3 \text{ r } 2$

'Fourteen divided into groups of four is equal to <u>three, with a remainder of two</u>.'
'So, the baker can make three boxes of cakes, with two left over.'

'The "14" represents the total number of cakes.'
'The "4" represents the number of cakes in each box.'
'The "3" represents the number of full boxes that can be made.'
'The "2" represents the number of cakes left over.'

Number line – removing groups of four:



$14 - 4 - 4 - 4 = 2$

$14 \div 4 = 3 \text{ r } 2$
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<tr>
<td>1:4</td>
<td>Before moving onto the next step, take a moment to examine the format of the division equation with a remainder, identifying the different parts of the equation.</td>
<td>
<table>
<tr><td>14</td><td>÷</td><td>4</td><td>=</td><td>3</td><td>r</td><td>2</td></tr>
<tr><td>dividend</td><td>÷</td><td>divisor</td><td>=</td><td>quotient</td><td>r</td><td>remainder</td></tr>
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<td>1:5</td>
<td>Now, spend some time following a similar process to that in steps 1:2 and 1:3, but now reflecting the partitive division structure. First present a partitive division context that will not result in a remainder; for example: 'Eighteen toy cars are shared equally between three children. How many cars does each child get?'</td>
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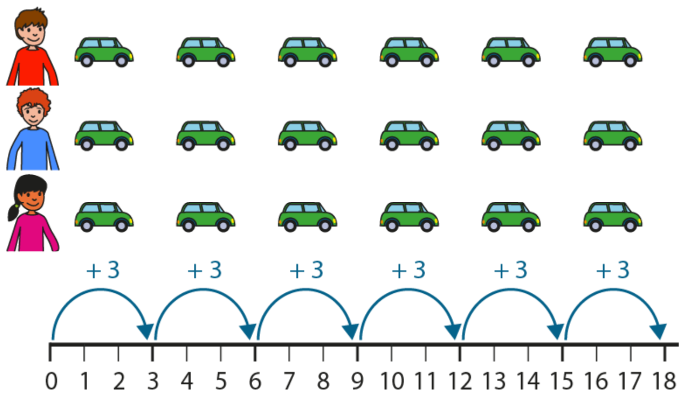
	Go through the process, introduced in segment *2.6 Structures: quotitive and partitive division*, of: • writing multiplication and division equations (with missing factor/quotient) to represent the problem • sharing out, in this case, three cars at a time (simultaneously) as you skip count according to the divisor to find the quotient (how many cars each child gets), supported by a number line with forward jumps of three • describing the solution in terms of the context, as shown on the next page; note that the language *'…divided between…'* contrasts with the quotitive division language of *'…divided into groups of…'* (for more on this, see segment *2.6, Overview of learning*) • representing backward jumps of three on a number line, alongside a repeated subtraction equation and the division equation.	Partitive division *without* remainder – presenting the problem: *'Eighteen toy cars are shared equally between three children. How many cars does each child get?'*  $18 = 3 \times \square$ $18 = \square \times 3$ $18 \div 3 = \square$ Partitive division *without* remainder – skip counting to find the solution:  • *'One three is one each. That's three.'* • *'Two threes is two each. That's six…'* • *'…Six threes is six each. That's eighteen.'*

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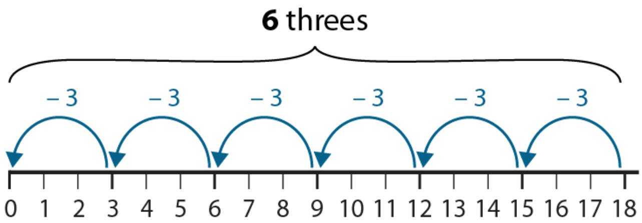
Partitive division without remainder – describing the solution:

$18 = 3 \times 6$ $18 = 6 \times 3$
$18 \div 3 = 6$

• 'Eighteen divided between three is equal to <u>six</u> each.'
• 'So the children get six cars each.'

Number line – removing groups of three:


$18 - 3 - 3 - 3 - 3 - 3 - 3 = 0$
$18 \div 3 = 6$
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<tr>
<td>1:6</td>
<td>
Now, in the same way as in step 1:3, present a related problem with a remainder, for example: 'Nineteen toy cars are shared equally between three children. How many cars does each child get?'

Work through the same process as in step 1:5, but now:
• when working pictorially (or with manipulatives) draw attention to the left-over cars that can't be made into a group of three
• as you skip count forwards, supported by the number line, emphasise the stopping point (i.e. we don't skip count to the next multiple of three (21) since there aren't enough cars to give the three children one more car each)
• include the remainder in the multiplication equations
• use the following stem sentence to describe the solution: '___ divided between ___ is equal to ___ each, with a remainder of ___.'
(note how the language here differs
</td>
<td>
Partitive division with remainder – presenting the problem:

'Nineteen toy cars are shared equally between three children. How many cars does each child get?'

$19 = 3 \times \boxed{}$ $19 = \boxed{} \times 3$
$19 \div 3 = \boxed{}$
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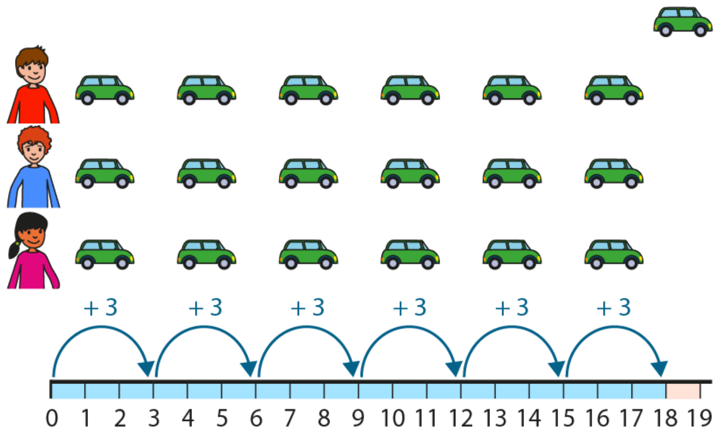
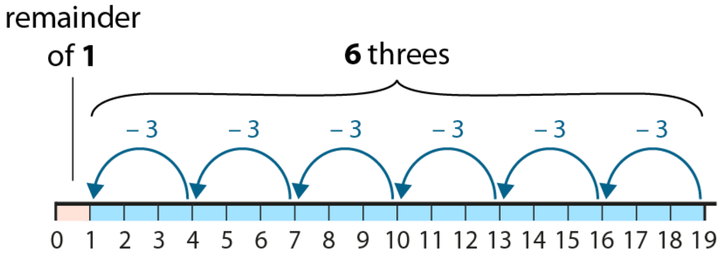
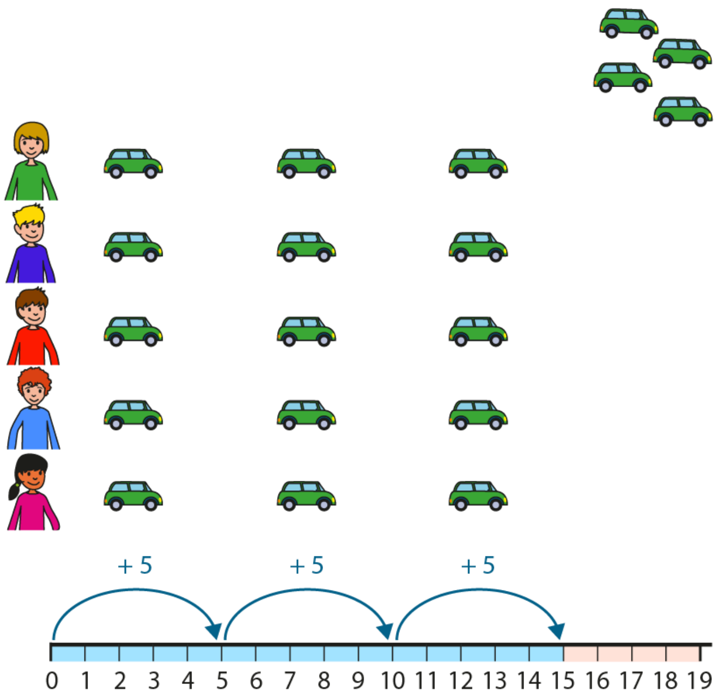
</table>

	slightly from that in step *1:3*, continuing to reflect the partitive structure of the problem) • demonstrate how the stem sentence is mirrored by the recording of the remainder as part of the division equation • ask children to describe what each part of the division equation represents, using full sentences, as exemplified opposite • highlight the remainder on the number line with backward jumps, and in the associated repeated subtraction and division equations.	Partitive division *with* remainder – skip counting to find the solution:  • *'One three is one each. That's three.'* • *'Two threes is two each. That's six…'* • *'…Six threes is six each. That's eighteen.'* • *'There is one car left over.'* Partitive division *with* remainder – describing the solution: $19 = 3 \times 6 + 1$ $19 = 6 \times 3 + 1$ $19 \div 3 = 6 \text{ r } 1$ • *'Nineteen divided between three is equal to six each, with a remainder of one.'* • *'So, the children get six cars each; there is one car left over.'* • *'The "19" represents the total number of cars.'* • *'The "3" represents the number of children the cars are shared between.* • *'The "6" represents the number of cars that each child gets.* • *'The "1" represents the number of cars left over.'*

		Number line – removing groups of three:  $19 - 3 - 3 - 3 - 3 - 3 - 3 = 1$ $19 \div 3 = 6 \text{ r } 1$
1:7	Work through more quotitive and partitive contexts with remainders, together as a class. Gradually move away from skip counting using language related to the context (as exemplified in previous steps), and towards identifying the relevant times table fact that corresponds to the largest multiple of the divisor that is equal to or less than the dividend (note that the ≤ symbol is introduced opposite). Use the following stem sentence: ***'The largest multiple of ___ that is less than or equal to ___ is ___.'*** Discuss: • The next multiplication fact ($4 \times 5 = 20$ in the example opposite) is not chosen (there are not enough cars for one more each). • We therefore use the fact related to the largest multiple less than or equal to the dividend (opposite, $3 \times 5 = 15$), and recognise that there are still some remaining, which we need to count to calculate the remainder (finding the difference of four cars by counting up from 15 to 19 or, preferably, using known facts: 19 is four more than 15 because nine is four more than five). Support children by making the times table charts available, allowing them to focus more fully on the structure of division with remainders, rather than	Partitive division *with* remainder – presenting the problem: *'Nineteen toy cars are shared equally between five children. How many cars does each child get?'* $19 = 5 \times \square$ $19 = \square \times 5$ $19 \div 5 = \square$ Partitive division *with* remainder – identifying the relevant times table fact: $0 \times 5 = 0$ | $5 \times 0 = 0$ $1 \times 5 = 5$ | $5 \times 1 = 5$ $2 \times 5 = 10$ | $5 \times 2 = 10$ $\mathbf{3 \times 5 = 15}$ | $\mathbf{5 \times 3 = 15}$ $4 \times 5 = 20$ | $5 \times 4 = 20$ • *'The largest multiple of five that is less than or equal to nineteen is fifteen.'* • *'Three fives are fifteen.'* • *'We can represent this as:'* $3 \times 5 \leq 19$ $3 \times 5 = 15$

	on calculation of the answers. Note that being able to: • identify the largest multiple of a divisor that is equal to or less than a dividend • then, find the difference between the dividend and the largest multiple are essential skills that children will need when they learn formal written division methods (short and long division).	Partitive division *with* remainder – describing the solution: • *'Three fives are fifteen…'* • *'… and nineteen is four more than fifteen.'* • *'So, nineteen divided between five is equal to three each, with a remainder of four.'*  $19 = 5 \times 3 + 4$ $19 = 3 \times 5 + 4$ $19 \div 5 = 3 \text{ r } 4$ • *'So, the children get three cars each; there are four cars left over.'*
1:8	Provide children with some abstract problems for practice, including: • identifying the largest multiple that is equal to or less than a given value • solving abstract division problems with a remainder. Keep to facts within the times tables, and make the times table charts available for support.	• *'Fill in the missing numbers.'* The largest multiple of 3 that is less than or equal to 28 is ___. This is ___ threes. $28 = \boxed{} \times 3 + \boxed{}$ $28 \div 3 = \boxed{} \text{ r } \boxed{}$

		• *'For each inequality, find the <u>largest</u> number that can go into the box.'* $\square \times 4 \leq 9$ $\quad$ $\square \times 8 \leq 7$ $\square \times 4 \leq 12$ $\quad$ $\square \times 8 \leq 25$ $\square \times 4 \leq 13$ $\quad$ $\square \times 8 \leq 56$ $\square \times 5 \leq 31$ $\quad$ $\square \times 9 \leq 39$ $\square \times 6 \leq 20$ $\quad$ $\square \times 11 \leq 42$ $\square \times 7 \leq 9$ $\quad$ $\square \times 12 \leq 125$ • *'Fill in the missing numbers.'* $8 \times 2 = \square$ $\quad$ $3 \times 9 = \square$ $9 \times 2 = \square$ $\quad$ $4 \times 9 = \square$ $17 \div 2 = \square$ r $\square$ $\quad$ $30 \div 9 = \square$ r $\square$ $8 \times 5 = \square$ $\quad$ $3 \times 11 = \square$ $9 \times 5 = \square$ $\quad$ $4 \times 11 = \square$ $43 \div 5 = \square$ r $\square$ $\quad$ $34 \div 11 = \square$ r $\square$ $13 \div 3 = \square$ r $\square$ $\quad$ $52 \div 10 = \square$ r $\square$ $32 \div 6 = \square$ r $\square$ $\quad$ $50 \div 12 = \square$ r $\square$

Teaching point 2:

If the dividend *is* a multiple of the divisor, there is *no* remainder; if the dividend *is not* a multiple of the divisor, there *is* a remainder. The remainder is always less than the divisor.

Steps in learning

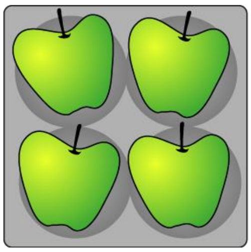
2:1 In this teaching point children will explore, through quotitive, partitive and abstract examples, why the remainder is always less than the divisor, and what conditions give rise to a non-zero remainder.

Use a contextual quotitive division problem to explore varying the dividend whilst keeping the divisor the same. Note that it is important to use an example with a divisor of about four or more here to allow for a reasonable exploration of the size of the remainder relative to the divisor.

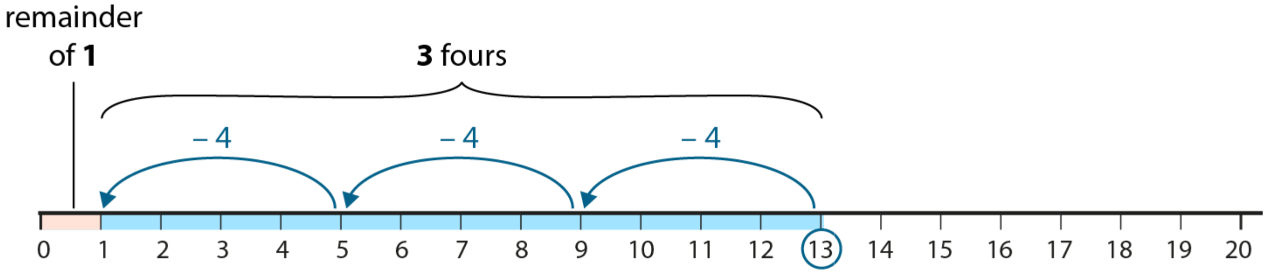
First present the context, for example, *'Apples are sold in trays of four. How many trays of apples can be made?'* Then set up a table, as shown below, and solve the problem for some different values of the dividend. For each dividend, use a number line with backward jumps to represent the solution and write the corresponding division equation.

Then ask children what they notice about the divisor and the remainder; they should notice that the remainder is always less than the divisor. Some children may also notice that when the dividend is a multiple of the divisor the remainder is zero/there is no remainder; if so, note this for later discussion. For now, focus on the relative value of the divisor and remainder, using the generalisation: ***'The remainder is always less than the divisor.'***

'Apples are sold in trays of four. How many trays of apples can be made?'



Example number line for 13 apples:



Comparison table:

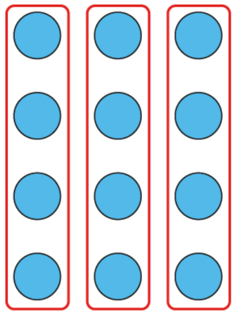
Total number of apples (dividend)	Number of apples in each tray (divisor)	Number of trays (quotient)	Number of apples left over (remainder)	Equation
12	**4**	3	**0**	$12 \div 4 = 3$
13	**4**	3	**1**	$13 \div 4 = 3$ r 1
14	**4**	3	**2**	$14 \div 4 = 3$ r 2
15	**4**	3	**3**	$15 \div 4 = 3$ r 3
16	**4**	4	**0**	$16 \div 4 = 4$
17	**4**	4	**1**	$17 \div 4 = 4$ r 1
18	**4**	4	**2**	$18 \div 4 = 4$ r 2
19	**4**	4	**3**	$19 \div 4 = 4$ r 3
20	**4**	5	**0**	$20 \div 4 = 5$

2:2

Now, use manipulatives to explore *why* the generalisation is true. Work through the calculations in the table again, now using counters to represent the apples. Begin by giving each child (or pair of children) 12 counters, and ask them to divide them into groups of four. As a class, represent the counters in an array, and describe the outcome and the division equation using the language from step *1:2*: *'Twelve divided into groups of four is equal to three.'* Then give children another counter, and repeat, using the language from step *1:3* to describe the outcome and equation: *'Thirteen divided into groups of four is equal to three with a remainder of one.'*

Continue working through the dividends in the table from step *2:1*. As you do so, children should start to realise that they don't need to create the groups again each time. For example, if children already have 13 counters arranged into groups of four and one more, when they are given another counter they may realise that they can simply add this to the existing 'one more'. During this process, draw attention to the fact that when we get to the stage where the 'extra counters' pile is equal to the value of the divisor (a group size of four), we can make a whole new group, and the remainder is zero once again. Reiterate the generalisation: ***'The remainder is always less than the divisor.'***

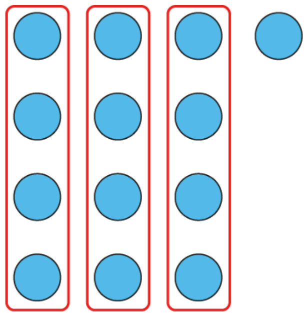
Dividend = 12, divisor = 4:



$12 \div 4 = 3$

- *'Twelve divided into groups of four is equal to three.'*

Dividend = 13, divisor = 4:

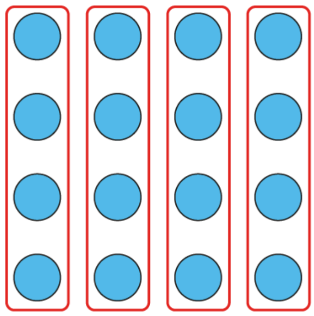


$13 \div 4 = 3 \text{ r } 1$

- *'Thirteen divided into groups of four is equal to three with a remainder of one.'*

⋮

Dividend = 16, divisor = 4:



$16 \div 4 = 3 \text{ r } 4$ ✗

- *'The remainder can't be equal to four, because we can make another group of four.'*

$16 \div 4 = 4$ ✓

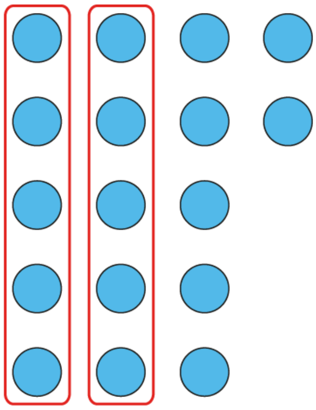
- *'Sixteen divided into groups of four is equal to four.'*

2:3

Work through some division equations, some correctly showing a remainder smaller than the divisor and others incorrectly showing a remainder greater to the divisor. For each equation, ask children to decide whether it is correct or not, reasoning about the relative size of the divisor and the remainder, referring to the generalisation, and using counters to demonstrate their understanding. For each incorrect equation, ask children to write the correct version. Also include examples with no remainder, with equations incorrectly showing a remainder equal to the divisor.

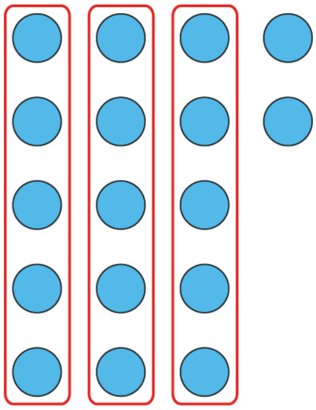
Gradually remove the scaffold of the counters, giving children practice until they can confidently identify incorrect equations and correct them.

Correcting equations – using counters for support:



17 ÷ 5 = 2 r 7 ✗

- *'This is incorrect, because seven is greater than five.'*
- *'The remainder should be less than the divisor.'*
- *'We can make another group of five.'*



17 ÷ 5 = 3 r 2 ✓

- *'This is correct, because two is less than five.'*
- *'The remainder is less than the divisor.'*
- *'We can't make another group of five.'*

Correcting equations – practice:

'Use a tick or a cross to show whether each equation is correct or not. Correct each incorrect equation.'

Equation	**✓ or ✗**	**If ✗, write the correct equation**
18 ÷ 4 = 3 r 6		
20 ÷ 5 = 3 r 5		
22 ÷ 6 = 3 r 4		
23 ÷ 7 = 3 r 2		
31 ÷ 8 = 3 r 7		

2:4

Now explore how we know whether there will be a remainder at all. The process in step *2:2* alluded to the fact that if the dividend *is* a multiple of the divisor, there *will not* be a remainder (and conversely, that if the dividend is *not* a multiple of the divisor, there *will* be a remainder). You can use the same example of the apples, but this time link to known multiplication facts, and use the following stem sentences:

- ***'___ is a multiple of ___, so when it is divided into groups of ___ there are none left over; there is no remainder.'***
- ***'___ is not a multiple of ___, so when it is divided into groups of ___ there are some left over; there is a remainder.'***

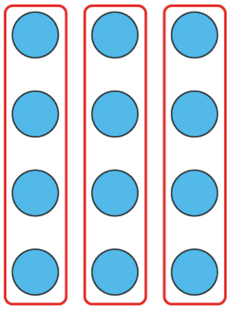
When identifying the relevant multiplication facts, link to the work done in step 1:8, by using the same language ***('___ is the largest multiple of ___ that is less than or equal to ___')***.

Work towards the following generalisations:

- ***'If the dividend is a multiple of the divisor, there is no remainder.'***

'If the dividend is not a multiple of the divisor, there is a remainder.'

Dividend = 12, divisor = 4:



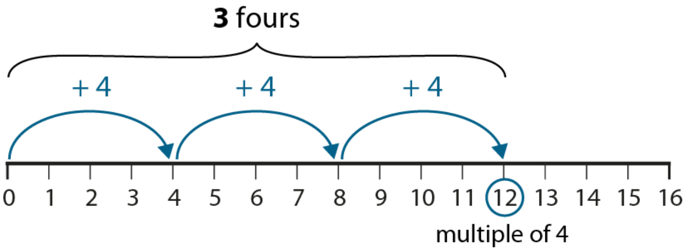
- *'Twelve is the largest multiple of four that is less than or equal to twelve.'*

$12 = 3 \times 4$ $\qquad$ $12 = 4 \times 3$

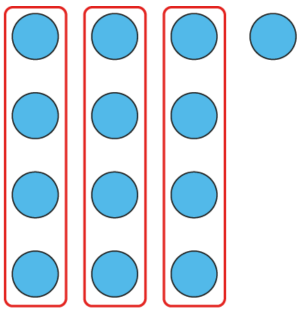
- *'Twelve is a multiple of four, so when it is divided into groups of four there are none left over; there is no remainder.'*

$12 \div 4 = 3$

- *'Twelve divided into groups of four is equal to three.'*



Dividend = 13, divisor = 4:



- *'Twelve is the largest multiple of four that is less than or equal to thirteen.'*

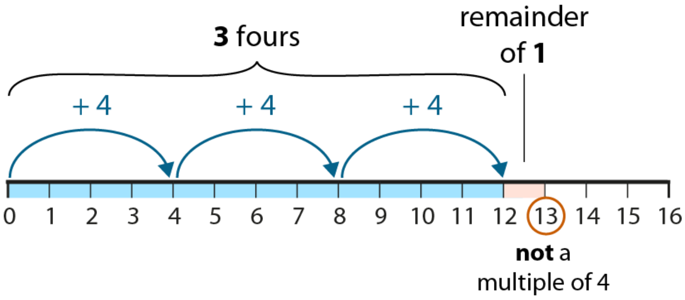
$12 = 3 \times 4$ $\qquad$ $12 = 4 \times 3$

$13 = 3 \times 4 + 1$ $\qquad$ $13 = 4 \times 3 + 1$

- *'Thirteen is not a multiple of four, so when it is divided into groups of four there are some left over; there is a remainder.'*

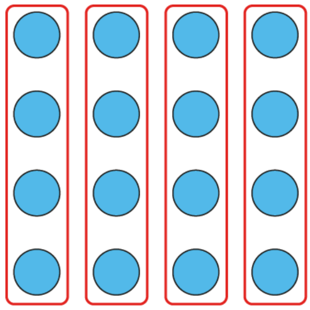
$13 \div 4 = 3 \text{ r } 1$

- *'Thirteen divided into groups of four is equal to three with a remainder of one.'*



⋮

Dividend = 16, divisor = 4:



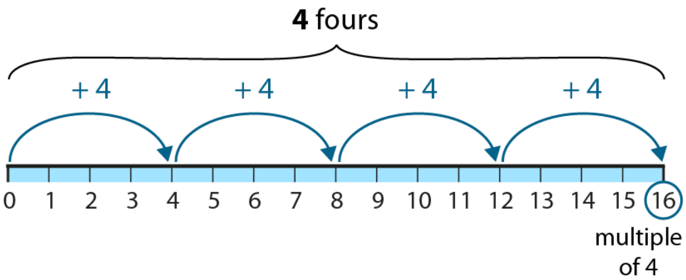
- *'Sixteen is the largest multiple of four that is less than or equal to sixteen.'*

  $16 = 4 \times 4$

- *'Sixteen is a multiple of four, so when it is divided into groups of four there are none left over; there is no remainder.'*

  $16 \div 4 = 4$

- *'Sixteen divided into groups of four is equal to four.'*



2:5

Once children are confident with the following generalisations, spend some time, as a class, applying them in a range of contexts, including statistics, measures, partitive and abstract examples:

- ***'The remainder is always less than the divisor.'***
- ***'If the dividend <u>is</u> a multiple of the divisor, there <u>is no</u> remainder.'***
- ***'If the dividend <u>is not</u> a multiple of the divisor, there <u>is</u> a remainder.'***

Statistics context:

'Paul records the quantities of different vehicles that drive past the school gate in an hour. Here are his results. Fill in the rest of the table.'

	Tally	**Total number**	**Will there be a remainder if we divide into groups of 5?**	**Division equation**
cars	卌 卌 卌 卌 ||||	24	yes	24 ÷ 5 = 4 r 4
lorries	卌 卌 ||	12		
vans	卌 卌			
motorbikes	卌 |			

Measures context:

'A dressmaker has a ribbon that is thirty centimetres long. She wants to cut it into equal lengths. Fill in the table to show how many lengths she could make and whether there will be any ribbon left over.'

Total length of the ribbon	**Length of each equal piece**	**How many equal-length pieces will there be?**	**Will there be any ribbon left over?**	**Division equation**
30	3	10	No	30 ÷ 3 = 10
30	4			
30	5			
30	6			
30	7			
30	8			
30	9			
30	10			

Partitive division context:

'Some carrots are shared equally between six horses. Fill in the missing numbers in the table, predicting whether there will be any carrots left over.

Total number of carrots (dividend)	Number of horses (divisor)	Is the dividend a multiple of the divisor? ✓ or ✗	Will there be any carrots left over? ✓ or ✗
12	**6**		
15	**6**		
18	**6**		
20	**6**		
25	**6**		
30	**6**		
60	**6**		

Abstract example:

'Find as many ways as you can to divide twenty-five into equal groups. When will there be a remainder? Write a multiplicaiton equation and a division equation to represent each of your answers. Use the number line to help you.'

Total value	Number of groups	Group size	Remainder	Multiplication equation	Division equation
25	25	1	0	$25 = 25 \times 1$	$25 \div 1 = 25$
25	12	2	1	$25 = 12 \times 2 + 1$	
25					
25					
25					
25					

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

<table>
<tr>
<td>2:6</td>
<td>
To complete this teaching point, provide children with practice, including:

missing-number problems (use intelligent practice, as shown opposite, to further reinforce the concepts learned in this teaching point)
true/false and 'always/sometimes/never' problems, such as those shown on the next page (encourage children to explain their reasoning)
word problems, for example:

'Nineteen children get into teams of five. How many teams will there be? How many children will be left over?'
Dòng nǎo jīn:
'I have fewer than fifty biscuits.'

'If I shared them equally between eight people, there would be no remainder.'
'If I shared them equally between three people, there would be no remainder.'

'I must have twenty-four biscuits. True or false?'
</td>
<td>
Missing-number problems:
$20 \div 4 = \square$ r $\square$
$21 \div 4 = \square$ r $\square$
$22 \div 4 = \square$ r $\square$
$23 \div 4 = \square$ r $\square$
$24 \div 4 = \square$ r $\square$
$16 \div 3 = \square$ r $\square$
$17 \div 3 = \square$ r $\square$
$18 \div 3 = \square$ r $\square$
$19 \div 3 = \square$ r $\square$
$20 \div 3 = \square$ r $\square$
$21 \div 3 = \square$ r $\square$
$20 \div 2 = \square$ r $\square$
$20 \div 3 = \square$ r $\square$
$20 \div 4 = \square$ r $\square$
$20 \div 5 = \square$ r $\square$
$20 \div 6 = \square$ r $\square$
$20 \div 7 = \square$ r $\square$
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True/false and 'always/sometimes/never' problems:

- *'Decide whether each calculation has a remainder or not.'*

	Has a remainder? ✓ or ✗
48 ÷ 7	
48 ÷ 8	
48 ÷ 9	
56 ÷ 7	
56 ÷ 8	
56 ÷ 9	

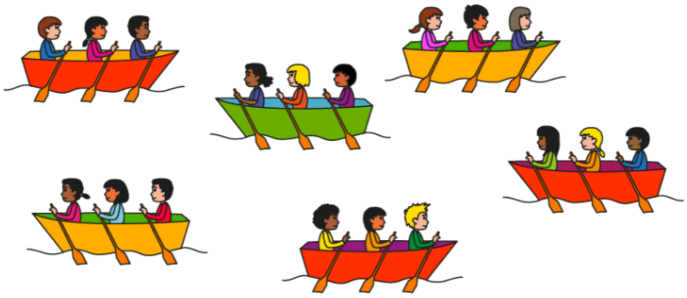
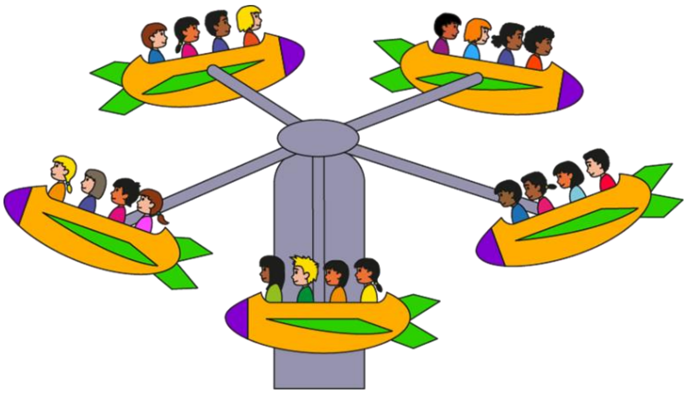
- *'Decide whether each sentence is **always** true, **sometimes** true, or **never** true. For each statement, explain your answer using a picture or an equation.'*
  - The remainder is less that the dividend.
  - The remainder is less than the divisor.
  - The remainder is greater than the divisor.
  - The remainder is zero.

Dòng nǎo jīn:

- ★ ÷ ♦ = 4 r 3
  'Circle all possible values of ♦.'
  0 1 2 3 4 5

- ● ÷ 5 = 6 r ■
  - *'Write down all the possible values of ■.'*
  - *'What is the largest possible value of ■?'*

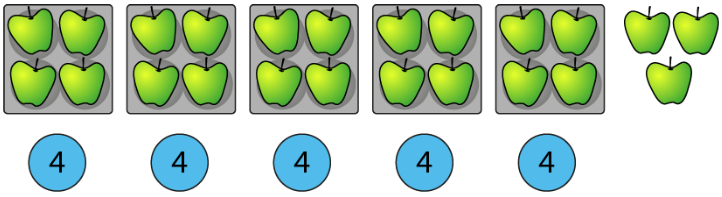
- 'Write a division calculation with this answer:'
  3 r 7
  - *'Can you write another?'*
  - *'And another?'*

		• *If the children on the boats swapped places with the children on the ride, what pair of expressions would describe the new situation?'* • $5 \times 3 + 5$ and $5 \times 3 + 3$ • $5 \times 3 + 3$ and $3 \times 6 + 2$ • $6 \times 3 + 2$ and $4 \times 4 + 2$ • None of these. Boats  Ride 

Teaching point 3:

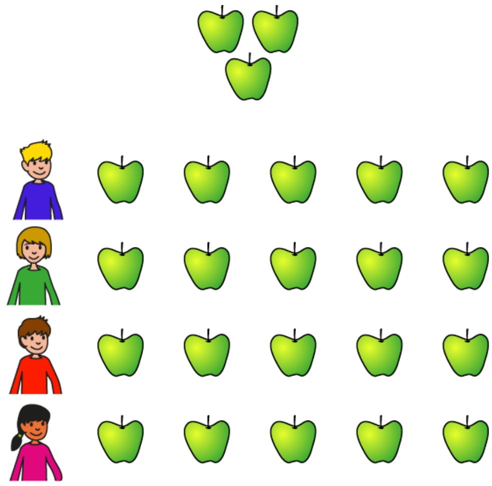
When solving contextual problems involving remainders, the answer to a division calculation must be interpreted carefully to determine how to make sense of the remainder.

Steps in learning

	Guidance	**Representations**
3:1	Now that children have a good understanding of remainders, it is important to spend some time, as a class, exploring the idea that the answer to the *calculation* is not always the answer to the contextual *problem*. The answer given, for any problem, must be both complete and correct. Consider the problem *'Four scouts can fit in each tent. How many tents will be needed for thirty scouts?'* Simply writing the calculation, despite it being correct, would correspond to an *incomplete* solution: $30 \div 4 = 7$ r 2 incomplete Some children may go a step further, but provide the incorrect answer to the original question, assuming that the answer to the problem is the same as the quotient: 7 tents are needed ✘ For the problem to have been solved, the answer provided must be complete and correct: $30 \div 4 = 7$ r 2 8 tents are needed ✓ Before exploring any situations like this, where the answer to the problem is not the same as the value of the quotient, spend a little time working through problems with familiar contexts. In particular, give children practice relating the division equations back to the context by describing what each number represents. Include both quotitive and partitive division context;	Relating division calculations to contexts – quotitive example: *'A grocer has twenty-three apples. He sells apples in trays of four. How many full trays of apples can he make? How many apples are left over?'*  $23 \div 4 = 5$ r 3 • *'The "23" represents the total number of apples.'* • *'The "4" represents the number of apples in each tray.'* • *'The "5" represents the number of full trays that can be made.'* • *'The "3" represents the number of apples left over.'* • *'So, the grocer can make five full trays of apples; he will have three apples left over.'*

<table>
<tr>
<td></td>
<td>note that, at this stage, the problems (as exemplified opposite) ask 'how many…' and 'how many are left over…' so there is no need for further interpretation, beyond linking the quotient and the remainder to each of these questions respectively.</td>
<td>Relating division calculations to contexts – partitive example:

'Twenty-three apples are shared equally between four children. How many apples does each child get? How many apples are left over?'



23 ÷ 4 = 5 r 3
'The "23" represents the total number of apples.'
'The "4" represents the number of children the apples are shared between.'
'The "5" represents the equal number of apples that each child gets.'
'The "3" represents the number of apples left over.'
'So, the children get five apples each; there are three apples left over.'</td>
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<tr>
<td>3:2</td>
<td colspan="2">Now, introduce a problem where the division calculation does not represent the complete solution, and for which the answer to the question posed is not the same as the value of the quotient. You can use the quotitive scout-camp example discussed in the previous step:
'Four scouts can fit in each tent. How many tents will be needed for thirty scouts?'

As a class, work through this problem following the same steps as in step 3:1. However, you will now need to continue further, since connecting the division calculation to the context reveals that with only seven tents, there are two scouts left without a tent. Once children have described what each number in the division calculation represents, ask them the following questions to draw attention to the fact that eight tents are needed rather than seven:
'Will all of the scouts fit into seven tents?'
(no)
'Why not?'
(Twenty-eight scouts fit into seven tents; there are two scouts left over.)
'So how many tents will we need to fit <u>all</u> of the scouts into tents?'
(Eight; we need seven tents of four scouts, and another tent for the two left-over scouts.)</td>
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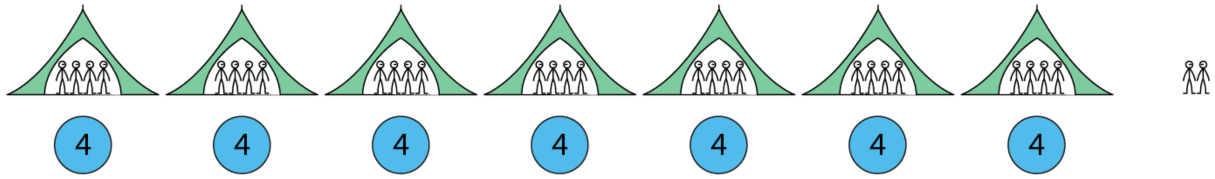
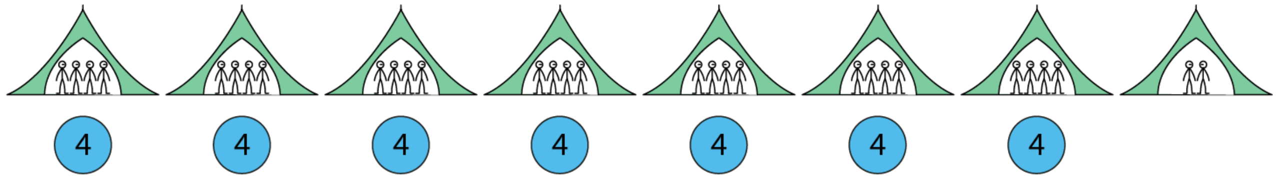
'Four scouts can fit in each tent. How many tents will be needed for thirty scouts?'



30 ÷ 4 = 7 r 2

- *'The "30" represents the total number of scouts.'*
- *'The "4" represents the number of scouts in each tent.'*
- *'The "7" represents the number of full tents.'*
- *'The "2" represents the number of scouts left over.'*

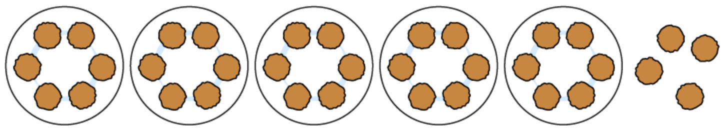
- *'We need another tent for the two left-over scouts. <u>Eight tents are needed</u>.'*



3:3

The example in the previous step involved 'rounding up' from '7 r 2' to '8'. Now work through a problem that involves 'rounding down' to the quotient (i.e. ignoring the remainder).

'Stephanie is having a party. She has thirty-four biscuits and wants to put them onto plates of six. How many <u>full</u> plates of six can she make?'



34 ÷ 6 = 5 r 4

- *'The "34" represents the total number of biscuits.'*
- *'The "6" represents the number of biscuits on each plate.'*
- *'The "5" represents the number of plates of biscuits.'*
- *'The "4" represents the number of biscuits left over.'*

- *'So, five <u>full</u> plates of biscuits can be made.'*

3:4	Work through a variety of quotitive and partitive division problems that involve either 'rounding up' or 'rounding down'. Gradually remove scaffolding, until children can confidently interpret their calculations to provide complete and correct answers without pictorial support and without needing to describe, in full, what each number in their calculation represents. Get children into the habit, whenever they have completed the division calculation and found it has a remainder, of always asking themselves, *'Is my answer complete?'* or *'Have I answered the question?'* (indeed, this is a good habit for any contextual problem).

3:5	To complete this teaching point, provide children with practice word problems. Include:
	• problems with a variety of different divisors, from 2 to 12, to ensure constant exposure across all times tables
	• both quotitive and partitive contexts
	• problems that involve both 'rounding up' and 'rounding down'.
	Example word problems:
	• *'If I share sixteen stickers between five children, how many stickers does each child get?'*
	• *'Fifty-eight people work at a factory. They all eat lunch together. If each table can seat six people, how many tables are needed?'*
	• *'It takes three sheets of paper to make a paper flower. How many flowers can be made with twenty-five sheets of paper?'*
	• *'A bottle of water can fill eight glasses. How many bottles are needed to fill sixty-seven glasses?'*
	• Dòng nǎo jīn: *'I need eight tables to seat all my friends, and me, for a birthday meal. Each table can seat six people.'*
	'Which of these is true?'
	  • I have 48 friends.
	  • I have more than 48 friends.
	  • I have less than 48 friends.
	'Explain your answer.'

