11 Geometry

Mastery Professional Development

11.1 Transformations and relative position

Guidance document | Key Stage 4

Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The fifth of the Key Stage 4 themes (the eleventh of the themes in the suite of Secondary Mastery Materials) is *Geometry*, which covers the following interconnected core concepts:

11.1 Transformations and relative position

11.2 Reasoning with the properties of a circle

11.3 Trigonometry

11.4 3D shapes

This guidance document breaks down core concept 11.1 Transformations and relative position into four statements of **knowledge, skills and understanding**:

11.1 Transformations and relative position

- 11.1.1 Work with bearings
- 11.1.2 Reason with vectors
- 11.1.3 Explore enlargement further
- 11.1.4 Combine transformations

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

- 11.1.1 Work with bearings
  - 11.1.1.1 Understand that a displacement has both magnitude and direction
  - 11.1.1.2 Understand and use the conventions of bearing notation to describe direction
  - 11.1.1.3 Use the language and notation of bearings to reason geometrically
- 11.1.2 Reason with vectors
  - 11.1.2.1 Understand that a displacement has both magnitude and direction and can be described by a vector
  - 11.1.2.2 Understand and use the conventions of vector notation and column vectors and interpret them geometrically
  - 11.1.2.3 Understand the multiplication of a vector by a scalar (including a negative value) both algebraically and geometrically
  - 11.1.2.4 Understand the addition and subtraction of vectors both algebraically and geometrically

11.1.2.5 Use vectors to construct geometric arguments and proofs

11.1.3 Explore enlargement further

11.1.3.1 Appreciate enlargement as continuous scaling

11.1.3.2 Understand that the scale factor affects both the dimensions of the image and the displacement of the image from the centre in the same way

11.1.3.3 Understand enlargement as a transformation of vectors

11.1.3.4 Understand the structures that underpin enlargement by a negative scale factor

11.1.3.5 Understand the relationships between lengths, areas and volumes in similar shapes

11.1.4 Combine transformations

11.1.4.1 Recognise and use the key characteristics of translations, rotation and reflection

11.1.4.2 Compare and contrast the key characteristics of translations, rotation and reflections with particular reference to the degrees of freedom

11.1.4.3 Use and apply the key characteristics of transformations to analyse situations where transformations are combined

Overview

This core concept is focused on the relative position of an object or point in relation to another, which is a core element of students' spatial reasoning. Students will build on existing knowledge of transformations and begin to describe displacement in new ways, including the use of vectors and bearings. The mathematics explored here is deep and interesting in its own right, but also provides rich and realistic contexts for students to apply their learning from other core concepts.

Geometry and spatial reasoning are of great importance to students both in their learning of mathematics, and in their everyday lives. Aside from the need to navigate the world, the necessity of describing displacements precisely and mathematically is important in fields as diverse as robotics, exploration and healthcare. Given the abstract nature of mathematics, it may surprise students to learn of some of the contexts where concepts are matched to spatial coordinates, such as the emerging Large Language Models (LLMs) in AI.

Underpinning the knowledge, skills and understanding in this core concept are the distinctions between distance, direction and displacement. Bearings are a precise way of describing the **direction** from one point to another. When combined with a **distance**, they can be used to identify the exact location of a point in relation to a starting position (i.e., a **displacement**). While this method for describing a direction is usually summarised as a set of rules, the introduction of bearings at Key Stage 4 provides students with an opportunity to think more deeply about displacement. It supports them in making connections between descriptions of displacement given as transformations and vectors. Bearings provide a practical, real-life application of angles and geometry and are often also used to solve multi-step problems involving other mathematical topics, such as the sine and cosine rules.

The ability to interpret mathematical relationships both algebraically and geometrically is developed at Key Stage 4. The use of formal vector notation to describe transformations is introduced, building on previous work on column vectors. Linking back to column vectors gives an opportunity to explore problems involving a change in position both numerically/algebraically and spatially/geometrically. At Key Stage 3, students will have developed an initial understanding of mathematical proof and started to recognise the difference between a demonstration for a few specific cases and a generalisation. It is important that students have an opportunity to construct proofs that rely on geometrical structure, and vectors provide a topic within which such geometric arguments can be explored.

Once the conventions for vector operations have been established, an understanding of enlargements as a transformation of vectors can also be developed. This provides an accessible way to investigate fractional and negative scale factors in a way that builds conceptual understanding, rather than a set of rules to be remembered (and likely confused).

At Key Stage 4, students build on their existing knowledge of transformations and start to consider combinations of rotations, reflections and translations. This includes realising that, in some cases, multiple transformations can be described by a single transformation. Recognising the changes and invariance achieved by combinations of transformations deepens understanding of translation, rotation and reflection, supporting students to progress from describing specific transformations to considering the general case.

Key to both describing and constructing transformations accurately is the understanding that every component of the object undergoes the same transformation. Teachers should support students to recognise what changes and what stays the same for an object when undergoing each of the four different transformations. Dynamic geometry software can support this understanding, as it enables students to see what happens when certain transformations are applied to an object. Students can then make conjectures about where the image of an object that has undergone a transformation will be, justify their conjecture and test it out.

Prior learning

It is important to check students' proficiency at using mathematical vocabulary to describe position, direction and movement, before introducing the new topic of bearings. Students' access to the key ideas involved also relies on a secure understanding of angles as a measure of turn, including an awareness of compass directions. This understanding has built gradually from Key Stage 1 and reinforced by learning in other subjects such as geography and design technology; it is important that teachers are alert to any gaps that have developed over this time. By the end of Key Stage 2, students should also be aware of some key angle rules – such as the sum of angles around a point, on a straight line, and within triangles and quadrilaterals. This learning is further developed at Key Stage 3 to include angles on parallel lines and within polygons. It is important that students are able to connect this knowledge to new areas of mathematics, such as bearings, so that they can apply angle rules in context.

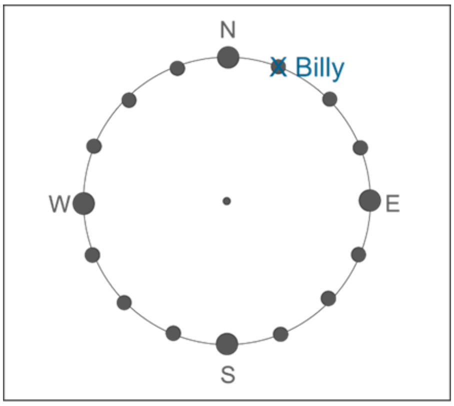
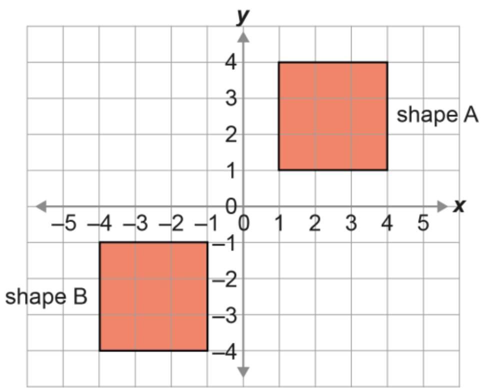
Students were introduced to transformations for the first time at Key Stage 2. At this stage, they will have translated objects using appropriate language; used the axes to reflect shapes; described rotations in terms of 'turn'; and explored enlargement problems involving similar figures. Their understanding of the four transformations is further developed at Key Stage 3, and so it is important that teachers work with their departments to ensure coherence in how the new learning in this core concept builds on the Key Stage 3 curriculum. For example, to check that students have a thorough understanding of what can vary and what remains invariant under each transformation. They also need to be clear about what information is required to describe each of the transformations precisely. Both of these things help students to establish ways in which translations, rotations and reflections can be combined and described using a single transformation at Key Stage 4.

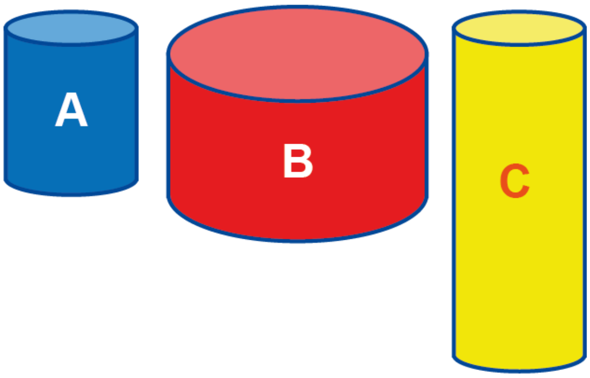
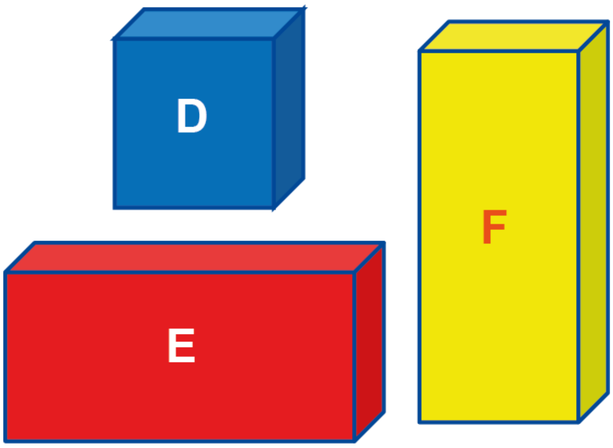
When working at Key Stage 3, students may have been introduced to column vectors as a way of describing translations. If so, this provides a good foundation on which to build when introducing vectors at Key Stage 4. The use of vectors to construct geometrical arguments and proofs is also explored at Key Stage 4. This relies on several different strands of learning from earlier in the curriculum – including algebraic manipulation, properties of shapes and interpreting geometric diagrams. Students' recall of these key ideas can be developed in the context of vectors and transformations to recognise relationships and structures, reason with them and prove results.

The core concept documents '*6.1 Geometrical properties*', '*6.2 Perimeter, area and volume'* and '*6.3 Transforming shapes'* from the Key Stage 3 PD materials explore the prior knowledge required for this core concept in more depth.

Checking prior learning

The following activities from the NCETM secondary assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Checkpoints '*Geometrical properties of polygons*', Checkpoint 9: Perranporth compass	This photograph shows a big compass that people can move around. Imagine Billy and his friends are playing on and around the compass. Billy's position is shown on the diagram below the photograph. The angle between Billy, the centre and Tom is 90°. a) Mark four different positions where Tom could be standing. The angle between Tom, the centre and Oscar is also 90°. b) Mark four different positions where Oscar could be standing. c) What is the angle between Billy, the centre and Oscar? Hannah comes to stand on the line through east and west. d) What might the angle be between her, the centre, and each of the other children?  
Secondary assessment materials, page 45	Shape B is a transformation of shape A. Alex says, 'Shape A has been reflected to make shape B.' Berenice says, 'Shape A has been rotated to make shape B.' Claudia says, 'Shape A has been translated to make shape B.' Deepak says, 'Shape A has been enlarged to make shape B.' Are any of them correct? If so, who? Explain your thinking. 

Key Stage 3 PD materials document '*6.1 Geometrical properties*', Key idea 6.1.1.2, Example 1	Emma and Samira each show that the angles in a triangle add up to 180°. Emma constructs a triangle using a pair of compasses and a ruler, measures each of the interior angles and adds them up. They have a sum of 180°. She repeats this for two different triangles and finds the same result. Samira cuts out a paper triangle, tears off all three corners and places them along the edge of a ruler to show that they fit together and lie on a straight line. Give reasons why Emma and Samira have not produced a convincing argument.
Checkpoints '*Perimeter, area and volume 2*', Checkpoint 28: Twice as wide, twice as tall	Below are three cylinders. Cylinder B is twice as wide as cylinder A. Cylinder C is twice as tall as cylinder A. a) Does B or C have the greater volume, or do they have the same volume?  Below are three cuboids. Cuboid E is twice as wide as cuboid D. Cuboid F is twice as tall as cuboid D. b) Does E or F have the greater volume, or do they have the same volume?  c) How would your answers change if the shapes were triangular prisms?
Key Stage 3 PD materials document '*6.2 Perimeter, area and volume*', Key idea 6.2.2.3, Example 9	Are all circles similar figures? Explain your answer.

Key vocabulary

Key terms used in Key Stage 3 materials

- enlargement
- orientation
- reflection
- rotation
- scale factor
- similar
- transformation
- translation

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
bearing	A direction obtained by measuring the angle, in degrees, clockwise from north and described using three-figure notation, e.g., 040°.
collinear	Three or more points are said to be collinear if they all lie on the same straight line.
displacement	A distance with a specified direction.
image	In transformations, the image refers to the object once it has undergone a transformation.
object	In transformations, the object refers to the figure that a transformation is applied to.
resultant vector	A resultant vector is the sum of two or more vectors.
scalar	A scalar quantity has magnitude but no direction.
vector	A vector quantity has magnitude and acts in a particular direction. A vector can be represented as a line segment with its direction labelled using an arrow. Vectors are considered equivalent if they represent the same displacement.

Knowledge, skills and understanding

Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a 🔍. These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

11.1.1 Work with bearings

Students are likely to work with bearings for the first time at Key Stage 4. They should have previously explored loci and therefore been introduced to the idea that, if we know an object is a specific distance away, then we know the object must lie somewhere on a circle with a radius of that distance. Combining this distance with a bearing allows us to know the exact position of the object on that circle. As a bearing is a measure of direction, when it is combined with a distance, a displacement is described. This means that the relative position of one object in relation to another (or the final position of an object relative to its starting position) can be determined.

Bearings are absolute directions, like compass points, and not relative directions, such as left and right; they do not depend on the orientation of the person receiving the directions. It is important for students to understand that the conventions around bearings mean that they provide unambiguous and precise directions, which are important when communicating directions. The conventions that students need to establish are: the clockwise direction of measure; the significance of north; and the use of three-figure notation. Students often fail to measure from north or to measure anticlockwise, especially when the bearing is greater than 180°. Ensuring that the north direction is not always represented as a vertical line helps to address the misconception that north is always pointing 'up' and provides an opportunity for students to think more deeply about the use of bearings to describe a change in position. Students can use local maps, and experiences of navigating in other subjects such as geography, to explore this

While the topic of bearings might be considered stand-alone, problems involving bearings often feature in other topics – such as those involving angle facts, constructions and trigonometry. Links with vectors can also be emphasised, as these two different methods for describing a displacement highlight the underlying idea of relative position. Problems involving bearings provide an opportunity for students to reason geometrically and problem solve. Moving beyond questions where students determine the bearing between two points, to scenarios where they are asked to provide possible positions for points that satisfy a particular bearing, as well as other additional constraints, supports students in thinking critically and giving justifications.

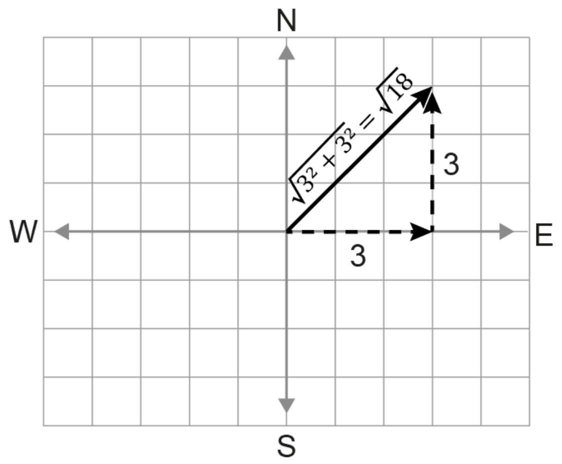
- 11.1.1.1 Understand that a displacement has both magnitude and direction
- 🔍 11.1.1.2 Understand and use the conventions of bearing notation to describe direction
- 11.1.1.3 Use the language and notation of bearings to reason geometrically

11.1.2 Reason with vectors

When an object has been displaced, its change in position has both magnitude (how far away it is) and direction (where the new position is in relation to the object's starting point). Vectors, which in Key Stage 4 begin to be represented as line segments with arrow labels, provide a means of describing a displacement, as they have both magnitude (represented by the length of the line segment) and direction (denoted by the arrow).

The formal use of vector notation is first introduced at Key Stage 4. However, students may have already been introduced to column vector notation in Key Stage 3, as part of work on identifying and describing translations. It is important that students recognise the relationship between the top number in a column vector and the change in the x direction, and the bottom number and the change in the y direction, when describing translations. Ensure that links with work on directed number are not overlooked. Students often develop a deeper understanding of negative numbers after working with vectors and, similarly, those who have difficulty with negative numbers might need extra support.

While the magnitude of a column vector when used to describe a translation is rarely considered, a column vector has both magnitude and direction. For example, $\binom{3}{3}$ has magnitude $\sqrt{18}$ and a north-easterly direction:



There are several ways to describe a vector between two points; it is important that students are familiar with them all and can use them interchangeably. The most common notation for a vector between the points A and B is $\overrightarrow{AB}$, but $\boldsymbol{a}$ and $\underline{a}$ may also be used and should be known to students.

Vectors can be operated on in the same way as numbers. It is important for students to explore how to add, subtract and multiply vectors by a scalar, to establish the geometric effect these operations have. A key understanding, when adding and subtracting vectors, is that two vectors are the same if they have the same magnitude and direction, and that the vector -$\underline{\boldsymbol{a}}$ is the vector with the same magnitude as $\underline{\boldsymbol{a}}$, but with movement in the opposite direction. As students' familiarity with vectors and their manipulation increases, their use of vectors can be developed to extend beyond the specific, to generalised proofs that depend on underlying geometrical structures.

Providing students with examples of vectors in the real world can help to deepen their understanding: two examples are vectors representing force and velocity. A force and a velocity are in a particular direction and the magnitude of the vector represents the strength and speed respectively.

11.1.2.1 Understand that a displacement has both magnitude and direction and can be described by a vector

11.1.2.2 Understand and use the conventions of vector notation and column vectors and interpret them geometrically

11.1.2.3 Understand the multiplication of a vector by a scalar (including a negative value) both algebraically and geometrically

11.1.2.4 Understand the addition and subtraction of vectors both algebraically and geometrically

11.1.2.5 Use vectors to construct geometric arguments and proofs

11.1.3 Explore enlargement further

At Key Stage 2, students developed an understanding that, when enlarging a shape, the image is similar to the original shape (i.e., the same shape, but a different size). At Key Stage 4, the relationship between lengths in similar shapes is extended to include the relationships between areas and volumes.

Students' work on similar shapes at Key Stage 2 should have already been developed at Key Stage 3 to include the idea of a centre of enlargement. When previously working with enlargements by considering only the scale factor, it can be easy to assume that the scale factor affects the dimensions of the image only. It is important for students to recognise that the scale factor affects the size of the image and its displacement from the centre of enlargement in the same way, as the position of the centre of enlargement in relation to the object determines the position of the image. Providing opportunities at Key Stage 4 to explore multiple enlargements with the same centre of enlargement can support students in appreciating enlargement as a continuous scaling.

When considering the displacement of an object during an enlargement, vectors can be used to support students' understanding of the process of transformation. The position of each vertex of an object in relation to the centre of enlargement can be expressed using vector notation. This provides a series of column vectors that can be operated on by multiplying each one by the scale factor. The resulting column vectors represent the displacement of each vertex from the centre of enlargement, and can therefore be used to identify the position of the corresponding vertices after the enlargement has taken place.

At Key Stage 3, enlargements involving positive scale factors are explored; at Key Stage 4, fractional and negative scale factors are introduced. It is important for students to recognise that an enlargement with a negative scale factor produces an image that is positioned on the other side of the centre of enlargement, with its orientation reversed. While enlargements can't be combined with other transformations in the same way as translations, reflections and rotations can. Identifying that an enlargement by a scale factor of −1 is the same as a rotation of 180°, is key to students understanding the geometrical structures that underpin enlargement by a negative scale factor.

11.1.3.1 Appreciate enlargement as continuous scaling

11.1.3.2 Understand that the scale factor affects both the dimensions of the image and the displacement of the image from the centre in the same way

11.1.3.3 Understand enlargement as a transformation of vectors

11.1.3.4 Understand the structures that underpin enlargement by a negative scale factor

11.1.3.5 Understand the relationship between lengths, areas and volumes in similar shapes

11.1.4 Combine transformations

Transformations describe different ways of mapping points on the plane to other points on the plane: connections can be made here with the concept of a function, explored in detail in theme 9. Each coordinate of the object represents an input that undergoes the same transformation: this results in a set of new coordinates, or outputs, for the image. Combining and reversing transformations can be linked to the idea of composite and inverse functions, to give geometric insight to the ideas explored in theme 9.

Prior to Keys Stage 4, students will have worked with translations, rotations and reflections and learnt to distinguish between them. Translations are the only transformation to maintain both congruence and orientation and so students are often introduced to this type of transformation first. As with translations, rotations also maintain congruence, but the orientation of the object is changed. At Key Stage 2, students will have explored $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{3}{4}$ turn rotations. This will have been generalised at Key Stage 3 to any angle where the size and direction of turn are specified. Students often struggle to visualise a rotation and it is important that they have had experience of hands-on activities, such as using tracing paper or dynamic geometry software, to familiarise themselves with how an object behaves when it undergoes a rotation.

If a student has a physical object to manipulate, they can rotate or translate it by sliding it along a surface (representing the plane). A reflection is the only transformation that affects the 'sense' of an object, in that the object is 'flipped' to form a mirror image. Transforming an object by reflecting it offers the full range of possible congruent shapes and a context in which congruence can be explored further.

A transformation can be followed by one or more further transformations, and the combination of transformations can often be described by a single transformation. The simplest case is that of a translation, followed by a translation, which can be combined and described as a single translation, known as the resultant vector. It is important that connections are made here with the ideas explored in 11.1.3. Exploring combinations of reflections and rotations can be helpful in establishing what happens to the general point (x, y) when reflected or rotated in different ways, and supports students in moving beyond an understanding of specific cases to being able to generalise combined transformations.

11.1.4.1	Recognise and use the key characteristics of translations, rotation and reflection
11.1.4.2	Compare and contrast the key characteristics of translations, rotation and reflections with particular reference to the degrees of freedom
11.1.4.3	Use and apply the key characteristics of transformations to analyse situations where transformations are combined

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

11.1.1.2 Understand and use the conventions of bearing notation to describe direction

Common difficulties and misconceptions
Bearings can often be treated as a 'standalone' topic in the mathematics classroom, and it is arguably rare that they are explored in any real depth. It can be tempting to simply view three-figure bearings as a quirk of notation, and the conventions taught as an easily forgotten set of rules. Encouraging students to think about the relationship between two points, and what information is needed to describe this accurately and precisely, can help them to understand **why** each element of the bearing is required and to build a deep and connected understanding. A key misconception with bearings is that they are the same as angles, and so students may give the angle formed between three points, rather than the angle clockwise from north. This is particularly likely to be the case when students have to use rules, such as complementary angles, to derive the angles first. Students need to attend to which direction is north and be aware that a diagram might not always be orientated so that north is vertically upwards. Ensuring that students experience a variety of tasks and contexts can help them, in time, to avoid such errors. It may be surprising to both teachers and students that one of the most common misunderstandings with bearings is not mathematical but linguistic. Students can tend to work in the order that the points are described in the question, rather than paying attention to the prepositions that are being used. In the instruction to measure the bearing 'of the lighthouse from the boat', for example, students may place their protractor on the lighthouse because it appears first in the instruction. Strategies such as 'underline the key words' can overlook the fact that the most important words can be seemingly insignificant prepositions such as 'to', 'of' and 'from'. There are several examples exploring the language of bearings in order to give students the best chance of developing a deeper understanding.

Students need to	Guidance, discussion points and prompts
Appreciate why bearings are measured from the same point (north) and in the same direction (clockwise) *Example 1:* *Fenn is trying to direct his friends to a picnic spot in the park. He says, 'Stand at the bandstand then turn 90 degrees and walk in that direction until you find me.'* *a) What might be the problem with Fenn's directions?* *Fenn tries again. He says, 'Stand at the bandstand facing the church, then turn 90 degrees and walk in that direction until you find me.'* *b) Will Fenn's directions be effective now? Why or why not?*	Students need to appreciate why certain information is needed to accurately and precisely describe direction. In this example, there is an opportunity to unpick the everyday **language** used to give directions. As the example develops, teachers should guide students to think about the necessary criteria to avoid ambiguity. Without a reference point for where to start their rotation (in this case, the church), Fenn's friends do not know where to measure their turn from. In the case of bearings, we fix this point of reference as north. Similarly, without an instruction about which direction to turn, Fenn's friends could walk in entirely the opposite direction. In the case of bearings, we fix the direction as always being clockwise so that we can express the turn in the most efficient way possible.
Example 2: *The map below shows the major airports of England and Wales.*  *Pick two airports to fill the gaps correctly. Is there more than one way to do this?* *a) _ is due east of _.* *b) _ is south west of _.* *c) The angle between _, _and _ is 90°.* *d) The angle between _, _and _is 225°.* *e) The bearing from _to _ is 090°.* *f) The bearing of _from _ is 225°.*	*Example* 2 connects compass points, angles and bearings, **deepening** students' understanding of what is the same and what is different about each of these. Throughout, the intention is for students to reflect on the information needed for each sentence. Draw attention to the fact that, when using angles, students need to refer to a third airport, rather than the two needed for all of the other parts. This is so that there is a line from which to measure the angle. Students may not have considered the definition of an angle for some time, although they will have been working with angle problems extensively. An angle can be defined both as a measure of turn, and as a figure formed by the meeting of two rays. Three points are needed to create an angle: a point on each ray and the single point where the two rays meet. In the case of bearings, one ray is always the same: the north line. Teachers might handle this task differently at different stages of students' learning. If it is being used to introduce bearings, then they might accept answers where the relationship between airports is approximated. However, three-figure bearings by their definition are designed to be more precise than worded compass directions. Teachers may be more rigid about what they accept as a bearing of 090° if students are already working accurately with bearings. Discuss with colleagues which answers they feel are appropriate for the point they use this task in your scheme of work; is there agreement among your team?

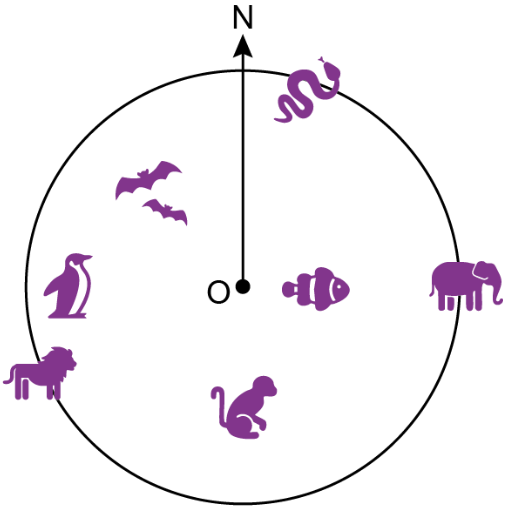
Example 3:

The head keeper of a zoo has a diagram showing where the different enclosures are in relation to his office (O).

a) Match the animal to the bearing.

b) For any unmatched bearings, suggest an animal and a possible position on the diagram.

Lion	*010*
Elephant	*350*
Penguin	*090*
Fish	*270*
Bats	*000*
Snake	*180*
Monkey	*250*
	060
	100
	320

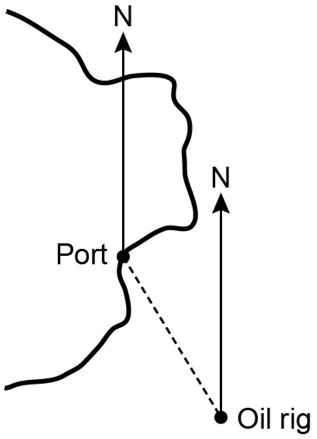
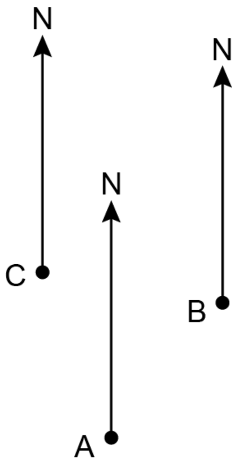


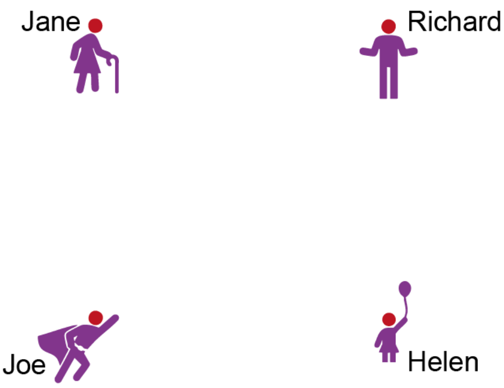
Before students can use bearings fluently to describe relationships between multiple points, it is important that they are secure in their understanding of what constitutes a bearing. Keeping the start point fixed simplifies the diagram and ensures that students practise working clockwise from the north line. Alongside these invariant aspects, the **variation** in different end points focuses students' attention on what the relationship between start and end points looks like at different bearing points.

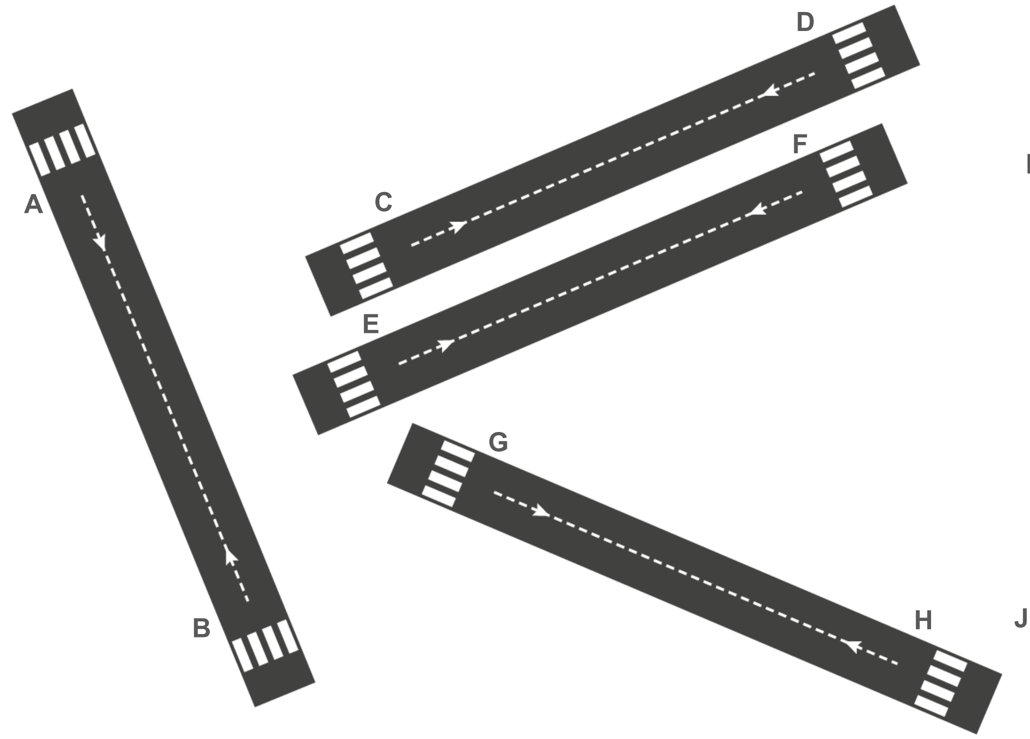
The **representation** of the circle, reminiscent of a compass diagram, is not often used in the diagrams of questions related to bearings. However, it is key in students' learning to understand the bearing as a measure of turn, and to associate angle measurements with compass points. Comparing the penguin to the fish or elephant can help reinforce this – they are all 90° from the north line, but only the fish and elephant are at a bearing of 090°.

Particularly important here is understanding that the fish and the elephant are on the same bearing, **deepening** students' understanding of bearings as a measure of direction, but not displacement. This can be further developed by comparing students' responses to part b – do they all position their respective animals the same distance from point O for the bearings 350°, 000°, 060° and 100? If so, do they actually understand that they could place their animal anywhere along each ray from the centre at those angles?

Some activities, such as *Examples 3* and *9,* arguably lose some of their value if students cannot physically interact with them. Students will interact differently with the diagrams based on whether they are presented to them as a large image on the classroom screen, or as physical printouts that they can move and annotate. Often, teachers are forced to be pragmatic about printing due to resource constraints, and so it will be necessary consider when the cost of printing can be justified by the learning benefits of having a printed resource. This may be a helpful discussion to have as a team, especially if colleagues have already tried the materials and have experience of the difference in how students interact with the mathematics when projecting or printing.

Understand how the relationship between start and end points can be expressed using a bearing *Example 4:* *The diagram below shows the position of an oil rig from a port.*  *a) What is the bearing of the oil rig from the port?* *b) Find the bearing to the oil rig from the port.* *c) On what bearing should a ship travel from the port to the oil rig?* *d) What is the same and what is different about parts a to c?*	*Example 4* is an opportunity to focus on the **language** used when describing displacement, and to ensure that students are clear about how to identify start and end points. This can help to tackle assumptions that students may have about the first location described in the statement being the 'start' point of the bearing. Draw attention to the prepositions 'from', 'to' and 'of' in these statements. Students may be accustomed to underlining key vocabulary but may not have considered that seemingly small and insignificant words could be so crucial to a question's meaning. The **variation** is such that all the statements are different ways of asking for the same bearing. Students might assume they have made a mistake, as this is an unusual structure for the task. Ensure that students are clear about the task's intention and are reassured by their answers. The task has been designed so that students can also practice using a protractor to measure the angles. However, teachers may instead choose to give the students either one of the angles (so that complementary angles are revised) or both of the angles (so that the focus is narrowed purely to the understanding of the language constructs).
Example 5:  *Use the diagram above to estimate the bearings and complete the following sentences with the words 'from' or 'to'.* *a) 050 is the bearing of B ____ A.* *b) 150 is the bearing of C ____ A.*	*Example 5* builds on the significance of the words 'to' and 'from', explored in the previous example. Focusing on **language** in this way helps to clarify how bearings are structured. Students cannot rely on assumptions that the letter listed first is the start point, and need to attend to the angle given. The expectation is that students will use a combination of estimation and deduction to work out which of the pairs of points is associated with the given angles, and through this process consolidate the information that constitutes a three-figure bearing. For example, if lines were drawn from A to both points B and C, then two acute angles would be formed. However, only the line from A to B forms a **bearing** of 050°, as the angle must be measured clockwise from north. Connecting pairs or groups of questions together will maximise the learning opportunities provided by the **variation** in this question. For example, parts a and d are identical, just phrased in opposite ways. Parts b and e both place point C first and point A second, so the emphasis is on the directional language. This pair of questions could also be used to explore complementary angles, as the bearing of 330° in part e means that an angle of 30° is

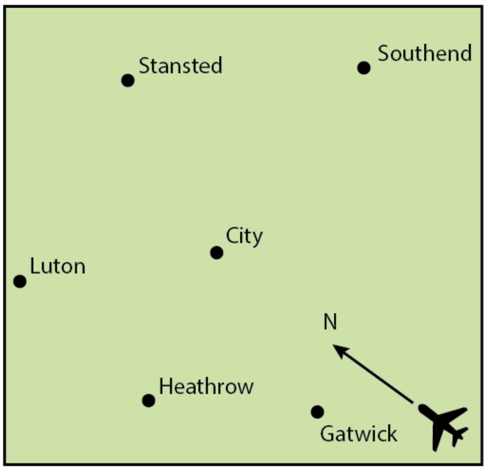
c) *100 is the bearing of C ____ B.* d) *050 is the bearing of A ____ B.* e) *330 is the bearing of C ____ A.* f) *280 is the bearing of C ____ B.* g) *230 is the bearing of A ____ B.*	created in the anticlockwise direction, which sums to 180° with the 150° bearing from part b. It is important that students are able to move fluently and flexibly between working with angle rules and expressing directions as bearings; this is explored further in subsequent examples.
Example 6: *Jane, Richard, Joe and Helen arrange themselves into the four corners of a square. Jane is due north of Joe.*  *Write names to complete these sentences. Is there more than one way to answer any of these?* a) *The bearing from ___ to ___ is 090°.* b) *The bearing from ___ to ___ is 180°.* c) *The bearing from ___ to ___ is 045°.*	*Example 6* may seem more straightforward than the previous two examples, but it is the first time in this sequence that the north line is not provided. The arrangement of a square, with familiar and predictable properties, is intended to support students to focus on the importance of **direction** when working with bearings. The **variation** inherent in the three questions exploits this further. Those less secure about this difference might assume, for example, that there are four ways to complete part a as there are four right angles in the triangle. Understanding that there are only two ways to complete parts a and b, and one way to complete part c, is essential for students to differentiate between bearings and angles. This example can be further developed by asking students to find the bearings not described in parts a to c, **deepening** their understanding of how to use angle properties to find bearings. Students who are developing a real fluency with angles and bearings might readily identify the values of 135°, 225° and 315° for the three other bearings formed by the diagonals. However, it may be counterintuitive to assign these angles to the correct people in the diagram.
Example 7: *Airport runways have numbers painted at either end. This represents the bearing that the plane is using to approach the runway. It is measured to the nearest 10 degrees, and then the 0 from the ones column is removed.* *For example, this horizontal runway has 09 for planes flying east and 27 for planes flying west.*  a) *Suggest values for each end of each of the runways below (labelled as A to J).* b) *What do you notice about the relationship for each pair of values at either end of the runway?*	The **variation** in *Example 7* has been designed so that particular properties of bearings can be identified. For example, students' attention could be drawn to the fact that bearings from points on parallel lines will be the same (i.e. that the same bearing should be used for parts c and e, and d and f, respectively). Part b can be used for **deepening** students' understanding of how bearings connect with angle rules. The difference between the two numbers is always 18, as the difference between the bearings is always 180°. Do students intuitively realise that this is because a runway is always a straight line? Draw attention to this if necessary. The intention is that students estimate the bearings required for each part, but teachers may choose to provide some angles (such as a and c) to support access to the task. Discuss with your team how providing angles changes the task, and what is gained and/or lost by reducing the element of estimation.



Express and find directions using bearings

Example 8:

A pilot is flying into the UK from Europe. The plane's position relative to the six London airports is shown below:



a) Measuring from the nose of the plane, which airports are on a bearing between:

(i) 000 and 090?

(ii) 090 and 180?

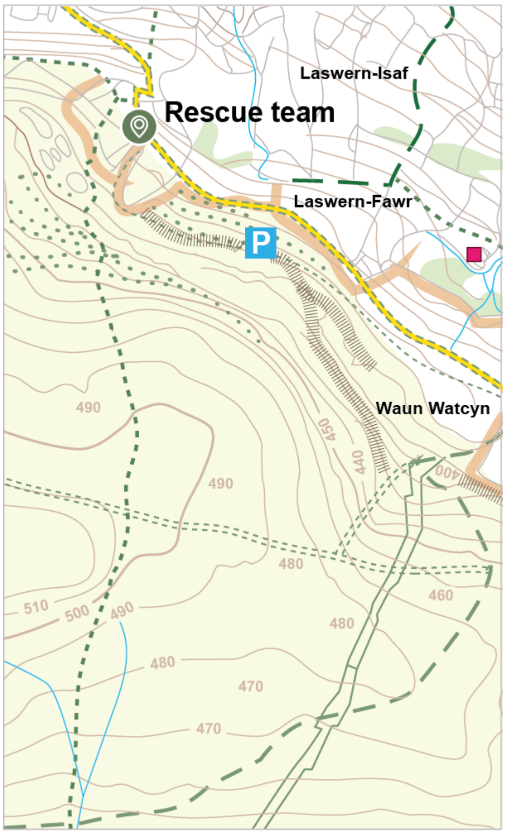
(iii) 180 and 270?

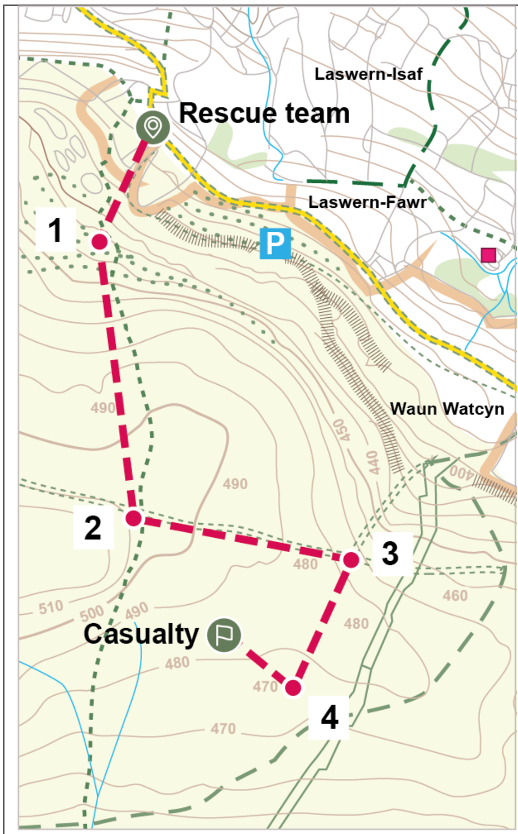
(iv) 270 and 000?

Safe air travel relies heavily on the use of bearings, and this provides a relatable context for students. Even if they themselves have not flown, they will have seen planes in the sky and be aware of airports. It also provides an opportunity to explore more realistic **representations** where north is not orientated vertically upwards.

In *Example 8*, part a is designed for **deepening** thinking about how the start point (here, taken as a point on the image of the aeroplane) is crucial for determining the bearing. Students may be surprised that there are no airports between the bearings given in part (ii) or (iii). Teachers could build on this by asking where possible airports would be situated, or where the plane would need to move to so that some of the airports in the diagram would fit the criteria. Such discussions can help reinforce the idea that bearings are relative, not absolute, measures of direction. There needs to be a point of reference, and the bearing would change if that point of reference changed.

Discuss with your team how students' thinking changes when they are asked to consider a region between two bearings, rather than a specific bearing. Do they engage differently with the diagram? It is important that students feel empowered to annotate and draw additional lines on any diagrams that they work with. These additional lines are often key to unlocking more complex geometry problems. What does your department do to support this as a routine part of working with mathematical imagery?

b) The plane is meant to land on a bearing of 330. Which airport might it be due to land at? *c) There is an issue with the airport, so the pilot is told to change to a bearing of 010. Which airport is the plane redirected to?*	
Example 9: *A mountain rescue team (marked on the map) needs to reach a casualty in a mountainous area where GPS does not work. North is at the top of the map.*  *The casualty is on a bearing of 170° from the rescue team.* *a) Suggest possible locations of the casualty on the map.* *The position of the casualty is marked by a flag on the second map, below. It is not possible to reach the casualty on a direct bearing, so the rescue team plan four safe points along the route (shown as 1 to 4).* *b) What are the five bearings that the rescue team need to take?*	Contexts are a form of **representation**, as they provide a physical or visual format to help reveal mathematical structure to learners. *Example 9* has been provided by a member of a real-life mountain rescue team, so teachers can see how bearings have genuine applications. Here, the focus is on measuring bearings at different points. Students should be encouraged to draw north lines on the diagram to replicate the experience of using a compass to find magnetic north. This question could also be used to exemplify a key advantage of vector thinking: as long as the rescue team's end point is at the casualty, the route taken is less important, i.e., the resultant vector of their path is the same as the displacement vector of the casualty from the rescue team. See the professional development prompt from *Example 3*, above, for some thinking points around supporting students to physically interact with this representation. Part a reinforces the idea of bearings being a measure of direction, but not distance, and so students might question how a rescue team find a casualty. This can be used as an opportunity for **deepening** their understanding by adding a measure of distance. Rescue teams need to work in conditions where visibility is poor, so they use paces as the basis for distance descriptions. One pace is the equivalent of two steps (for example, every time a person's right foot touches the floor), and 70 paces approximates to 100 m. Students can use this information to identify the number of paces required for each leg of the route. 'Not all students will have experience of the outdoors or map-reading, and it is important that they have the opportunity to gain this through school.' Discuss with your team how far you agree with this sentiment, and what you might need to put in place to ensure that this lack of experience is not a barrier to the mathematics. For example, you might consider liaising with the humanities department to discuss curriculum cross-over for map-reading; or allowing more lesson time to explore the map before working on the bearings.



Use angle facts to find bearings

Example 10:

Tyriq is in his maths exam but only has a 180° protractor.

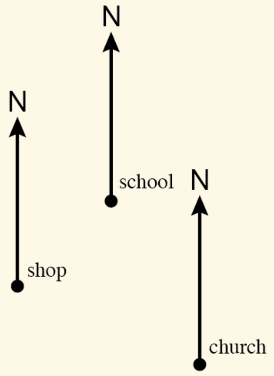
He thinks this means he needs to skip the question shown below, as the bearings are greater than 180 degrees.

Is Tyriq correct? Why or why not?

Some students practise orienteering using three village landmarks. The shop and school are both the same distance from the church.

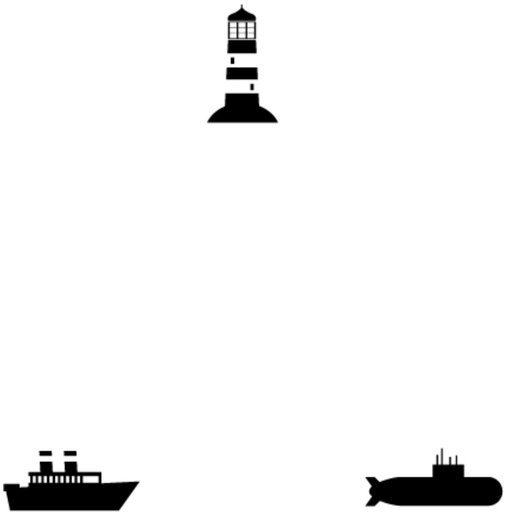
a) Find the bearing they need to take to navigate to the school from the church.

b) Find the bearing they need to take to navigate from the school to the shop.



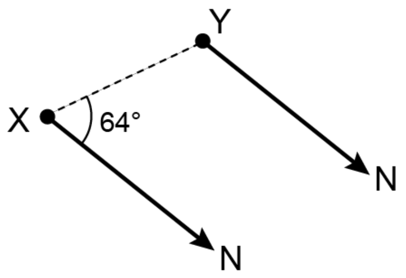
Example 10 reminds students that they can build up components of bearings that are greater than 180°. Ensure they know that they can engage with the **representation**, for example by drawing on a 'South' line so they are measuring on from 180°. It may seem obvious to teachers that this is the case, but students may only work with protractors a few times each year, and may not have built the fluency needed to make such connections.

Viewing mistakes as learning opportunities is a valuable skill, and it can be helpful for students to work with fictional students' mistakes to de-personalise the errors. Reflect on your classes. Is there a culture of embracing mistakes? If not, how might questions like this help to build one?

Example 11: *A submarine and a ship are both the same distance from a lighthouse, but in different directions. The submarine is due east of the ship.*  *a) If the **angle** between the ship, the lighthouse and the submarine is 30°, then what is the **bearing** of:* *(i) The submarine from the lighthouse?* *(ii) The ship from the lighthouse?* *b) If the **bearing** of the ship from the lighthouse is 200°, then what is the **angle** made at:* *(i) The submarine between the lighthouse and the ship?* *(ii) The lighthouse between the ship and the submarine?* *c) If the **bearing** of the lighthouse from the submarine is 300°, then what is the **bearing** of:* *(i) The ship from the lighthouse?* *(ii) The submarine from the ship?*	In this example, the **language** of ‘angle’ and ‘bearing’ is explicitly compared: ensure students are aware of the similarities and differences between the two terms. In part b, attention could be drawn to the fact that there are now three points of reference, rather than two, reinforcing the understanding that the third point that creates the angle in a bearing is always north. Drawing the **representation** of a circle (centred on the lighthouse, with the ship and submarine on the circumference) may help students to identify the salient information in the question. If the submarine were not due east of the ship, creating a horizontal base to the isosceles triangle that creates all three points, then we would not have enough information to answer this question. This unpicking of the rubric could be used for **deepening** students’ understanding of when angle rules can and cannot be used in such questions. If we only had the first sentence, then we would not know where on the circle each of the vessels was situated. However, regardless, they would form an isosceles triangle. Dynamic geometry software can be particularly powerful for representing such situations as the points can be moved until enough information is given to define them more firmly.
Example 12: *In each of the examples below this question, one angle has been given.* *Find the bearing of X from Y each time.*	*Example 12* may look most like a ‘classic’ bearings question, with students asked to use their knowledge of angle rules to identify bearings. The **variation** has been planned so that the amount students have to attend to builds gradually each time. In part a, north is no longer orientated vertically upwards, but the bearing can be found in one simple step – it is the complement of 64°. In part b, north is returned to its ‘usual’ position, but students need to take two steps to find an angle at point Y and then

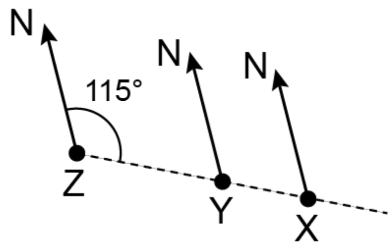
calculate the bearing. Parts c and d again change the orientation, but the focus is now on the properties of parallel lines. In part c, students simply need to identify that the bearing required corresponds with the angle given; in part d, they need to both identify the alternate angle and subtract this from 360°.

a)

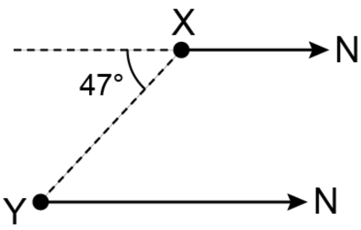


b)

c)

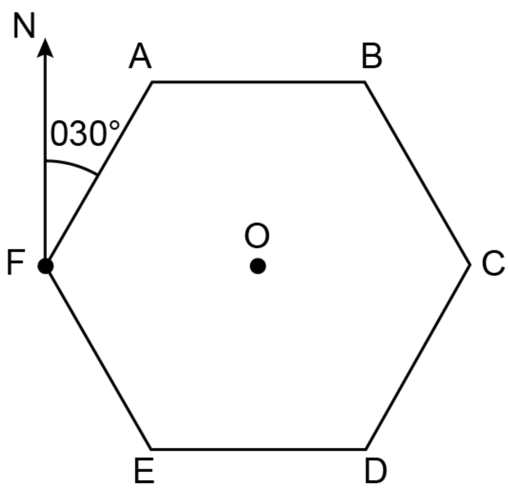


d)



Example 13:

Anya, Bailey, Cordelia, Dawn, Ethan and Faith stand around a regular hexagon. Oz stands in the centre.



Faith throws a ball on a bearing of 030° to Anya.

a) Who else could throw the ball on a bearing of 030° so that their friend could catch it?

Example 13 is the final one in the sequence of bearings questions. It uses the **representation** of a regular hexagon, so students need to connect their understanding of the angle properties of regular shapes with their new learning on bearings. The first bearing is modelled on the diagram, but subsequent bearings in the question are not. Teachers should encourage students to be in the habit of annotating diagrams and drawing extra lines if required.

The **variation** in this question pulls together students' learning on bearings. There is a particular focus on parallel bearings in part a, and bearings in opposite directions in part b. This variation extends to the use of the same diagram to explore vectors (exemplified key ideas 11.1.2.4, below). By directly contrasting how bearings and vectors describe particular movements, students can build a keener understanding of what is the same and what is different about them.

Parts c and d offer a final exploration of two common misconceptions about bearings, **deepening** students' understanding of the difference between angles and bearings. In part c, students need to be clear that, although the angle formed between north, D and E is 90°, only the bearing of D from E is 090°. In part d, students need to be able to articulate that the angle formed at D by the line

Oz throws the ball to Dawn, on a bearing of 150°. Dawn throws it back to Oz. *b) On what bearing has Dawn thrown the ball?* *c) Using the bearings 090° or 180° who could throw the ball to whom?* *Giles wasn't playing, but he knows that Cordelia threw the ball to Ethan. He says, 'The ball was thrown on a bearing of 120°.'* *d) Is Giles correct? Give reasons for your answer*	segments ED and DC is not the same as the bearing of C from D. Compare each part of this example with the equivalent parts of *Example 5* from key idea 11.1.2.4 Discuss your observations with a colleague. How might looking at these two tasks together help to reinforce the difference between bearings (which have direction, but no magnitude) and vectors (which have both magnitude and direction)? For example, in part a, students could choose Ethan throwing the ball to either Bailey or Oz, and both throws would be described as being on a bearing of 030°. How does this compare to the possible answers for Ethan in the vector version of this question?

11.1.2.4 Understand the addition and subtraction of vectors both algebraically and geometrically

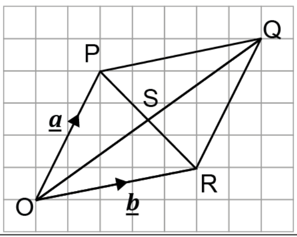
Common difficulties and misconceptions

Students often first meet vectors in the context of transformations and are likely to have experience of expressing a translation as a column vector. They can confuse column vectors with coordinates as they generally both refer to x and y. It is essential that teachers are clear that coordinates specify the **position** of a point, while column vectors specify its **movement**. The x- and y-values of coordinates relate a point's position to the **axes**; the x- and y-values of a column vector identify how far a point has been moved to the right (or left) and up (or down) in relation to its **starting point**.

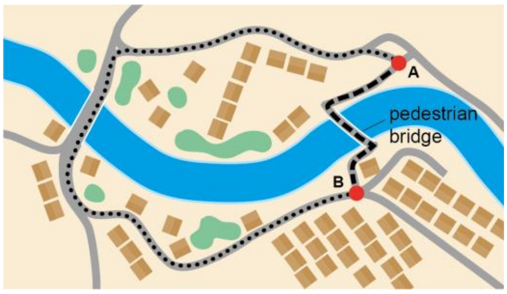
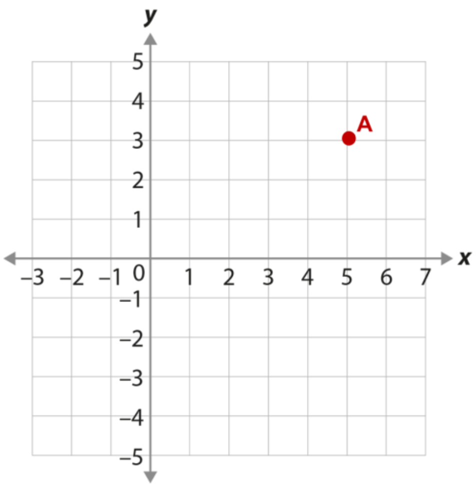
New learning about vectors in Key Stage 4 can cause further confusion, as students need to grapple with unfamiliar notation and a whole new way of thinking about movement and position. The vector notation $\overrightarrow{AB}$ can be relatively intuitive for students as it clearly represents the movement from point A to point B. However, more abstract notation such as $\underline{\boldsymbol{a}}$ or $\boldsymbol{a}$ makes the connection to movement much less obvious and students can mistakenly think of the vector as a description of the length of a line segment.

Students therefore need a thorough grounding on what a position vector is (and what it is not) before they begin to operate with vectors. A vector describes the magnitude and direction of a displacement in two- or three-dimensional space and can be represented as a **directed** line segment. When a vector, such as $\underline{\boldsymbol{a}}$, is defined, it can be placed at any starting point, and so there are infinite parallel directed line segments described by the same vector. When the starting point (tail) of the vector is placed at an origin and the end (tip) at a specific point, a position is described.

Geometric problems can be solved using the rules for vector addition: when two vectors are added (or subtracted), another vector (the resultant) is created. In order to be able to add two vectors, the tail of the second vector needs to coincide with the tip of the first. Without the understanding that directed line segments with the same magnitude and direction are the same vector, this can confuse students, as they may think that only vectors that are physically touching in a diagram can be added. It is common to think of combining vectors along a 'route' from one point to another. However, before students can do this fluently, they need to readily recognise parallel directed line segments.



To subtract two vectors, the tails of the vectors can be placed together and the resultant vector (the difference of the two vectors) found. In the diagram to the right, the vector $\overrightarrow{PR}$ is the resultant of $\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$ and the vector $\overrightarrow{RP}$ is the resultant of $\underline{\boldsymbol{a}} - \underline{\boldsymbol{b}}$. Students can sometimes make the mistake of placing the tips of the vectors together, when adding and subtracting vectors. Reversing the direction of the second vector and so adding the inverse vector rather

than subtracting the original can help to address this (for example, writing the vector $\overrightarrow{PR}$ above as $-\underline{\boldsymbol{a}}+\underline{\boldsymbol{b}}$. This also emphasises a consistent tail to tip method for both the addition and subtraction of vectors, and can be further supported by representing the vectors as both column vectors and directed line segments. Teachers should draw attention to the links between the two representations when vectors are added.	
Students need to	**Guidance, discussion points and prompts**
Understand that a displacement can be described by a vector *Example 1:* *Mr Caldicott teaches at the school his son Ben attends, but they travel to and from school separately. Mr Caldicott drives from school (A) to home (B), while Ben walks using the pedestrian bridge.*  *When they both arrive home:* *a) Whose journey covered the longest distance?* *b) Who is further from where they started?*	*Example 1* clarifies the difference between distance and displacement. Central to students' understanding of vectors is an awareness that displacement depends on start and end point, but the 'route' between these is arbitrary. In vector terms, it does not matter whether a journey consists of $\underline{\boldsymbol{a}}$ then $\underline{\boldsymbol{b}}$ or a direct route $\underline{\boldsymbol{a}}+\underline{\boldsymbol{b}}$; in both cases the displacement is the same. It is important that students have a secure understanding of this **language**, and that teachers do not allow the two terms to be used interchangeably. The image in this example was originally used in Core Concept document *9.4 Sequences, functions and graphs*, to illustrate the difference between the distance-time graphs of two different journeys to the same point. A graphical **representation** of displacement may help students to appreciate the concept: Mr Caldicott's displacement from point A will vary considerably at different stages of his long route home, but he will end up with the same displacement as Ben. Comparing the distance-time graph from 9.4.1.1 with a displacement-time graph may also reinforce the differences.
Example 2: *An ant travels from point A to three different places on the grid.*  *Its movements to/from each consecutive point are recorded as column vectors:*	*Example 2* further exemplifies the concept of displacement, this time in the context of a point being translated on a cartesian coordinate grid. Students work between two **representations** – the grid and column vector notation – both of which should be familiar to them from Key Stage 3. How students interact with this question may help teachers to plan their next steps for teaching. Some students may rely entirely on physically mapping out the movements on the grid. Others may recognise that they could surmise the overall displacement by totalling the movements in x (the top value in the column vector) and the movements in y (the bottom value). It is important that, whatever their initial approach, they are supported to appreciate the connection between both approaches. The **variation** in this question is such that the vectors in parts b and d, $\overrightarrow{AB}$ and $\overrightarrow{AC}$, are identical, even though there is a longer sequence of movements to achieve the latter vector. The intention, as with the map example above, is to reinforce the point that an overall vector is only defined by its start and end points.

$$\begin{pmatrix}-2\\0\end{pmatrix}, \begin{pmatrix}0\\-2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}$$

a) Find the coordinates of the ant's final destination, point B.

b) Write the column vector $\overrightarrow{AB}$.

Another ant travels from point A to five different places on the grid:

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\-7\end{pmatrix}, \begin{pmatrix}-8\\2\end{pmatrix}, \begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}$$

c) Find the coordinates of the ant's final destination, point C.

d) Write the column vector $\overrightarrow{AC}$

e) What do you notice?

Understand that directed line segments with the same magnitude and direction are the same vector

Example 3:

The grid is made up of congruent parallelograms.

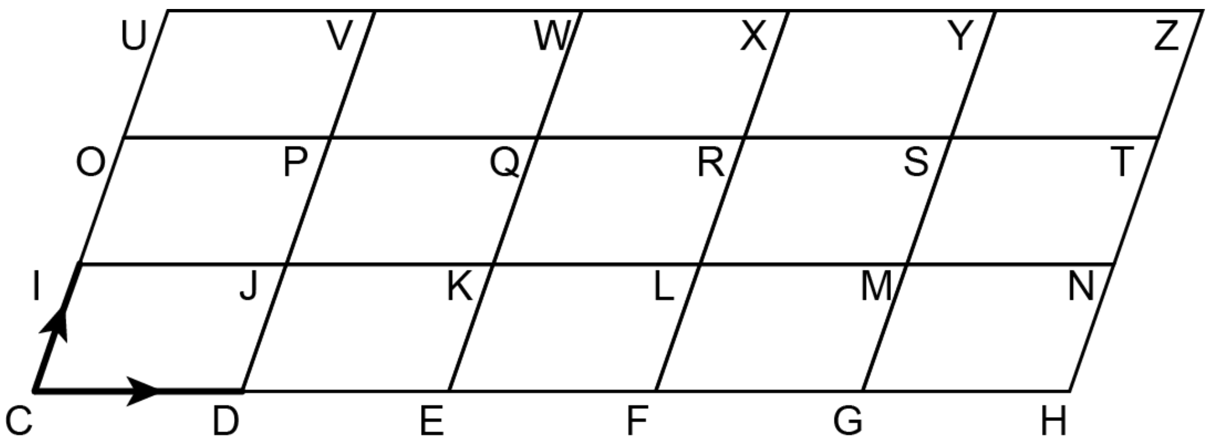
If $\overrightarrow{CD}$ = $\underline{\boldsymbol{a}}$ and $\overrightarrow{CI}$ = $\underline{\boldsymbol{b}}$, write in terms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$:

a)	$\overrightarrow{DE}$	*g)*	$\overrightarrow{NZ}$
b)	$\overrightarrow{RS}$	*h)*	$\overrightarrow{CJ}$
c)	$\overrightarrow{JP}$	*i)*	$\overrightarrow{JQ}$
d)	$\overrightarrow{QR}$	*j)*	$\overrightarrow{CQ}$
e)	$\overrightarrow{QW}$	*k)*	$\overrightarrow{JR}$
f)	$\overrightarrow{KW}$	*l)*	$\overrightarrow{JW}$

In *Examples 1* and *2*, routes of differing levels of complexity resulted in the same displacement. *Examples 3* and *4* introduce the idea of parallel vectors also being the same. The emphasis moves from considering fixed start and end points to the relationship between the start and end points, **deepening** students' understanding of vectors as entities in their own right.

The **variation** offers a logical progression of ideas, starting with vectors that are equivalent to $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$, and building up to multiples of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$. At this stage, it is important that students are secure in the idea of what constitutes the 'same' vector, and so all of the required movements are in the same direction. In part g onwards, students also begin to explore the additive structure of vectors. Parts j to k are the only without counterpart answers within the questions; ask students to generate their own congruent vectors.

What further questions might you ask using this grid? If teachers are not confident that students have understood, how might further questions be structured to help them build insight into the underlying mathematical structures?



Know that $-\underline{a}$ is a vector with the same magnitude as $\underline{a}$, but pointing in the opposite direction

Example 4:

Using the diagram from Example 3,

If $\overrightarrow{CD}$ = $\underline{\mathbf{a}}$ and $\overrightarrow{CI}$ = $\underline{\mathbf{b}}$, write in terms of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$:

a)	$\overrightarrow{DC}$	h)	$\overrightarrow{LS}$
b)	$\overrightarrow{JI}$	i)	$\overrightarrow{LG}$
c)	$\overrightarrow{JD}$	j)	$\overrightarrow{SL}$
d)	$\overrightarrow{JP}$	k)	$\overrightarrow{SX}$
e)	$\overrightarrow{PJ}$	l)	$\overrightarrow{XS}$
f)	$\overrightarrow{VJ}$	m)	$\overrightarrow{GL}$
g)	$\overrightarrow{DP}$	n)	$\overrightarrow{UP}$

Example 4 uses the same **representation** as *Example 3*, offering continuity as students build their understanding of the structure of vectors. This time, the emphasis is on direction, so that students are directly comparing vectors with the same magnitude but different directions. This should lead to the understanding that a negative vector is a movement in the opposite direction to its positive equivalent.

The **variation** in this question is subtly different from that in the previous example. Previously, the emphasis was on finding examples with the same answer to draw out the idea of congruence. This time, teachers need to draw students' attention both to pairs that are congruent and to pairs that are reversals of each other (such as parts k and l, which and then in turn each be compared with parts m and n respectively).

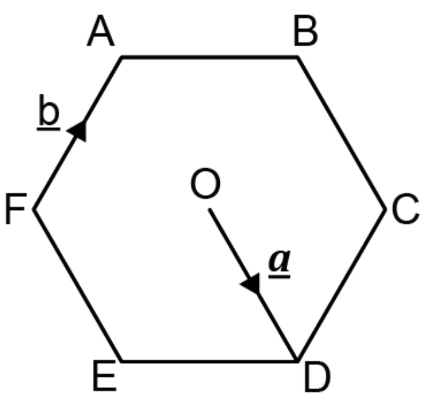
As with *Example 3*, the latter parts of this question start to explore the addition of vectors. Students may readily appreciate the similarities with the additive structure for numbers and algebraic terms, and so there is a risk that they start to replicate processes without making the connection with the resultant movement. It is important that they have time to consolidate the movements described by increasingly complex vectors. Consider if the fourteen examples here are sufficient for **deepening** students' understanding before moving on.

Discuss with colleagues how you will know whether students are using superficial patterns or connecting the additive structure with movement on the diagram. It may be helpful to draw on the prompts suggested below to think about different ways to assess students' understanding:

- *"Suggest another vector with the same answer as part i…and another…and another…"*
- *"How many possible vectors are there with the same answer as part a? How about part f? And part j?"*
- *"Add a vector to your answer to part i, so that the resultant vector is horizontal. Is there more than one way to do this?"*
- *"Write three vectors that are parallel to your answer in part a/e/h. Can you write a parallel vector that nobody else can?"*

Example 5:

Anya, Bailey, Cordelia, Dawn, Ethan and Faith stand around a regular hexagon. Oz stands in the centre.



Faith throws a ball along vector ***b*** *to Anya.*

a) *Who else could throw the ball along vector* ***b*** *so that their friend could catch it?*

Oz throws the ball to Dawn, along vector ***a****.*

Dawn throws it back to Oz.

b) *What vector has Dawn thrown the ball along?*

c) *Using one of vectors* ***a****, –****a****,* ***b*** *or –****b****, who could throw the ball to whom?*

Giles wasn't playing, but he knows that Anya threw the ball to Bailey. He says, 'I think that the ball was thrown along the vector ***a*** *+* ***b****.'*

d) *Is Giles correct? Give reasons for your answer*

In *Example 5*, students begin to describe a vector joining two points, by creating a path along vectors that are known. The **representation** of a regular hexagon, and the scenario of a ball being thrown, is used to support students as they build up their understanding of vectors as entities in their own right. For students at this stage, a vector may still seem somewhat intangible, whereas a ball being thrown is something they can readily imagine.

Students and teachers may recognise the premise from *Example 13* of 11.1.1.2, and the **variation** between the questions is intended to draw out the similarities and differences between bearings and vectors. It is important that students can recognise under what conditions two bearings are the same (where the line segments form the same angle measured clockwise from north, regardless of the length of the line); and under what conditions two vectors are the same (where the line segments are at the same angle and are equal in length). This can help students build the essential understanding that parallel vectors are the same, and that routes between points can therefore be created if required vectors, or any line segments parallel to them, are labelled.

Describing $\overrightarrow{AB}$ in terms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ relies on students' understanding of the structure of a regular hexagon, including that a regular hexagon is made up of six congruent equilateral triangles, with opposite sides being parallel to each other. This means that $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$**.** It is important to check students' understanding of the terms 'congruent' and 'equilateral' and encourage the precise use of **language** when discussing the mathematical structure of regular hexagons.

Discuss with teachers the importance of asking students to explain how they know that $\overrightarrow{AO} = \underline{\boldsymbol{a}}$ and explore possible explanations that students might give. For example, students may recognise that the points AOD lie on a straight line and O is the midpoint of the line AD, so $\overrightarrow{AO}$ must therefore equal $\underline{\boldsymbol{a}}$. Or they may take an alternative approach, identifying, for example, that $\overrightarrow{AO} = \overrightarrow{AF} + \overrightarrow{FE} + \overrightarrow{EO}$. Emphasise that determining that $\overrightarrow{FE} = \underline{\boldsymbol{a}}$ and $\overrightarrow{EO} = \underline{\boldsymbol{b}}$ relies on an understanding that two vectors with the same magnitude and direction are equal.

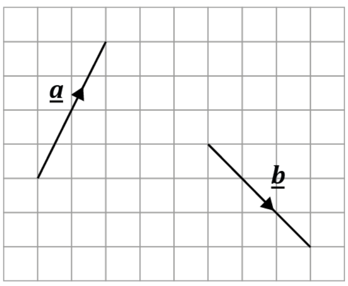
Know that the sum of two (or more) vectors is described as the resultant

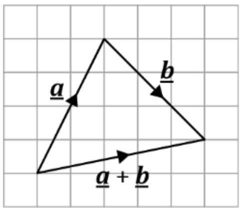
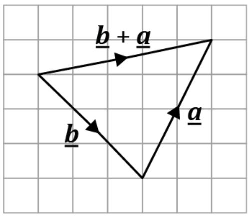
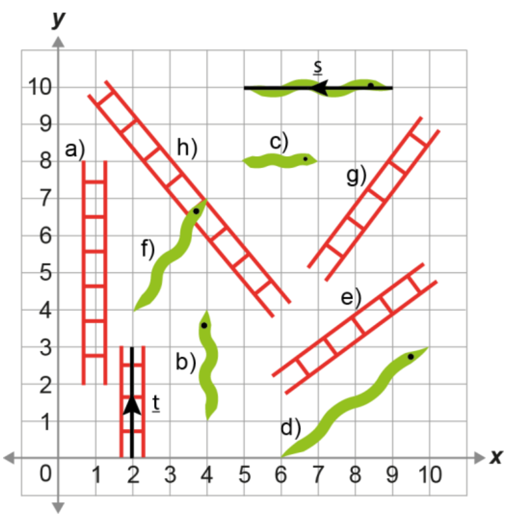
Example 6:

$$\binom{2}{4} + \binom{3}{-3}$$

a) *Which of the following three column vectors is the resultant of the above vector calculation?*

Example 6 provides an opportunity for students to consider the addition of two column vectors and aims to establish that when two vectors are combined, a single vector known as the resultant vector is created. Multiple possible column vectors for the resultant vector are provided, with carefully-designed **variation**. From this, students are required to identify the correct column vector, which encourages them to think about the addition and exposes some possible errors when finding the resultant vector.

A $\begin{pmatrix}5\\7\end{pmatrix}$ *B* $\begin{pmatrix}5\\1\end{pmatrix}$ *C* $\begin{pmatrix}1\\7\end{pmatrix}$ b) *Explain your answer to part a. How does your explanation compare to others in your class?* c) *Determine* $\begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}3\\-3\end{pmatrix}$	Students should be encouraged to sketch their own **representations** of the given vectors, to support the formal addition of vectors and to help them understand the practical benefits of building it up piecewise. Highlight to teachers the importance of asking students to explain what someone who selected $\begin{pmatrix}5\\7\end{pmatrix}$ or $\begin{pmatrix}1\\7\end{pmatrix}$ may have been thinking, and checking that they recognise the distinction between finding the difference in corresponding components of the vectors ($\begin{pmatrix}1\\7\end{pmatrix}$) and subtracting the column vectors as in part b.
Example 7: a) *Complete the gaps in these column vectors. Is there more than one way to do this? Can you think of a way that nobody else has?* *(i)* $(\ \ ) + (\ \ ) = \begin{pmatrix}8\\9\end{pmatrix}$ *(ii)* $(\ \ ) + (\ \ ) = \begin{pmatrix}2\\-5\end{pmatrix}$ *(iii)* $(\ \ ) + (\ \ ) = \begin{pmatrix}-2\\-5\end{pmatrix}$ *(iv)* $(\ \ ) + (\ \ ) = \begin{pmatrix}0\\0\end{pmatrix}$ b) *How would your answers to part a change if the questions were subtractions instead?* c) *How might your answers to part a change if the question asked for three vectors* $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$ *and* $\underline{\boldsymbol{c}}$, *with* $\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}} + \underline{\boldsymbol{c}}$ *resulting in the same vectors as given in parts i to iv?*	The focus of *Example 7* is on highlighting that multiple different vectors can be combined to give the same resultant vector, which contributes to students' growing appreciation of the structure of vectors and vector addition. Here, students use the **representation** of blank column vectors to help them consider the numerical structures, but they should be encouraged to also draw and compare their different solutions to provide insight into what the resultant vector represents spatially. Students should be encouraged to think beyond horizontal and vertical vectors ($\begin{pmatrix}8\\0\end{pmatrix} + \begin{pmatrix}0\\9\end{pmatrix} = \begin{pmatrix}8\\9\end{pmatrix}$ for part a, for example. The prompts for part a address this by asking students to provide more than one possibility for each of the four parts. This means that there is opportunity for **deepening** students' understanding by collating and responding to the solutions that they generate. In part iv, the zero (or null) vector is introduced, and students need to identify two equal vectors that are pointing in opposite directions to each other. It is important to establish that the zero vector has no direction and no magnitude. Discuss with teachers, how, once combinations of two vectors have been explored for each of the in part, students might approach parts b and c. What questioning might be needed alongside to ensure that students' understanding is deepened?
Example 8: *A teacher asks three students to add vectors* $\underline{\boldsymbol{a}}$ *and* $\underline{\boldsymbol{b}}$*:*  *Dani says the vectors can't be added as they don't touch.*	*Example 8* highlights that a vector describes magnitude and direction but not position, unless a particular start or end point is specified. Charlie's method for adding vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ applies the triangle law of addition, which states that when two vectors are represented as two sides of a triangle, the third side of the triangle represents the resultant vector. The vectors in the original image do not form a triangle. Teachers need to ensure that students understand why the two diagrams are valid. Consider the **language** used to explain this: students may initially find it easier to think of vector $\underline{\boldsymbol{b}}$ as simply being translated, before building up to the more sophisticated idea of vectors as an infinite set of parallel directed line segments. Using the **representation** of a square grid background means that students can count the squares and think of

Charlie disagrees and draws the following diagram:  *a) Comment on what Charlie has done.* *Lou says, 'I got a different solution.'*  *b) Is Lou's solution different? Explain your answer.*	the vectors in column vector terms if helpful. In thinking of $\underline{a}$ as the column vector $\begin{pmatrix}2\\4\end{pmatrix}$ and $\underline{b}$ as $\begin{pmatrix}3\\-3\end{pmatrix}$, students may more readily see that, when we take a vector and translate it to a new position, the vector we obtain is the same vector we started with. Part b explores the additive identity in the context of vectors, demonstrating that the commutative structure students have learnt in the context of number and algebra applies for vectors too. Building on this, teachers might like to ask students to draw a diagram that represents $\underline{a} - \underline{b}$ and/or $\underline{b} - \underline{a}$ and comment on what they notice, to support them in **deepening** their understanding. This example explores knowledge of the triangle law of addition and the understanding that a vector's direction and magnitude is unchanged when it is translated. Discuss with your team whether these two concepts are best introduced together or separately. If the latter, in which order do your colleagues feel they should be introduced? It is important that, regardless of the conclusion your team come to, students are supported to notice the connections to consolidate their understanding.
Example 9: *Hazel and Dave are playing a game of snakes and ladders, using vectors.* *Two of the vectors are labelled $\underline{s}$ and $\underline{t}$.* *Which of the other vectors (labelled as parts a to g) can you now describe in terms of $\underline{s}$ and $\underline{t}$? You can do parts a to g in any order.* 	Students are asked here to work with the **representation** of a snakes and ladders board game to express other vectors in terms of the two vectors that have been defined. The snakes and ladders context further reinforces the idea of vectors having direction, as students are likely to be familiar with the idea that players can only travel in one direction along each image (down a snake/up a ladder). The **variation** is designed to progress students' thinking: from using a negative vector to represent movement in the opposite direction, through vectors of different lengths (introducing the idea that scalar multiplication changes the magnitude) and on to adding vectors to form a resultant. The visual juxtaposition of parts d, e and g is intended for **deepening** students' thinking around what is and is not possible in using given vectors to describe a resultant vector. Students should recognise that the vector in part e is parallel to that in part d, but that the movement in both directions is reversed (so $\underline{s} - \underline{t}$ becomes $\underline{t} - \underline{s}$, or $-\underline{s} + \underline{t}$). In part g, there is still a movement of 3 and a movement of 4, but the directions have been switched, and so a fractional scalar is required for both component vectors: $\frac{4}{3}\underline{t} - \frac{3}{4}\underline{s}$, or $-\frac{3}{4}\underline{s} + \frac{4}{3}\underline{t}$. Students less fluent in multiplicative reasoning may find the introduction of scalar multiplication more challenging. A return to representations such as double number lines or bar models may support them to visualise the scalar structure and connect to prior learning on ratio and fractions. Discuss how you might approach this.

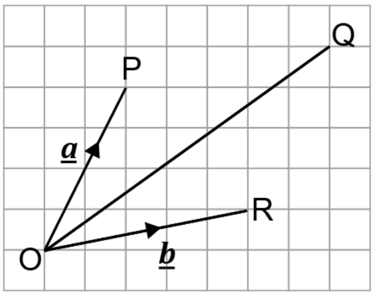
Apply the properties of shapes to vector calculations

Example 10:

Callum and Priya are doing their homework on vectors.

$\overrightarrow{OP} = \underline{\boldsymbol{a}}$ and $\overrightarrow{OR} = \underline{\boldsymbol{b}}$.

They need to find $\overrightarrow{OQ}$.



Callum says, 'We can't work out $\overrightarrow{OQ}$ as there is no way of getting from O to Q along the lines in the diagram.'

*Priya says, 'We could draw the vector $\overrightarrow{PQ}$ but we still don't know what that is in terms of **a** and **b**.'*

What might help Callum and Priya to solve this problem?

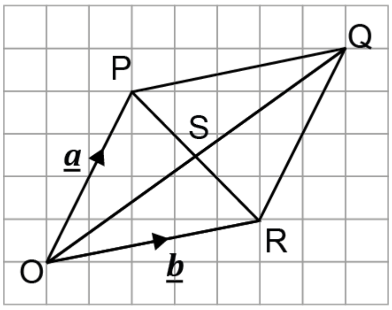
Examples 10 and *11* explore the intersection of vector addition with students' prior work on the properties of shapes. The diagram in *Example 10* shows two adjacent sides of a parallelogram and its diagonal. The remaining two sides of the parallelogram are omitted from the graphical **representation**, so that students apply their understanding that directed line segments with the same magnitude and direction are the same vector.

Students may approach this in several ways. They might think about translating vector $\underline{\boldsymbol{b}}$ so that it connects points P and Q; draw the missing sides of the parallelogram and recognise that OR and PQ are parallel line segments; or take either of those approaches with vector $\underline{\boldsymbol{a}}$ instead. Comparing different students' explanations may help with **deepening** their understanding of the underlying structures of vector addition.

The reasoning explored in this example can be articulated as the parallelogram law of vector addition, which states that if two vectors are represented by the adjacent sides of a parallelogram, then the diagonal of the parallelogram represents the sum of the two vectors. The parallelogram law is effectively the triangle law (explored in *Example 8*) applied twice. It can be used to show that vector addition satisfies the commutative law such that $\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}} = \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}}$. Formal naming of these laws is not required at Key Stage 4. Discuss with colleagues to gain a collective understanding of your team's subject knowledge for vectors.

Example 11:

OPQR is a parallelogram. S is the point at which the diagonals of the parallelogram intersect:



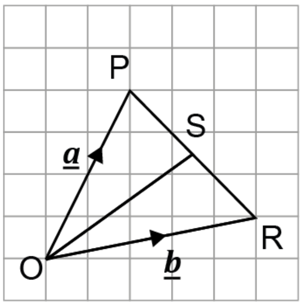
$\overrightarrow{OP} = \underline{\boldsymbol{a}}$ and $\overrightarrow{OR} = \underline{\boldsymbol{b}}$.

Write down an expression for $\overrightarrow{OS}$ in terms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.

Example 11 is an opportunity for further **deepening** students' understanding of vector addition, focusing on the properties of the shape alongside the rules for adding and subtracting vectors. The point S is the midpoint of the lines PR and OQ and so students can either think of $\overrightarrow{OS}$ as $\frac{1}{2}\overrightarrow{OQ}$, $\overrightarrow{OP} + \frac{1}{2}\overrightarrow{PR}$, or $\overrightarrow{OR} + \frac{1}{2}\overrightarrow{RP}$. Recognising that all three are valid and equivalent is important, and time should be spent comparing the different ways that students think about this vector. Ensure they are aware that all three approaches will result in the same expression in terms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.

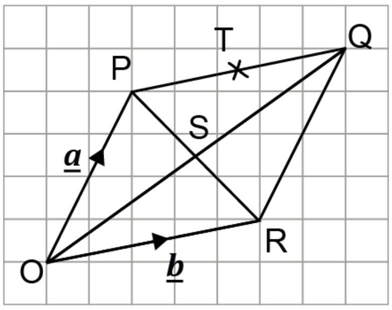
Plan how teachers intend to work with the diagram to help students understand the connections between the algebraic and geometric **representations**. It may be helpful, for example, to highlight different paths in different colours, and to model the relevant expressions being derived in the same colour.

To work in terms of $\frac{1}{2}\overrightarrow{OQ}$, students need to be fluent with vector addition and may find $\overrightarrow{OQ}$ by either adding vectors $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ $(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$ or $\overrightarrow{OR}$ and $\overrightarrow{RQ}$ $(\underline{\boldsymbol{b}} + \underline{\boldsymbol{a}})$. Expressing $\overrightarrow{OS}$ in terms of $\overrightarrow{OP} + \frac{1}{2}\overrightarrow{PR}$ or $\overrightarrow{OR} + \frac{1}{2}\overrightarrow{RP}$ requires an understanding of the subtraction of vectors and students should be

	encouraged to explore all three paths to deepen their understanding of vector addition and subtraction. Ask teachers to consider the relationships that should be emphasised to students when working with $\overrightarrow{OS}$ expressed in terms of $\overrightarrow{OP}+\frac{1}{2}\overrightarrow{PR}$ and $\overrightarrow{OR}+\frac{1}{2}\overrightarrow{RP}$ and explore some possible prompts together. They may, for example, like to ask students: • *'Are there any other possible paths that have not been explored?' For example, $\overrightarrow{OS}=\overrightarrow{OP}+\overrightarrow{PQ}+\frac{1}{2}\overrightarrow{QS}$.* • *'Are there any advantages or disadvantages to using a path such as this?'* • *'Can you use this path to verify that $\overrightarrow{OS}=\frac{1}{2}(\underline{\boldsymbol{a}}+\underline{\boldsymbol{b}})$?'*
Example 12: *OPR is a triangle. The point S is the midpoint of the line PR:*  $\overrightarrow{OP}=\underline{\boldsymbol{a}}$ *and* $\overrightarrow{OR}=\underline{\boldsymbol{b}}$. *Write expressions in terms of* $\underline{\boldsymbol{a}}$ *and* $\underline{\boldsymbol{b}}$ *for:* *a)* $\overrightarrow{PR}$ *b)* $\overrightarrow{RS}$ *c)* $\overrightarrow{SO}$	*Example 12* makes explicit the relationship between inverse vectors and provides an opportunity for students to consider carefully the direction of vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$. The geometric **representation** focuses on one half of the parallelogram diagram used in *Example 11* and looks at exploring paths that combine known vectors, with derived vectors constructed from combinations of known vectors, with a specific focus on direction. Understanding that $\overrightarrow{PO}=-\overrightarrow{OP}$ is key to establishing $\overrightarrow{PR}$ as being equal to $-\underline{\boldsymbol{a}}+\underline{\boldsymbol{b}}$ and part a provides an opportunity for **deepening** students' thinking about vector direction. $\overrightarrow{RS}$ can either be thought about in terms of $\overrightarrow{RP}$ (in which case the relationship $\overrightarrow{RP}=-\overrightarrow{PR}$ can be identified and the result for part a used to determine part b). Alternatively, the relationship $\overrightarrow{RS}=\overrightarrow{RO}+\overrightarrow{OS}$ can be explored and the relationship $\overrightarrow{SO}=-\overrightarrow{OS}$ used to establish an expression for $\overrightarrow{SO}$ based on the workings for part b. Do all colleagues feel equally confident teaching vectors? Do they all approach vector problems in the same way? How flexible is their thinking when presented with other solutions? Having a departmental culture where teachers are open to learning from each other can be very beneficial, especially for topics such as vectors which some colleagues might not teach every year. Highlight to teachers that it is important that both they and their students recognise the interconnected nature of the vectors, so that that students develop an awareness of ways in which new paths can be built on previously-established results.

Example 13:

OPQR is a parallelogram. S is the point at which the diagonals of the parallelogram intersect, and T is the midpoint of the line PQ:



$\overrightarrow{OP} = \underline{\boldsymbol{a}}$ and $\overrightarrow{OR} = \underline{\boldsymbol{b}}$.

Which of the following (A, B or C) describes the vector $\overrightarrow{RT}$*?*

A. $\underline{\boldsymbol{a}} + \frac{1}{2}\underline{\boldsymbol{b}}$

B. $\underline{\boldsymbol{a}} - \frac{1}{2}\underline{\boldsymbol{b}}$

C. $\frac{1}{2}\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$

Explain your answer.

Example 13 exposes some common mistakes when working with vectors. It is important that discussions with students identify assumptions that could have been made for the incorrect expressions, to support with **deepening** their understanding.

There are several routes that could be explored when expressing the vector $\overrightarrow{RT}$ in terms of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$. Students should be encouraged to compare more than one approach as part of the solution process. The **variation** in the possible answers will draw out some likely misconceptions. A student selecting A as the correct expression may have incorrectly assumed that $\overrightarrow{RP} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$, rather than $\underline{\boldsymbol{a}} - \underline{\boldsymbol{b}}$, adding the two vectors instead of subtracting them. A student selecting C as the correct expression has identified $\overrightarrow{TR}$ rather than $\overrightarrow{RT}$.

Using students' responses to distinguish between error and misconception can be a valuable use of departmental time. Emphasise to teachers the importance of distinguishing between calculation errors, that can occur when collecting like terms, and misconceptions or gaps in understanding related to the addition and subtraction of vectors. When solving geometrical problems that involve vectors, a deep understanding of vector arithmetic, inverse (or negative) vectors and the properties of shapes is required.

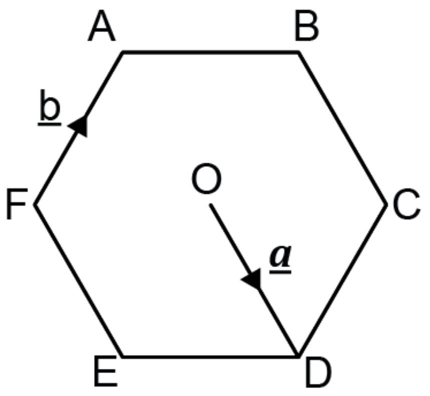
11.1.2.5 Use vectors to construct geometric arguments and proofs

Common difficulties and misconceptions

Students often assume that two vectors cannot be the same if they are not in the same place. Establishing that, when we take a vector and translate it to a new position, the vector we obtain is the same vector we started with, is key to developing students' understanding of parallel vectors. Parallel vectors with equal magnitude and direction represent the same vector: students need to understand that it is a mathematical object in its own right, and that it can be placed in any position. Another way of thinking about vectors is as an infinite set of parallel, directed line segments. This is likely to be a step beyond what students might be ready for at Key Stage 4, but it may be helpful for teachers to be aware, for co-planning discussions, that colleagues may define parallel vectors in subtly different ways.

The recognition that two vectors are parallel if they are multiples of each other relies on an understanding that when we multiply a vector by a (positive) scalar, the direction of the vector does not change. This is because the scalar has magnitude but no direction. The resultant vector therefore has the same direction as the original vector and so is parallel to it. Understanding the relationship between parallel vectors provides a means of proving that two vectors are parallel and can also be used to determine whether a set of points lie on a straight line, which students often struggle do.

Geometric arguments and proofs rely on a secure understanding of geometrical properties as well as a fluency with algebraic manipulation. Teachers should be ready to identify the root of any potential misconceptions – when students are working between different mathematical themes, any gaps in knowledge or understanding can become exacerbated.

Students need to	Guidance, discussion points and prompts
Use their knowledge that directed line segments with the same magnitude and direction are the same vector *Example 1:* *A regular hexagon ABCDEF has centre O:*  $\overrightarrow{OD} = \underline{\boldsymbol{a}}$ and $\overrightarrow{FA} = \underline{\boldsymbol{b}}$. a) *Explain how you know that:* (i) $\overrightarrow{DC} = \underline{\boldsymbol{b}}$ (ii) $\overrightarrow{BC} = \overrightarrow{FE} = \underline{\boldsymbol{a}}$ (iii) $\overrightarrow{OC} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$ (iv) $\overrightarrow{AE} = \underline{\boldsymbol{a}} - \underline{\boldsymbol{b}}$ (v) $\overrightarrow{BE} = -2\underline{\boldsymbol{b}}$ b) *What other lines can be expressed as* $\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$*? What do you notice about these lines?*	*Example 1* revisits the **representation** of a regular hexagon to explore vectors. The familiar shape again provides an opportunity for students to make connections between vectors and properties of shapes. This idea is explored in more depth in *Example 5* of exemplified key idea 11.1.2.4, and here as the examples that follow hinge on this understanding. It is key to establish that, although a vector may be defined as going from one point to another, it can be used anywhere. While a vector has both magnitude and direction, it does not have position. When a directed line segment has the same magnitude and is parallel to another, they are the same vector. Students are asked to consider five vectors, **deepening** their understanding of the geometric properties of a regular hexagon. Students should recognise that a regular hexagon has six equal sides; is constructed from three pairs of parallel lines; and can be divided into six congruent equilateral triangles. The **language** of part a is 'explain how you know'. The open prompt allows teachers to use their discretion around what they expect students' answers to involve. At an earlier stage in students' learning, teachers may expect them to be verbally explaining, or perhaps visually demonstrating, how each of these vectors can be found. Later, students should be constructing formal proofs, with clear, logical steps, and any calculations supported by mathematically-sound factual statements. Discuss some possible prompts to support students who are struggling to recognise that a regular hexagon can be divided into six congruent equilateral triangles. For example: • *'If the centre of the hexagon (O) is joined to vertex C to form a triangle, what is the size of the angle at O?'* • *'What can you say about lengths OD and OC?'* • *'What can we conclude about triangle DOC?*
Demonstrate that vectors are parallel if one is a scalar multiple of another *Example 2:* *Look at the diagram from Example 1.* a) *How do you know that* $\overrightarrow{BE}$ *is parallel to* $\overrightarrow{AF}$*?* b) *Use vectors to convince a friend that* $\overrightarrow{FC}$ *is parallel to* $\overrightarrow{ED}$.	*Example 2* continues with the **representation** of the regular hexagon, now introducing the idea of parallel vectors. The two parts are, at first glance, pretty similar, but the intention is to support students to develop their thinking from informal reasoning to the beginnings of geometric proof. Students' answers to part a are likely to be revealing in terms of assessing how secure they are with the structure of a regular hexagon, and the ways in which such a shape can be constructed.

Example 3:

$\overrightarrow{AB} = 3\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$

$\overrightarrow{CD} = 12\underline{\boldsymbol{a}} + 4\underline{\boldsymbol{b}}$

$\overrightarrow{EF} = 15\underline{\boldsymbol{a}} + 4\underline{\boldsymbol{b}}$

a) *Show that $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are parallel.*

b) *Is $\overrightarrow{EF}$ parallel to $\overrightarrow{CD}$? How do you know?*

Example 3 explores parallel vectors described algebraically. Looking at a geometrical idea with a numerical/algebraic lens is something that students need to become increasingly comfortable with, particularly if they continue to study mathematics beyond Key Stage 4. An example such as this provides an opportunity for students to move freely between the spatial/geometrical and numerical/algebraic worlds and **deepening** understanding of the connections between the two.

Teachers need to support students to gain confidence in the mathematical **language** they need to justify their answers, including where there is more than one possible approach. Students can either multiply $\overrightarrow{AB}$ by 4 to show that $4\overrightarrow{AB} = \overrightarrow{CD}$ or factorise $12\underline{\boldsymbol{a}} + 4\underline{\boldsymbol{b}}$ to give $\overrightarrow{CD} = 4(3\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}) = 4\overrightarrow{AB}$. Similarly, students can multiply $\overrightarrow{AB}$ by 5 to show that this does not generate the same vector as $\overrightarrow{\mathrm{EF}}$, or demonstrate that the expression for $\overrightarrow{\mathrm{EF}}$ cannot be factorised.

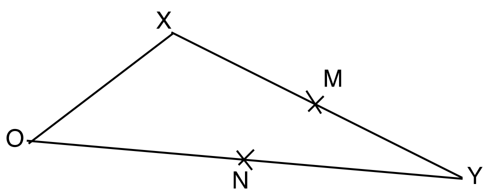
Discuss what further prompts teachers might use to take this example further, and to ensure students have understood that vectors are parallel if, and only if, one is a multiple of the other. They may, for example, ask more open questions such as:

- *'Vectors $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are parallel. What can we say about them?'*

'If $\overrightarrow{EF} = 2\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}$, what vector could $\overrightarrow{GH}$ be?'

Example 4:

YXO is a triangle:



M is the midpoint of XY and N is the midpoint of OY.

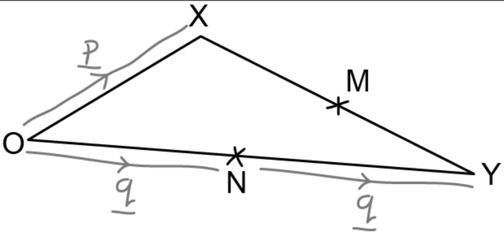
$\overrightarrow{OX} = \underline{\boldsymbol{p}}$ *and* $\overrightarrow{ON} = \underline{\boldsymbol{q}}$

A Key Stage 4 exam question asks students to prove that $\overrightarrow{OX}$ and $\overrightarrow{NM}$ are parallel vectors.

The first step of a student's working is shown below:

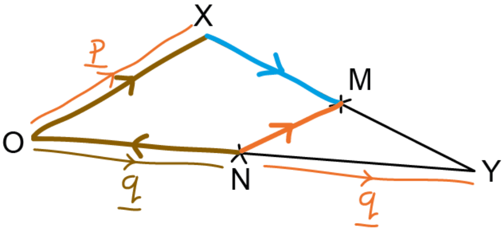
Example 4 explores vectors within a triangle, using a question that might look similar to an exam question. It is helpful for students to get into the habit of annotating diagrams that they are working with, and the workings shown here are designed to support students to see how they might do this. By providing the information about the vectors $\overrightarrow{OX}$ and $\overrightarrow{ON}$ below the diagram, students should also notice that they need to engage with the information provided in a more holistic way, as they interpret the **language** of the question alongside the algebraic descriptions and determine how they connect with the diagram provided.

Teachers may find it helpful to further annotate the **representation** given to help students to fully understand the situation. It is perhaps easy to overlook the fact that a triangle is half a parallelogram and so, often, drawing in the parallelogram can help students to 'see' the vectors. If students find it challenging to get started on part a, sketching a full parallelogram for the two vectors $\underline{\boldsymbol{p}}$ and $2\underline{\boldsymbol{q}}$ may help them to recognise the movements that are identified in the question. Links could then be made with the diagram from *Examples 3* and *4* of key idea 11.1.2.4, so that students can see how an exam question like this is still built on the same foundational principles.



a) *What information has the student used to annotate the diagram? Why is this valid?*

Next, the student highlights their diagram to show a path from point N to point M:



$$\overrightarrow{NM} = \overrightarrow{NX} + \overrightarrow{XM}$$

b) *Explain the student's reasoning for finding vector $\overrightarrow{NM}$.*

c) *How might the student find vector $\overrightarrow{XM}$?*

The final stage in the student's working is shown below:

$$\overrightarrow{NM} = \overrightarrow{NX} + \overrightarrow{XM}$$

$$\overrightarrow{XM} = \tfrac{1}{2}(\overrightarrow{XY}) \qquad \overrightarrow{NX} = -\underline{q} + \underline{p}$$

$$\overrightarrow{XY} = -\underline{p} + 2\underline{q}$$

$$\overrightarrow{XM} = -\tfrac{1}{2}\underline{p} + \underline{q}$$

d) *Complete their workings to show that $\overrightarrow{NM} = \frac{1}{2}\underline{\mathbf{p}}$*

e) *Why does this mean that $\overrightarrow{OX}$ and $\overrightarrow{NM}$ are parallel vectors?*

Students will build on their understanding that the vector joining two points can be identified by creating a path along vectors that are known. They also need to recognise the significance of M and N being midpoints and how this can be reflected in any expressions that they generate, **deepening** their understanding of what vectors are known based on the information provided. A question such as this can be intimidating at first. By working through the stages of a fictional student's thinking, it is hoped that your students can grow in both confidence and understanding.

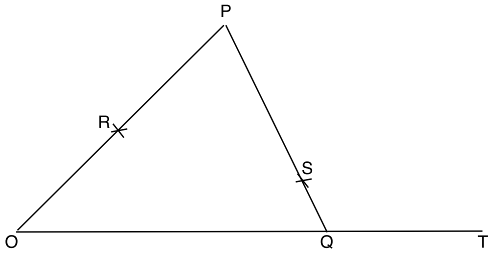
There is opportunity for further **variation** of the position of M and N, which would allow for connections between different concepts to be made, and the relationship between ratios and fractions explored. Teachers might, for example, ask students to consider what would happen if M and N were moved, so that they divided XY and OY respectively in the ratio 1:2. In this case, it would be important to ask students to predict whether or not vectors $\overrightarrow{OX}$ and $\overrightarrow{NM}$ would still be parallel before checking, and to see if they can explain why the multiplier between vectors $\overrightarrow{OX}$ and $\overrightarrow{NM}$ is $\frac{2}{3}$ rather than $\frac{1}{2}$ now.

The workings shown here are one example of how a student might approach this problem and communicate their thinking. Discuss with your team the different approaches they take, and what you feel it is important to include in the worked examples that students experience. What features of your modelled examples do you want them to take forward into their own written work? Consider also how you might use colour, annotation and algebra to best explain to students the geometric and algebraic structures that are being exploited. Note that students with a visual impairment might not be able to access explanations that rely on contrasting colours; the orange, blue and brown used in this example are chosen as the colour combination that is least likely to cause issues, but you may want to consider using dashed/dotted lines instead.

Know how to prove three points lie on a straight line

Example 5:

OPQ is a triangle. OQT is a straight line.



R is the midpoint of OP and S is the point on PQ such that PS:SQ = 3:1. Q is the point on OT such that OQ:QT = 2:1.

$\overrightarrow{OR} = \underline{x}$ and $\overrightarrow{OQ} = \underline{y}$

Use a vector method to show that the points R, S and T lie on a straight line.

Example 5 focuses on **deepening** students' understanding of parallel lines, now using vectors to show that three points lie on a straight line. As with *Example 3*, it is very similar in structure to a standard vectors examination question, but this time students work with the question rather than a hypothetical answer. To show that points R, S and T lie on a straight line, students can demonstrate either that vector $\overrightarrow{RS}$ is a multiple of (and therefore parallel to) $\overrightarrow{RT}$, or that vector $\overrightarrow{RS}$ is a multiple of (and therefore parallel to) $\overrightarrow{ST}$. If students compare vectors $\overrightarrow{RS}$ and $\overrightarrow{RT}$, they will identify the relationship $\overrightarrow{RS} = \frac{1}{2}\overrightarrow{RT}$. The two parallel vectors start at the same point (R), so R, S and T lie on a straight line. Teachers might also ask students what the relationship $\overrightarrow{RS} = \frac{1}{2}\overrightarrow{RT}$ tells us about position of the point S on the line RST.

When comparing vectors $\overrightarrow{RS}$ and $\overrightarrow{ST}$, students will discover that $\overrightarrow{RS} = \overrightarrow{ST} = -\frac{1}{2}\underline{x} + \frac{3}{4}\underline{y}$, which provides an opportunity to discuss what a multiple of 1 tells us about the vectors $\overrightarrow{RS}$ and $\overrightarrow{ST}$. This relationship is not immediately obvious from the **representation**, a useful reminder not to make assumptions from diagrams but rather rely on rigorous proof to draw conclusions.

As a stimulus for a professional development discussion, *Example 4* can be taken in several directions. For example, the next step for variation described in *Example 3* could be the starting point for a departmental workshop about adapting and extending tasks. Discuss possible ways to develop students' understanding or connect to other areas of mathematics, by employing similar variation in *Example 4*. It could also be useful opportunity to discuss which language and definitions are being actively taught across your team, so that you ensure all students have familiarity with common terms. Are students aware that the term 'collinear' can be used to describe points that lie on a single straight line?

11.1.3.3 Understand enlargement as a transformation of vectors

Common difficulties and misconceptions

Using vectors to describe an enlargement relies on an understanding of multiplying vectors by a scalar. Multiplication by a scalar is a way of changing the magnitude (and/or direction, if the scalar is negative) of a vector. Once the position vectors of the vertices of the object have been established, it is possible to determine the resulting position vectors for the vertices of the image and therefore identify the image following the enlargement.

Students can sometimes find the multiplication of a vector by a negative number confusing: a negative scalar changes the **direction** of the vector. Relating this to an enlargement with a negative scale factor, students need to understand that a negative scalar will produce an image that is positioned on the other side of the centre of enlargement.

While students have worked with centres of enlargement at Key Stage 3, the introduction of fractional scale factors often exposes the common misconception that after an enlargement, the image will always be larger than the object. Contrary to the implication of the term 'enlargement', this transformation can involve making a shape smaller as well as bigger. Establishing this is key to students' understanding. It is important that students recognise that it is fractional scale factors between 0 and 1 rather than negative scale factors that decrease the size of the object. Students commonly muddle the two transformations.

Students need to

Understand the effect the centre of enlargement has on the position of an enlarged image

Example 1:

A triangle ABC (labelled object) is enlarged by a scale factor of 2, from three different centres of enlargement.

Images of each of these three enlargements are shown below this example.

a) *What is the same and what is different about the three enlarged images?*

Below are three sets of vectors, describing each vertex of the original and enlarged triangle as translations from point Q:

A	$\overrightarrow{QA} = \binom{-2}{0}, \overrightarrow{QB} = \binom{-2}{3}, \overrightarrow{QC} = \binom{0}{0}$. $\overrightarrow{QA'} = \binom{-4}{0}, \overrightarrow{QB'} = \binom{-4}{6}, \overrightarrow{QC'} = \binom{0}{0}$.
B	$\overrightarrow{QA} = \binom{1}{2}, \overrightarrow{QB} = \binom{1}{5}, \overrightarrow{QC} = \binom{3}{2}$. $\overrightarrow{QA'} = \binom{2}{4}, \overrightarrow{QB'} = \binom{2}{10}, \overrightarrow{QC'} = \binom{6}{4}$.
C	$\overrightarrow{QA} = \binom{-1}{-1}, \overrightarrow{QB} = \binom{-1}{2}, \overrightarrow{QC} = \binom{1}{-1}$. $\overrightarrow{QA'} = \binom{-2}{-2}, \overrightarrow{QB'} = \binom{-2}{4}, \overrightarrow{QC'} = \binom{2}{-2}$.

b) *Match the description of the enlargement, to the correct image.*

c) *Determine the positions of the three centres of enlargement.*

Guidance, discussion points and prompts

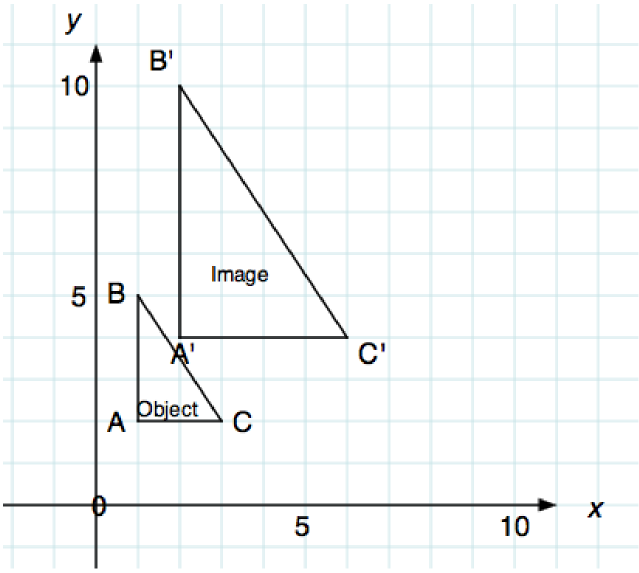
Example 1 uses **variation** to draw attention to the effect that the centre of enlargement has on the position of an enlarged image. By keeping the scale factor consistent, (equal to 2 in this example), students can identify that the size and orientation of the image is the same for all three enlargements, but their position changes. Students should be encouraged to notice that the position of vertices of the image is determined by the relationship between the vertices of the object and the centre of enlargement. For example, if a vertex lies on the centre of enlargement (as in enlargement 3) then the corresponding point on the image will also lie on the centre of enlargement. When the centre of enlargement lies within the object (and the scale factor is positive and greater than 1), the object becomes enclosed within the enlarged image (as in enlargement 2).

It is likely that students will be most familiar with the type of enlargement shown in enlargement 1. When centres of enlargements are first introduced, they are often positioned outside of the object and, if a Cartesian coordinate grid is used, this is commonly at the origin. It is important that students experience visuals of enlargements where the centre is positioned differently, so that this does not become their sole mental **representation** for what an enlargement can look like. The emphasis should be on the centre of enlargement being the only truly invariant point, with all other points being subject to the transformation described.

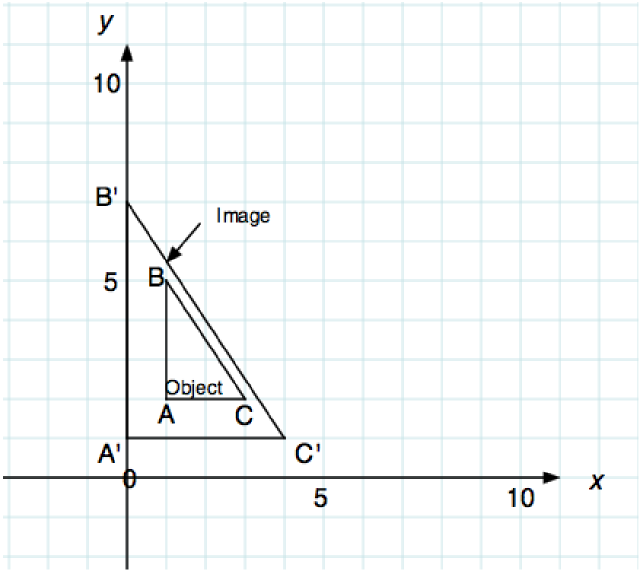
Parts b and c focus specifically on **deepening** understanding of enlargement as a transformation of vectors. Recognising that $\overrightarrow{QA}$ describes the position vector of A from Q (the centre of enlargement), and so $\overrightarrow{AQ} = -\overrightarrow{QA}$ describes the relative position of the centre of enlargement from A, is a fundamental step in this.

Ask teachers to reflect on and compare the methods their students normally use to determine the position of a centre of enlargement given an object and corresponding image. They may be familiar with drawing lines through corresponding vertices of the object and image, to see where these lines intersect. Is your team confident with how position vectors can be used instead to determine the centres of enlargement? Can teachers see the advantages of this approach over others (such as tracing rays) to build coherence and show the usefulness of vectors in this context?

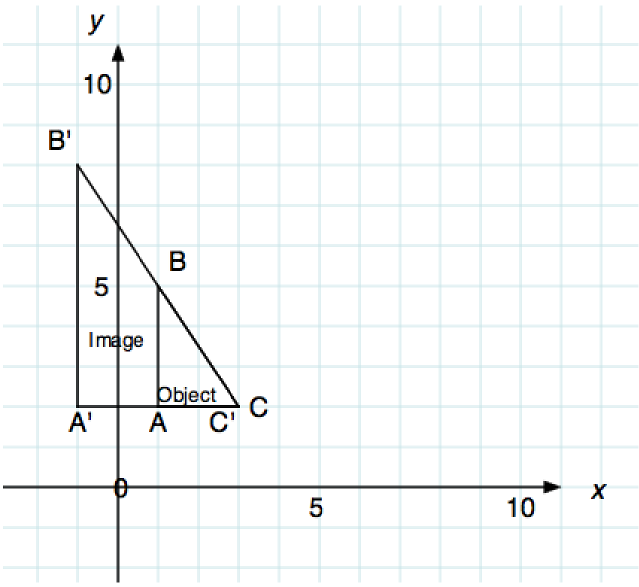
Enlargement 1:



Enlargement 2:



Enlargement 3:



Appreciate the effects of an enlargement by scale factor between 0 and 1

Example 2:

Below is a diagram showing a crescent shape (the object) which has been enlarged by a scale factor of $\frac{1}{3}$ (images 1 to 4).

Several different possible centres of enlargement are also given.

a) *Identify the correct centre of enlargement for each image.*

b) *Pick one of the unused centres of enlargment, and describe the position of the new image.*

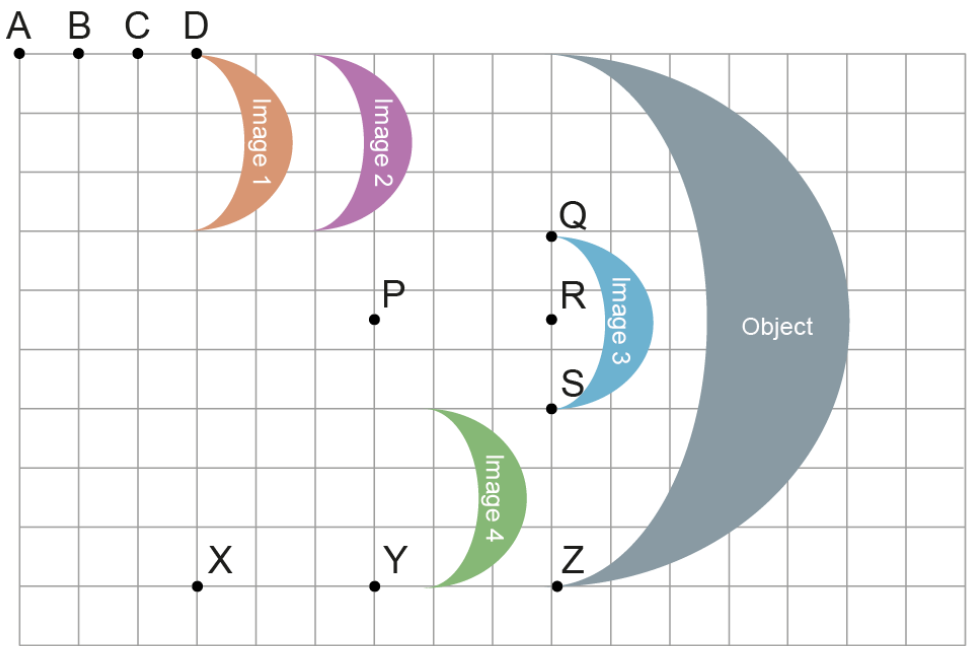
c) *Repeat part b for each of the other unused centres of enlargement.*

Example 2 uses the **representation** of a crescent shape so that students have fewer points to consider – just the two points at each extreme of the crescent. This supports them in developing their understanding of enlargement as vectors from a centre of enlargement. There is a dual emphasis on fractional scale factors and centres of enlargement.

The **variation** draws attention to the mathematical structure of vector transformation. For each image, at least one pair of corresponding points lies on the same horizontal or vertical line, so that one of the values in the relevant column vector will be 0. This means that students can focus on the interaction between the scale factor and a single value in the vector.

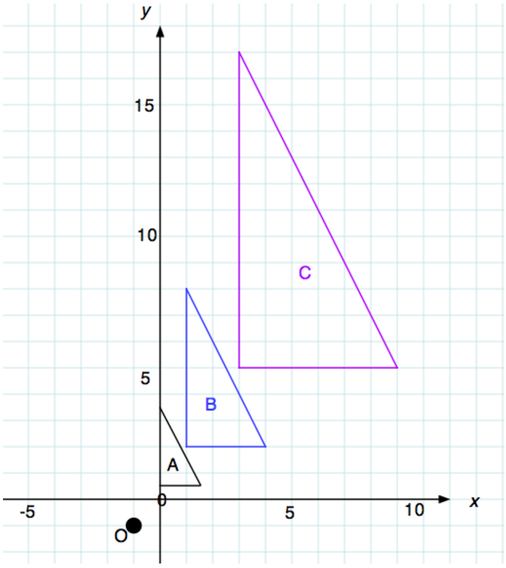
Students' awareness of vectors in this context is further developed in part b, where they start to think about how the position of the images would change were the other centres of enlargement used. Plan ahead to anticipate what kind of **language** you will likely hear from students, particularly when interacting with a static example such as this. Students should recognise whether the image would be further away from or closer to the object, and under what conditions the image would move both vertically and horizontally. This could be usefully explored using dynamic geometry software.

In this example there is a dual emphasis on centres of enlargement and fractional scale factors. Depending on students' prior experience of enlargements, it might be that teachers would prefer to focus on just one aspect. Discuss with your team how this task could be adapted so that it uses the more familiar positive scale factors instead, allowing students to focus solely on centres of enlargement.



Example 3:

The centre of enlargement at $(-1, -1)$ is marked O:



A point (P) on one of these triangles corresponds to the point P' on the enlarged image.

The vector $\overrightarrow{OP}$ describes the position of P relative to the centre of enlargement (O).

The vector $\overrightarrow{OP'}$ describes the position of the corresponding point P' relative to the centre of enlargement.

a) Match the enlargement to its description:

(i) A to B	*(A)* $\overrightarrow{OP'} = \frac{1}{4}\overrightarrow{OP}$
(ii) B to A	*(B)* $\overrightarrow{OP'} = \frac{1}{2}\overrightarrow{OP}$
(iii) A to C	*(C)* $\overrightarrow{OP'} = 2\overrightarrow{OP}$
(iv) C to A	*(D)* $\overrightarrow{OP'} = 4\overrightarrow{OP}$
(v) B to C	
(vi) C to B	

b) Comment on what you notice.

Example 3 further exposes the misconception that an enlargement always makes an object bigger and provides an opportunity for **deepening** students' thinking about scale factors. They need to establish that a fractional scale factor between 0 and 1 will cause the image to be smaller than the object, without building any misconceptions about fractional scale factors greater than 1.

Enlargements are often presented to students in terms of an object and an 'image'. Students may be given an object and asked to perform an enlargement, or to describe the enlargement that transforms a given object to a given image. In this example, each of the triangles A, B and C can be the object of the enlargement, or the resulting image. This ambiguous **representation** provides an opportunity for students to consider the two way relationship between an object and an image and to establish that an enlargement by a scale factor of 2 can also be described as an enlargement with scale factor $\frac{1}{2}$, if the object and image are interchanged.

The **variation** between triangles A, B and C has been designed so that triangle C is an enlargement of both triangles A and B.

- Triangle A is an enlargement of triangle B scale factor $\frac{1}{2}$, and of triangle C scale factor $\frac{1}{4}$.
- Triangle B is an enlargement of triangle A scale factor 2, and of triangle C scale factor $\frac{1}{2}$.
- Triangle C is an enlargement of triangle A scale factor 4 and of triangle B scale factor 2.

It is important that enlargements B to A, C to A and C to B are explored sufficiently to emphasise the inverse relationships (through consideration of enlargements A to B, A to C and B to C). Once established, the concept of a fractional scale factor between 0 and 1 becomes more straightforward, as the freedom to interchange the object and image allows the effects of a fractional scale factor of $\frac{1}{2}$ and $\frac{1}{4}$ to be related to the scale factors 2 and 4.

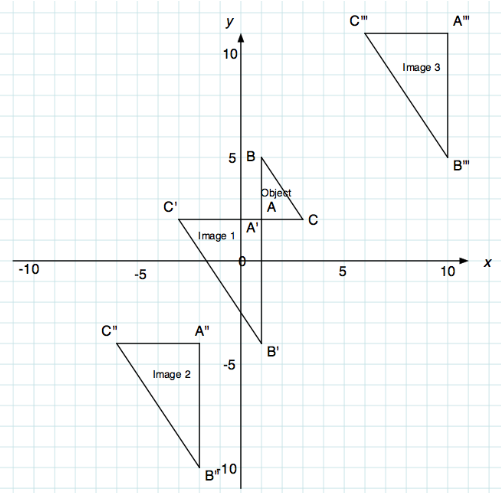
Discuss with teachers the importance of emphasising to students that a fractional scale factor may be greater than 1 or negative. They may ask questions such as:

- *'How does the effect of a fractional scale factor, greater than one ($\frac{3}{2}$ for example) differ from a positive scale factor with an integer value?'*
- *'What effect would a scale factor of $-\frac{1}{2}$ have on an object?'*

Appreciate the effects of an enlargement by a negative scale factor

Example 4:

A triangle ABC (labelled object) is enlarged by a scale factor of −2, from three different centres of enlargement):



(A larger version of this image is provided on the next page).

The object is enlarged with centre of enlargement $(1, 2)$ *to give Image 1.*

a) *Using O to denote the centre of enlargement, complete the table below with column vectors:*

$\overrightarrow{OA}$	( )
$\overrightarrow{OB}$	( )
$\overrightarrow{OC}$	( )
$\overrightarrow{OA'}$	( )
$\overrightarrow{OB'}$	( )
$\overrightarrow{OC'}$	( )

b) *Locate the centres of enlargements for images 2 and 3.*

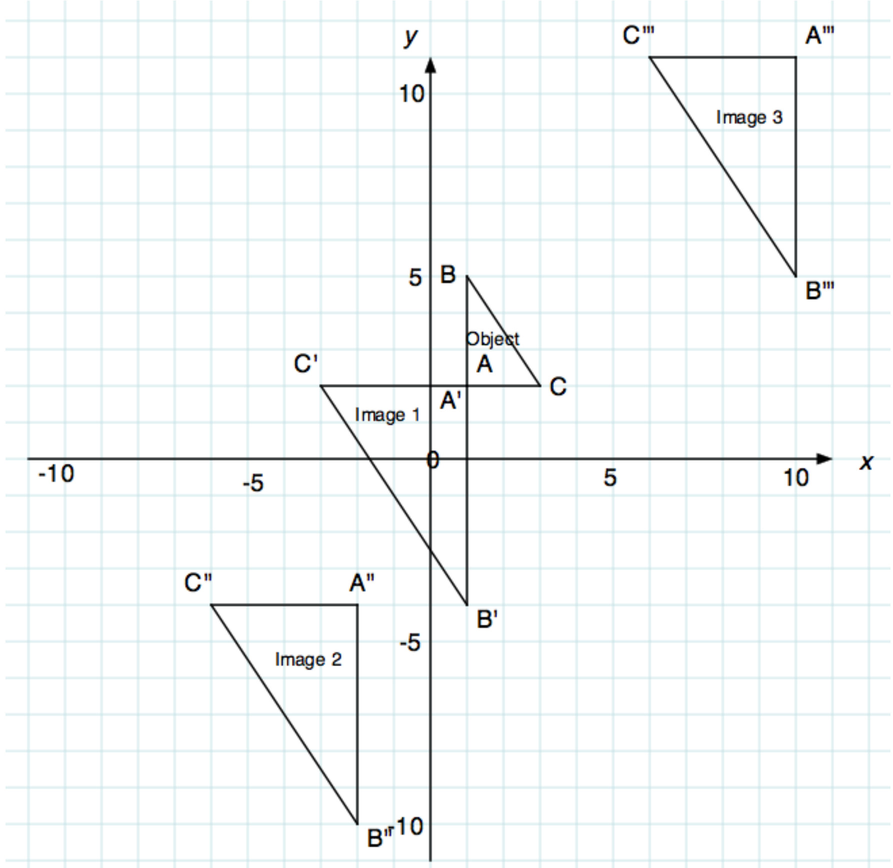
Example 4 focuses on the effect of negative scale factors. It provides an opportunity for students to identify that an enlargement with a negative scale factor produces an inverted image on the other side of the centre of enlargement. Students can often confuse the effects of negative and fractional scale factors: the **variation** in this example draws attention to the properties of negative enlargement by exploring multiple enlargements with the same negative scale factor (−2).

Using vector **representation** helps to expose the effect of negative component of the scale factor. When determining the column vectors $\overrightarrow{OB}$ and $\overrightarrow{OB'}$ as $\binom{0}{3}$ and $\binom{0}{-6}$ respectively, this relationship can be expressed in terms of the scale factor of −2: $\binom{0}{-6} = -2\binom{0}{3}$ and $(\overrightarrow{OB'} = -2\overrightarrow{OB})$. Students should identify the corresponding relationship between column vectors $\overrightarrow{OC}$ and $\overrightarrow{OC'}$, and understand the same relationship between $\overrightarrow{OA}$ and $\overrightarrow{OA'}$ exists (albeit this is not visible in the column vectors due to the zero property of multiplication). It is important for students to recognise that the distance from the centre of enlargement to any point on the image is twice the distance from the centre of enlargement to the corresponding point on the object.

When locating the centres of enlargement for images 2 and 3, it is likely that students will connect corresponding vertices. When finding the centre of enlargement for image 2, for example, the vector between A and A'' can be described as $\binom{-3}{-6}$. A scale factor of −2 means that point A'' is twice as far from the centre of enlargement as point A. This can be used to establish that $\overrightarrow{AO}$ must be $\frac{1}{3}\binom{-3}{-6} = \binom{-1}{-2}$ and so the centre of enlargement is located at the origin. Once the centre of enlargement for image 3 has been established as the point $(4, 5)$, you may ask questions that help with further **deepening** students' understanding of the relationships between the object, image and centre of enlargement. For example:

- *'What do you notice about the position of the image in relation to the centre of enlargement?'*
- *'Can you find a centre of enlargement that would result in the image lying in the second or fourth quadrants?'*
- *'What happens to the image when the centre of enlargement lies inside the object?'*

Discuss with your team the intended learning point of part b. Students' focus is likely to be on correctly identifying the centres of enlargement, so teachers will play an important role in using this question to help students generalise about enlargement by a negative scale factor. Even though students may not have used vectors to locate the centres of enlargement, taking time to establish the vectors between vertices on the object and the images can help to develop this understanding.



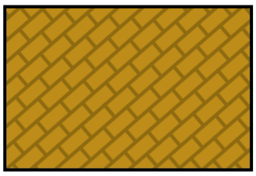
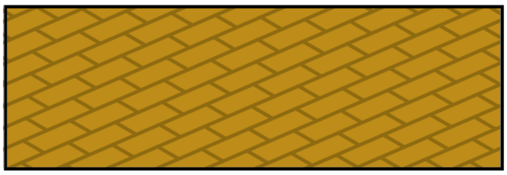
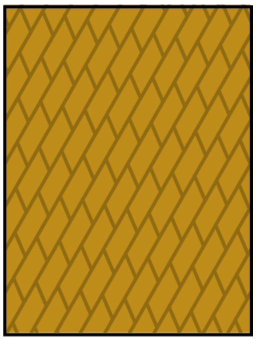
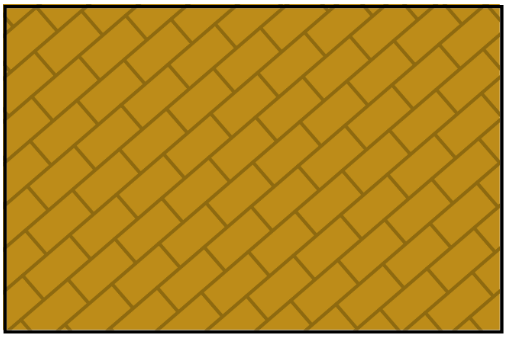
11.1.3.5 Understand the relationship between lengths, areas and volumes in similar shapes

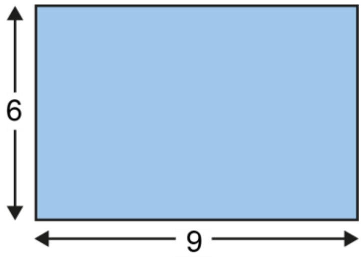
Common difficulties and misconceptions

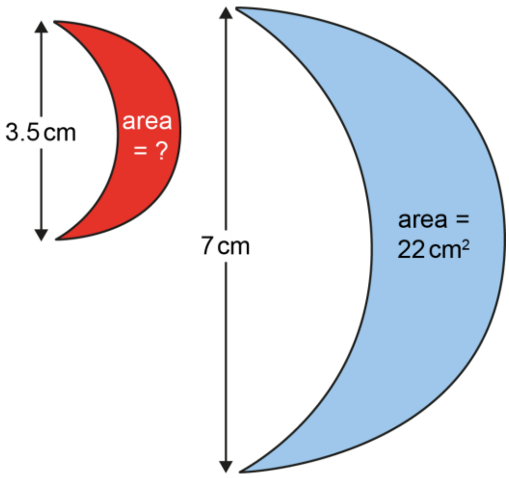
Understanding the relationship between lengths, areas and volumes in similar shapes relies on an appreciation of similar shapes being enlargements of one another, and a recognition of the significance of the scale factor when describing enlargements. Students' understanding that, for example, a scale factor of 2 means that one shape is 'twice the size of the other' can often lead to the incorrect assumption that the area of one shape is also twice the area of the other. It is important that teachers are specific in their language and relate the scale factor to the **lengths** of the sides of the shapes.

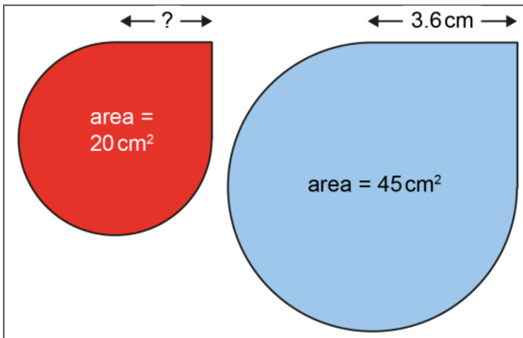
Similarly, when working with 3D similar figures, students can assume that the volume scale factor is the same as the length scale factor. This assumption highlights the importance of exploring the relationships between area and length and volume scale factors. When addressing the misconception that the scale factors for length, area and volume are the same, teachers should ensure students are connecting their reasoning to earlier work on powers and indices. Pay particular attention to the values used in examples – using lengths like 2 or 3 (as suggested below) is helpful in terms of the manageability of representations and calculations, but common errors (6 is often mistakenly given as the result of both 2^3 and 3^2) can manifest into misconceptions. This can be mitigated against either by following up with further examples using different values, or by using physical manipulatives alongside.

It is important to explore the effects on area and volume when one, two, or all three dimensions are scaled, in order to develop students' understanding of mathematical structure and support them in making generalisations. Manipulatives can help students to appreciate the way in which scale factors work across dimensions – for example, making a simple $1 \times 2 \times 3$ cuboid and then enlarging it by a scale factor of 2 so that it becomes a $2 \times 4 \times 6$ cuboid. Using a table to record the change in length (of one or more sides), area (of one or more faces) and volume can support students to notice pattern and structure. This can help them to build their own understanding of why, for a given scale factor n, the area will change by a scale factor of n^2 and the volume by scale factor n^3.

What students need to understand	Guidance, discussion points and prompts
Appreciate the connection between similarity and proportionality *Example 1:* *A picture is resized using three different methods.* *Original picture:*  *Resize 1:*  *Resize 2:*  *Resize 3:*  *Describe the three different resizing methods. What is the same/different?*	*Example 1* will reveal students' understanding of similarity, and is an opportunity to identify whether the connection between similarity and proportionality has been made. The original picture makes obvious the disproportion that occurs when one dimension is varied and the other one remains fixed, **deepening** students' understanding that similar shapes are proportional to each other. Notice the **language** used by students when describing the three resizing methods and think about what this reveals about their understanding. For example, using the term 'proportion' in their description suggests that they have made the connection between enlargement, similarity and proportionality. Students may also try to identify the scale factor. While this isn't strictly necessary, it provides an insight into their understanding of scaling, and whether they recognise that scaling all the lengths of the original picture creates a similar shape. Students may describe the three resizing methods without referring to similarity, proportionality or enlargements. Discuss prompts that may help to expose students' current understanding. For example: • *'How has the picture been transformed to get resize 1/resize 2?'* • *'Describe the relationship between resizes 1 and 2 and resize 3'* • *'How has the picture been transformed to get resize 3?'* • *'How do the pictures in resizes 1 and 2 differ to both the original picture and the picture in resize 3?'*

Understand that the 'area scale factor' is the square of the 'length scale factor' *Example 2:* *A rectangle has base length 9 cm and height 6 cm.*  *Which of statements A, B and C are correct?* *A: If the base length is doubled to 18 cm, the area is doubled.* *B: If the height is doubled to 12 cm, the area is doubled.* *C: If the base length and the height are both doubled, the area is doubled.* *Explain your answer.*	In *Example 2,* the effects of doubling one or both dimensions on the area of a rectangle are explored. Students often assume that doubling the dimensions of a rectangle results in the area being doubled also. This example provides an opportunity for **deepening** their thinking about the difference between the 'length scale factor' and the 'area scale factor', as well as the relationship between the two. Students may calculate the area when the base length is doubled to 18 cm and establish that it doubles to 108 cm^2. Similarly, they may calculate the area when the height is doubled to 12 cm to confirm that the area doubles to 108 cm^2 again. Once students have identified that statements A and B are correct, it is important for them to think about the area of the rectangle when both the base length and height are both doubled, before performing any calculations. Once it has been established that the area of the rectangle is four times bigger when both the base length and the height are doubled, check students' understanding of why this is the case. It may be helpful to partition a 12 cm by 18 cm rectangle into four 6 cm by 9 cm rectangles as a **representation** of what is happening and to draw students' attention to the structure of the enlargement. Discuss with teachers the potential misconception that can occur when a scale factor of 2 is used, as 2^2 and 2×2 both result in 4. Ensure teachers understand the importance of students recognising that the 'area scale factor' is equal to the square 'length scale factor'^{2}and not 2 × 'length scale factor'. They may like to ask students what they think would happen to the area of the rectangle if all lengths are multiplied by 3, for example.
Example 3: *Two picture frames are mathematically similar. One frame is 8 cm wide and the other is 20 cm wide. The area of the smaller frame is 12 cm^2.* *Which of the following describes the area of the larger frame?* *A: 24 cm^2 B: 30 cm^2 C: 75 cm^2* *Explain your answer.*	In *Example 3*, students are given the corresponding widths of two similar figures, alongside the area of the smaller figure, and presented with some possible options for the area of an enlarged figure. The **variation** in the options for students to select from exposes common misconceptions, and it is important to discuss these with students. When students select one of the two distractors, it can help teachers to identify a need for **deepening** understanding of different aspects of multiplicative relationships. Those who select option A may be adopting an additive approach and so need more support with understanding how two values can be connected multiplicatively. Students selecting option B are likely to have correctly identified a scale factor of 2.5 for the widths and then used this to determine the area, failing to recognise that the area of the larger frame will be 2.5^2 × the area of the smaller frame. Discuss with teachers any other possible values for the area of the larger frame that their students are likely to generate. Is your team clear about the difference between a calculation error and a common misconception based on a lack of understanding?

Example 4: *The two figures below are mathematically similar.*  *Explain why the missing area is* ***not*** *11 cm².*	In *Example 4*, the misconception that the length and area scale factors are the same is addressed explicitly. By asking students to explain why the missing area does not equal 11 cm^2, rather than asking them what the missing area is, this common assumption is exposed as incorrect. The choice of shape for the similar figures is such that students cannot apply a familiar area formula to find the missing area. Requiring students to think about the relationship between the two areas to establish that the missing area is 5.5 cm^2 means that they should be **deepening** their understanding of scale factors across different dimensions. Discuss with teachers how to support students who think the missing area is equal to 11 cm^2 and are struggling to explain why this is not the case. They may, for example, ask students to consider two squares with side lengths 3.5 cm and 7 cm and compare the areas. Explore ways of developing students' understanding of the relationship between the two scale factors from specific examples to the general case.
Example 5: *Circle A has an area of 63 cm^2 and circle B an area of 7 cm^2.* a) *Which of statements A to D below are correct?* *A: The area scale factor for enlargement is $\frac{1}{9}$.* *B: The area scale factor for enlargement is 9.* *C: The radius of circle A is 9 times bigger than the radius of circle B.* *D: The radius of circle B is $\frac{1}{3}$ of the length of the radius of circle A.* b) *Rewrite any incorrect statements, so that they are correct.*	*Example 5* is intended to draw students' attention to the relationship between scale factors for length and area, as well as the bi-directional nature of an enlargement. The focus on the **language** of enlargement helps to make connections between an integer scale factor and its fractional inverse. For statements A and B, it is not specified which image is being enlarged, and so both statements can be argued to be true. Teachers should ensure that students can identify which shape is being enlarged for each scale factor. Any two circles are similar, and so they are a useful **representation** for exploring scale factors. Students may not make this connection and it should be articulated explicitly to support their reasoning. Can they identify other shapes that will always be similar? The structure of this example is replicated in *Example 9* below, in that instance exploring the relationship between scale factors for volume and length. Discuss with your team how else this structure could be used to draw attention to features of enlargement.
Example 6: *The two figures below are mathematically similar.*	*Example 6* provides an opportunity for further **deepening** students' thinking about the relationship between the lengths and areas of two similar figures. By presenting the areas of the two figures and asking students to consider a length, the mathematical structure of the relationships can be explored. It is likely that students will use the areas of the two similar figures to identify a scale factor of 2.25. However, they may not recognise that this is the 'area scale factor', which means they need to find the square root of this to find the scale factor for length. Asking students to show that the missing length is 2.4 cm, rather

 *Show that the missing length is 2.4 cm.*	than to determine the missing length, means that the focus is on the relationship between the areas and lengths, in particular the relationship between 2.25 and 1.5. This example contains a number of complexities, such as a decimal scale factor; the given scale factor being for area; and the smaller of the two lengths being required. Discuss possible adaptations that could be made if any of those complexities present a barrier to the intended learning point of this example. How can you achieve a balance between managing students' cognitive load, but ensuring the example remains sufficiently challenging to move their thinking on?
Understand that the 'volume scale factor' is equal to the cube of the 'length scale factor' *Example 7:* *A 'super cube' is a cube that can be created from other cubes. An example is a cube with a volume of 8 000 cm^3.* *a) For a cube with volume 8 000 cm^3:* *(i) What is the length of each side?* *(ii) What is the area of each face?* *An 8 000 cm^3 cube can itself be made up of eight cubes that have a volume of 1 000 cm^3.* *b) For a cube with volume 1 000 cm^3:* *(i) What is the length of each side?* *(ii) What is the area of each face?* *c) What do you notice about your answers to parts a and b?* *The 8 000 cm^3 cube can also be made up of one thousand cubes that each have a volume of 8 cm^3.* *d) Compared with the 8 000 cm^3 cube, how many times smaller is the:* *(i) Area of one face of the 8 cm^3 cube?* *(ii) Length of one side of the 8 cm^3 cube?*	*Example 7* uses the **representation** of a cube to explore the relationship between lengths, areas and volumes in similar 3D shapes. Teachers need to ensure that students are clear that all cubes are similar shapes, but that not all cuboids are, and that cubes are therefore a useful starting point for exploring these relationships in more depth. It is likely to be helpful for students to have physical cubes to see and compare. While 8 000 multilink cubes might be a rare commodity in schools, teachers and students coud construct $2 \times 2 \times 2$ cubes and identify that each cube represents 1 000 cm^3. Teachers should attend to the **language** used throughout this example, as it is very easy to lose precision and create confusion. For example, students should be clear about whether they are referring to a length or an area and that these correspond to sides and faces respectively. Be clear about the equivalence of a cube being, for example, '1 000 times smaller' and 'one-thousandth the size'. While there is potential for confusion when so much meaning rests on subtle differences in language, drawing attention to these subtleties provides opportunity to emphasise connections and strengthen conceptual understanding.
Example 8: *A set of packing cubes are mathematically similar.* *The smallest type of cube has a volume of 8 000 cm^3.*	*Example 8* again uses cubes to explore the relationship between similar lengths, areas and volumes. The **representation** explored is of cubes that can fit inside one another. The area scale factor of 4 is implied rather than explicitly given – as four of one cube fit exactly into the base of another, students need to imagine that these four cubes are arranged as a square. From here, they should be able to surmise that another layer of four will form a

Four of the smallest cubes can fit exactly in the bottom of the middle-sized packing cube.

a) What is the volume of the middle-sized packing cube?

b) What is the relationship between the height of the smallest cube and middle-sized packing cube?

The largest packing cube has a height of 60 cm.

c) What is the relationship between the volume of the smallest and largest cubes?

d) How about the middle-sized cube and the largest cube?

cube, thus creating a volume scale factor of 8. Either of these scale factors can then be used to identify the scale factor for length as 2.

Parts c and d offer the opportunity for **deepening** students' understanding of these relationships. Students may readily be able to imagine three of the smallest cubes lined up to form a length of 60 cm, which leads to an area scale factor of 3^2 and a volume scale factor of 3^3. Realising that a whole number of middle-sized cubes will not fit into the larger cube, but that this does not mean that the shapes are not similar, requires a more sophisticated level of mathematical reasoning. Students need to appreciate that the same relationships apply: so if 1.5 of the middle-sized cubes create a length of 60 cm, then the area scale factor is 1.5^2 and the scale factor for volume is 1.5^3.

Students may push back against the idea of cubes fitting exactly inside each other, as this does not take into account the thickness of whatever material the cubes are made from. Discuss with your team how you will handle such challenges. Are students willing to accept that mathematical modelling can involve a degree of assumption? Could the situation actually be feasible if the cubes were made from a thin fabric, with capacities of 8 000 cm^3, 64 000 cm^3 and 216 000 cm^3 once filled?

Example 9:

Sphere A has a radius of 6 cm and sphere B a radius of 18 cm.

a) Which of statements A to D are correct?

A: The volume of sphere A is $\frac{1}{9}$ of the volume of sphere B.

B: The volume of sphere B is 27 times the volume of sphere A.

C: A circle with the same radius as sphere A would have an area that is $\frac{1}{9}$ that of a circle with the same radius as sphere B.

D: A circle with the same radius as sphere B would have an area that is 6 times that of a circle with the same radius as sphere A.

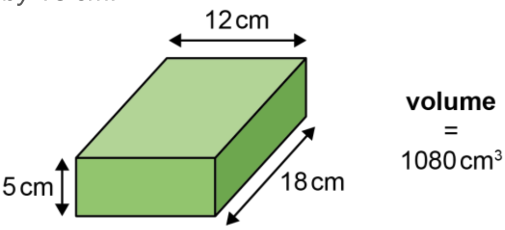
b) Rewrite any incorrect statements, so that they are correct.

In *Example 9*, the structure of *Example 5* is revisited, but this time with the intention of exposing misconceptions around the relationship between similar volumes. Classroom examples often use length scale factors of 2 or 3, so that the volume scale factors remain recognisable to students. However, this can give rise to some common misconceptions: for example, students commonly confuse $2^3, 2 \times 3$ and 3^2. The **variation** between the statements and the selected values for scale factors is designed to support teachers to expose these issues.

The **language** of statements C and D may take some unpicking, but it is important that students are able to visualise the cross-section at the centre of a sphere, so that the radius of the 2D circle is the same as that of the 3D sphere. It may be helpful for students to organise the calculations for the length of the radius, area of the cross-sectional circle and volume of the sphere into a table, as shown below.

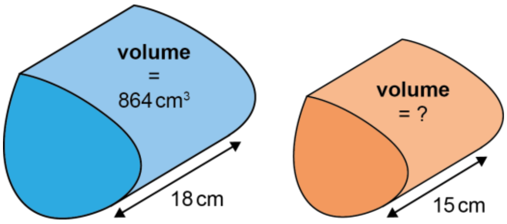
	Length of radius	**Area of cross-sectional circle**	**Volume of sphere**
A	6 cm	x cm^2	y cm^3
B	$6 \times 3 = 18$ cm	$x \times 9$ cm^2	$y \times 27$ cm^3

This **representation** can then act like a ratio table to help students identify the scale factor between each shape in

	each dimension, supporting them to draw conclusions about the general relationships from this particular example.
Example 10: *A range of parcel boxes come in four different sizes: small, medium, large and extra large.* *The small box measures 5 cm by 12 cm by 18 cm:*  *The medium box has the same length and width as the small box but is twice as tall.* *The height of the large box is 5 cm, but the other two dimensions are double the size of the small box.* *a) Calculate the volumes of the medium and large boxes.* *The dimensions of the extra-large box are double the dimensions of the small box.* *Phoebe says, 'Doubling one dimension doubles the volume; doubling two dimensions makes the volume four times bigger; so doubling all three dimensions will make the volume six times bigger.'* *b) Do you agree with Phoebe? Explain your answer.*	The **variation** of *Example 10* provides an opportunity to explore the effects of doubling one, two and all three dimensions of a cuboid on its volume. It also exposes the misconception when using a scale factor of 2 that the 'area scale factor = 2 × length scale factor', rather than 'area scale factor = square of the length scale factor'. Scale factors of 2 are useful for the manageability of the numbers involved in any calculations, but run the risk of such misconceptions occurring. Check that students can explain why the volume of the smallest box is 1 080 cm^2. Notice whether they then rely on using dimensions to determine the volumes of the medium and larger boxes, or whether they demonstrate an understanding of the effects of scaling. Those with **deepening** understanding should be able to identify the relationship between the different sized boxes and use the scale factors appropriately. Phoebe's statement links to a common misconception, and can be easily checked by carrying out the multiplication: $10 \times 24 \times 36 = 8\,640$ cm^3. It is important to discuss with students the thinking behind Phoebe's statement and to establish that the 'volume scale factor = cube of the length scale factor', and why this is the case. Students who are able to reason without calculating, referring instead to the relationships between scale factors, are likely to have a more secure understanding. Pay attention to the **language** students use in their explanations, ensuring that they reference relevant dimensions and scale factors precisely. In this example, the dimensions of the small box and its volume are given. Discuss with teachers what possible variations there might be for how the information is presented, and what purpose any changes would have. For example, what is the minimum information that could be given about the boxes? How would providing students with one of the small box's dimensions and its volume affect the question's accessibility and cognitive demand?

Example 11:

Below are two similar figures.



Show that the missing volume is 500 cm³.

In *Example 11,* the relationship between the 'volume scale factor' and the 'length scale factor' is explored. Students are often more comfortable when working with scale factors that are greater than one, and in decimal rather than fraction notation. It is likely that they will identify the 'length scale factor' as 1.2 rather than $\frac{5}{6}$. Recognising the need to cube this scale factor and then divide 864 by it, demonstrates a **deepening** understanding of the multiplicative relationship between the two figures.

It is unlikely that students will be able to show that the missing volume is 500 cm³ using a trial and improvement approach, so it is important to explore ways to make this example accessible to those who struggle to get started. For example, might it help to swap the order in which the two figures are presented? Or to first explore an example with an integer scale factor? It is important to find a balance between offering examples where the complexity of calculations does not get in the way of understanding, and ensuring students are exposed to non-integer scale factors as standard.

Example 12:

Plastic pipes come in two different diameters: 6 cm and 18 cm. They can be cut to any length.

a) What conditions would need to be in place for any two given lengths of pipe to be similar?

A 9 cm length of 6 cm diameter pipe has a volume of 81π.

The table below this example shows the relationship between the lengths, areas and volumes of the two pipes.

b) Complete the first column of the table, to show how many times bigger the volume is for the different lengths of 6 cm diameter pipe shown.

c) Use the information provided to complete the header cell of the right-hand column of the table.

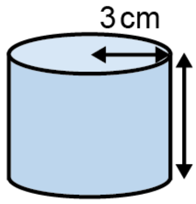
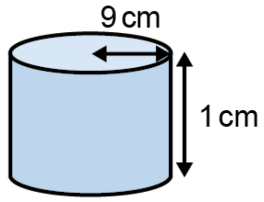
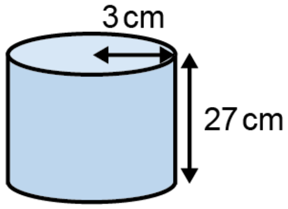
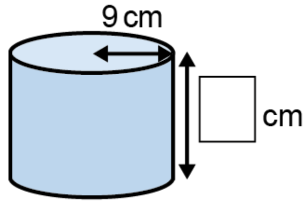
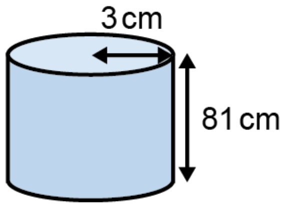
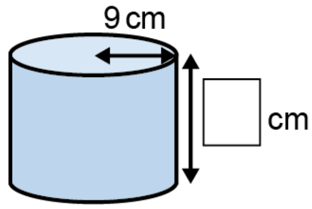
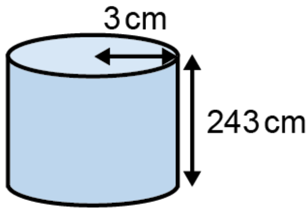
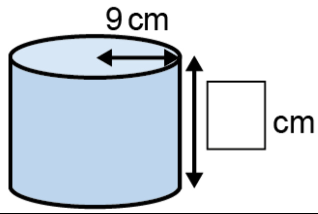
d) What lengths of 18 cm diameter pipe would need to be cut for the volumes given? Complete the labels on the three pipes in the right-hand column.

It is important that students are aware where similarity exists, even where that **language** has not been explicitly used. The focus of *Example 12* is on the effects of scaling two or three dimensions of a 3D shape on the area of the cross-section and its volume

Whilst the two sizes of pipe have been described in terms of their diameter, the table **representation** labels the radii instead. This is to make the cross-sectional area calculation visible to students, but it will be important to draw students' attention to this to check that they have noticed this distinction. Although completing the missing values in the first column of the table may seem fairly straightforward, it highlights the important recognition that multiplying just one of the cylinder's dimensions by three, trebles the volume.

Asking students to complete the lengths of the three pipes in part d is intended for **deepening** their understanding of the mathematical structure of the connection between the two types of pipe. This emphasises the multiplicative relationship of the scale factors.

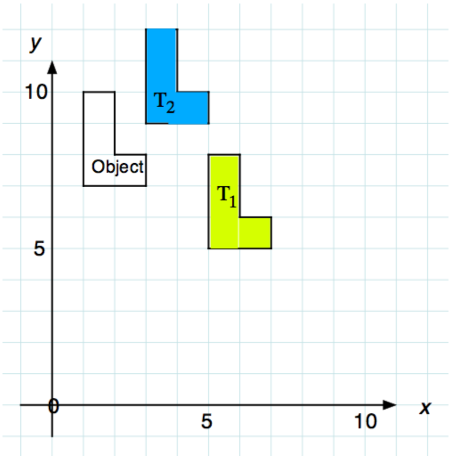
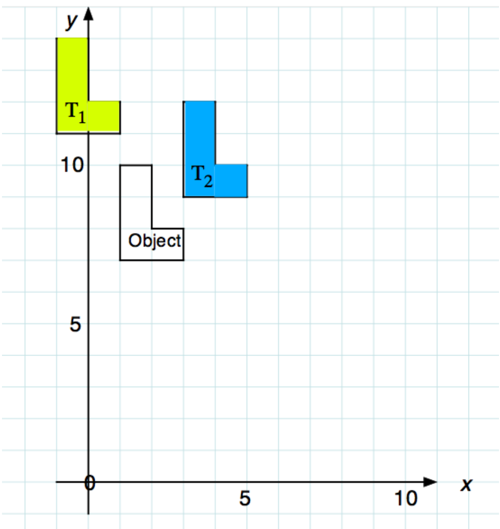
Discuss with teachers how the use of a cylinder helps to focus students' attention on the 'area' and 'volume' scale factors. Explore how the table would differ if a cuboid was used instead, for example. In what ways might this benefit students? What complications might this introduce?

	Cross-sectional area = 9π cm²	**Cross-sectional area = ☐ times bigger**
Volume = 81π cm³		
Volume = ☐ times bigger		
Volume = ☐ times bigger		
Volume = ☐ times bigger		

11.1.4.3 Use and apply the key characteristics of transformations to analyse situations where transformations are combined

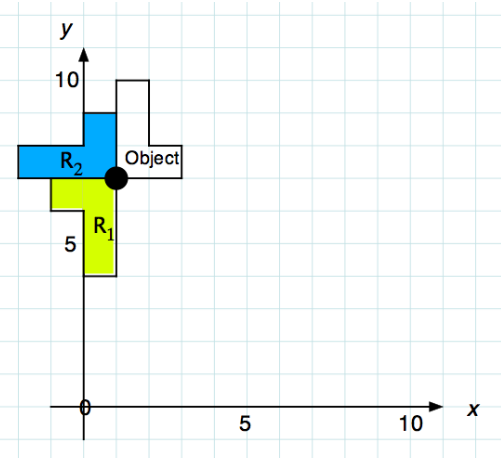
Common difficulties and misconceptions

When performing more than one transformation, students often assume that the order of the transformations does not matter; this is not always the case. A combination of translations provides a clear and accessible illustration of multiple transformations that can be carried out in any order – as does a series of rotations (with the same centre of rotation, and no other transformation in the sequence). However, the inclusion of more than one type of transformation – or of any combination of reflections – warrants further exploration to establish what happens when these transformations are combined. It is important, when working with reflections, that students consider the reflection of an object in lines that are not vertical or horizontal. This helps to expose a possible misconception – that, having undergone a reflection, the image of the original object will remain in the same vertical (or horizontal) plane. Exploring specific cases can help students to think about the characteristics of transformations and analyse situations where transformations are combined. Moving beyond the specific to the general case is an important step in identifying why the order matters in some cases and not in others. It can help students to consider real-life situations as a reference point – the use of ‘people maths’ can illustrate when order matters for combinations of transformations.

Students need to	Guidance, discussion points and prompts
Know how the commutative property applies when combining two transformations of the same type *Example 1:* *Catilin translates an L-shape (the object) using the column vectors* $\begin{pmatrix}4\\-2\end{pmatrix}$ *and* $\begin{pmatrix}-2\\4\end{pmatrix}$*. The image after the first vector is labelled* T_1 *and the final image is labelled* T_2*:*  *Heidi uses the same vectors in a different order. Her images are also labelled* T_1 *and* T_2*:*  a) *What is the same and what is different about Caitlin and Heidi's work?* b) *Find two more vectors that would also result in the object ending up in the same position. What do you notice?*	Exploring combinations of translations helps with **deepening** understanding of the structure of translation and gives a context to consolidate vector addition. In *Example 1*, students should recognise that the final image, T_2, is in the same position in both cases, and may add the two column vectors to establish the resultant vector as being $\begin{pmatrix}2\\2\end{pmatrix}$. Further **variation** to part b could involve asking students to find even more translations that will result in the final image being in the same position as T_2. Can they generalise to identify that any two column vectors $\begin{pmatrix}a\\b\end{pmatrix}$ and $\begin{pmatrix}c\\d\end{pmatrix}$ such that $a + c = 2$ and $b + d = 2$ will result in T_2? Students often confuse the terms 'translation' and 'transformation' and so it is important to check that their use of **language** is accurate, and to emphasise that a translation is an example of a transformation in the same way that rotations, reflections and enlargements are.

Example 2:

Amy rotates an L-shape (the object) 180° about the point $(1, 7)$ and labels the image R_1. She then rotates R_1 90° clockwise about the same point and labels this R_2.



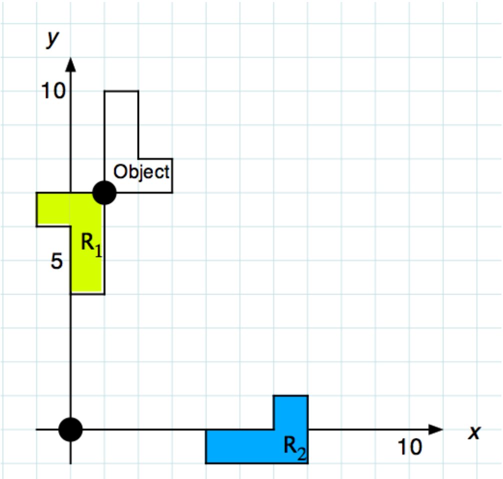
When she does the rotations in a different order, R_2 is in the same place.

a) Where would R_1 have been?

Amy says, 'It doesn't matter what order rotations are performed in, the image will be the same at the end.'

b) Do you agree? Why or why not?

Ben rotates the L-shaped object 180° about the point $(1, 7)$ and then rotates the resulting image 90° clockwise about the origin.



c) What is the same and what is different about Amy and Ben's final images?

d) Will Ben get the same final image if he swaps the order of the two rotations? Explain your answer.

Example 2 highlights the circumstances where two rotations produce the same image regardless of order, and where they do not. Students need to understand that the order of transformations does not matter when the centre of rotation remains constant, but that it does when different centres of rotation are used. The **variation** in this example draws attention to this property: every other feature (shape, angle, direction) is fixed. Positioning a centre of rotation on a vertex of the object can be helpful in highlighting that this point does not move under the transformation.

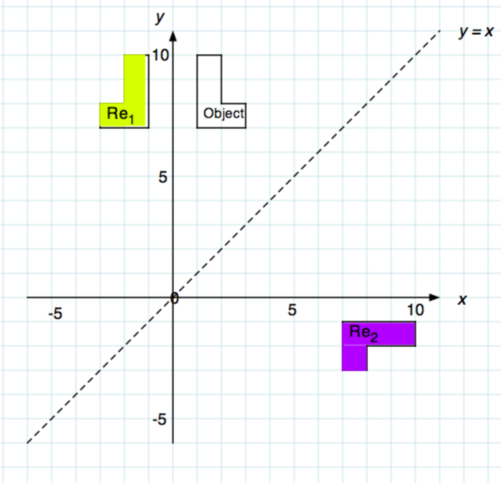
Students should explore the **language** of rotation and have a sense of what information is required to describe unambiguously such transformations. They may intuitively recognise that combining two rotations with the same centre of rotation results in a rotation with an angle of rotation equal to the sum of the two initial rotations. It might be helpful to highlight the reason why a direction of turn is not required when describing a 180° rotation, and also to consider what happens when the total angle of rotation from the combined transformations is equal to, or greater than 360°.

Students should be able to recognise that changing the centre of rotation between the two rotations will change the position of the image. They should also be able to surmise that Ben will therefore not achieve the same final image when the two rotations happen in a different order: the orientation will be the same, but the position will be different. Can students identify the quadrant of the graph the final image will be in after a rotation of 180° about the point $(1, 7)$, without carrying out the rotation? Further questions for **deepening** understanding could involve asking them to specify the third rotation required to transform the L shape from part c back to the position of R_2. This particular centre of rotation has coordinates that are not integer values; this can be used to emphasise that the centre of rotation can be any point in the plane. Students should recognise that a centre of rotation can lie within, outside, or on a vertex of the shape that is being rotated and how the position of the centre of rotation affects the position of the resulting image.

In this and the previous example, static images are used to represent a dynamic mathematical concept. Discuss the strategies teachers use to demonstrate movement in the classroom; for example, dynamic geometry software, cut-outs of shapes or even the students themselves? What are the benefits and limitations of each approach? It is important to find a balance between trying out different combinations of translations efficiently, and ensuring students understand whether the order in which multiple transformations are carried out matters.

Example 3:

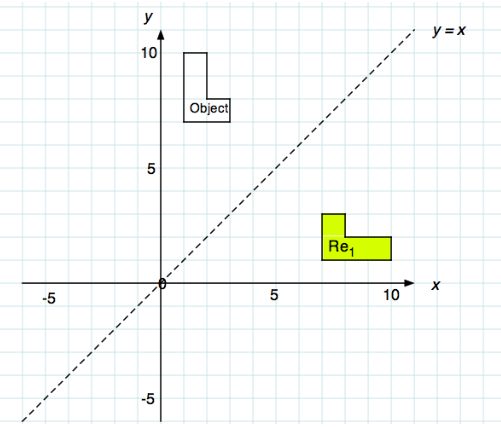
Dev reflects an L-shape in the y-axis and then reflects the resulting image in the line $y = x$:



Dev notices that his final image could also be described as a rotation from the original object.

a) Describe the single rotation to transform the object to Re_2.

He decides to do the two reflections in a different order. First, he reflects the L-shape in the line $y = x$ to create Re_1:



Next, he reflects Re_1 in the y-axis.

b) Will his L-shape end up in the same position as Re_2 did previously? Why or why not?

Example 3 focuses on the combination of two reflections, and the different **language** that can be used to describe this overall transformation. It also highlights the importance of order when combining multiple reflections. Students may initially think that the final image will be in the same place when the order of the reflections is reversed. They may need prompting to recognise that Re_1 would actually need to be reflected in the x-axis rather than the y-axis for this to be the case. There are a number of scenarios where the combination of two reflections can be described as a single rotation. This is the focus of subsequent examples, but *Example 3* is an opportunity for students to begin to explore this.

Consideration of the angle between the two lines of reflection and the angle of rotation for the single rotation can be used to explore combinations of reflections in lines of symmetry that have an angle of ±45° between them. Support students with **deepening** their understanding of these transformations by recording and comparing the outcomes in a table, as shown below. Can they predict the rotation for the combination of reflections that has not been explored in this example?

Two reflections	**Single rotation**
y-axis, then $y = x$	90° clockwise about the origin
$y = x$, then y-axis	90° anticlockwise about the origin
$y = x$ then x-axis	90° clockwise about the origin
x-axis then $y = x$	

It is possible that students will draw inaccurate generalisations from particular cases. Discuss the importance of discussing with students whether or not 'this is always the case'. Explore together some possible prompts that could be used to support students in moving beyond specific examples to the general case, for example:

- *'What is the same and what is different about the angles formed between the y-axis and the line $y = x$, and the x-axis? How does this relate to the single rotation that transforms the object to the final image Re_2?'*
- *'Are there any other combinations of two reflections that result in a rotation of the object 90° anticlockwise to give the (final) image?'*

Understand when a composition of two transformations can be combined as a single transformation

Example 4:

An L-shaped object undergoes some transformations, shown below this example.

The object is rotated 90° clockwise about the origin to give image A.

Different transformations are then applied to image A to create images B, C and D.

a) Match the transformation to the image:

(i) Image A is translated $\binom{-6}{8}$ to give Image…	*B*
(ii) Image A is reflected in the x-axis to give Image…	*C*
(iii) Image A is rotated 90° clockwise about $(7, -1)$ to give Image…	*D*

b) Determine whether a single transformation can be used to describe the transformation of the object to image:

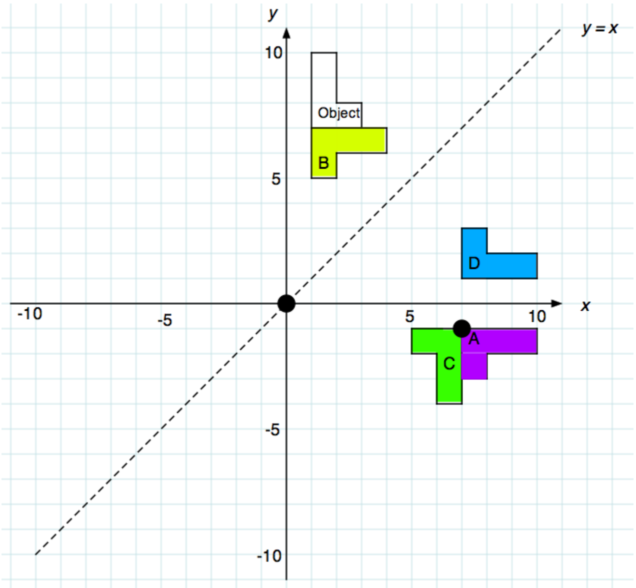
(i) B *(ii) C* *(iii) D*

In the previous example, students began to explore how the combination of two transformations of the same type can often be described as a single transformation and this is now developed further. In *Example 4*, the **variation** explores all possible combinations of transformation, where the first transformation remains fixed as a rotation.

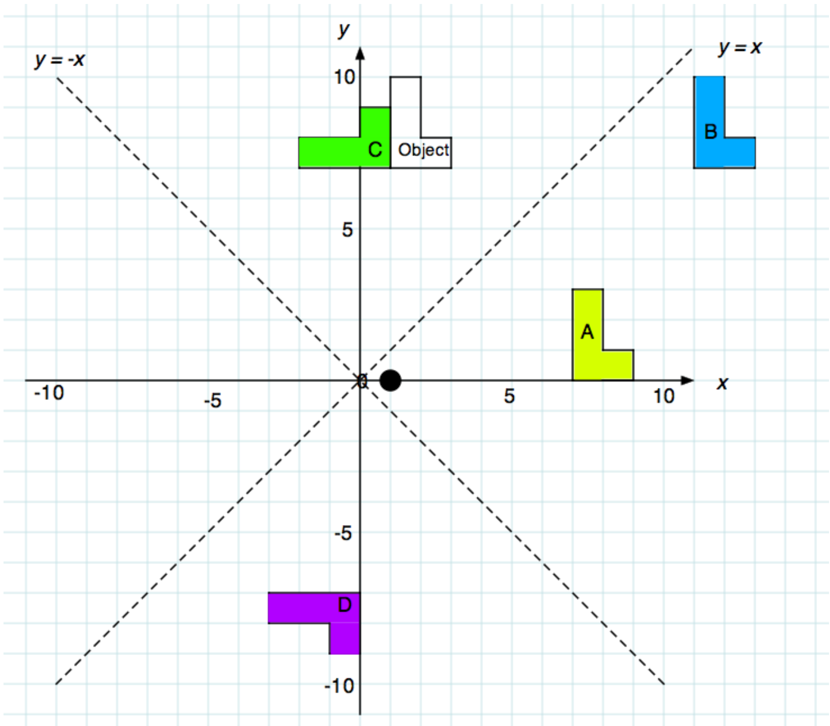
It Is Important that students are familiar with the **language** associated with each transformation, and understand which pieces of information are required to fully describe them. For example, when working on part a, students should describe all three parts of the rotation (angle, direction of turn and centre of rotation).

Students may identify the transformations based on the orientation of the final image, but struggle to identify details. Teachers' prompts have a significant role to play in **deepening** student understanding. For example, in part b students should be encouraged to move beyond a trial and improvement approach, instead identifying the relationship between the centre of rotation, object and image. Students should recognise that only two corresponding points are need to be connected for this. In part c, they should explore how the line of reflection (the x-axis) relates to the centre of rotation for the initial rotation. Teachers could ask students to consider what would happen if the reflection line did not intersect the centre of rotation.

Work with teachers to explore the importance of students making generalisations, and how these generalisations are articulated. What does this demonstrate about their understanding? Teachers could consider the general point (x, y), for example, and ask students to explain why a rotation of 90° clockwise about the origin results in new coordinates $(y, -x)$. Discuss which other transformations explored in *Example 4* may be helpful to consider in general terms.



Example 5: *An L-shaped object undergoes some transformations. These are shown in the image below this example.* *a) Which of the following column vectors describes the translation of the object to create image A?* *(i)* $\binom{4}{-4}$ *(ii)* $\binom{6}{-7}$ *(iii)* $\binom{-7}{6}$ *Image A then undergoes three different rotations so that it is mapped to three different images. It is:* • *translated* $\binom{4}{7}$ *to give image B* • *rotated 90° anticlockwise about a centre of rotation at* $(1, 0)$ *to give image C* • *reflected in the line* $y = -x$ *to give image D.* *b) Is it possible to create each of the above images (B, C and D) with a single transformation of the object?* *c) For those that are possible, describe the single transformation of the object that maps onto each of the images.*	There are multiple layers of **variation** in *Example 5* – firstly, the options presented in part a are designed to expose some common misconceptions. Teachers should reflect upon what students who select column vectors (i) or (iii) might be thinking and how they could suppport them to identify their mistake. This image is then kept as a constant, so that all of the combinations of transformations have the same first step, so that students can more easily identify the pertinent features of the subsequent transformations. In part b, students are simply asked if it is possible to create each of the images with a single transformation from the original object. This relies on students having a secure mental **representation** of the effect of each transformation. Students may initially assume that all three are possible: while it is clear that the transformation of the object to image D cannot be a translation, they may assume that it is a rotation or reflection. It is important that students are encouraged to compare carefully both the position (to rule out reflection) and orientation of image D. Once students have identified that the transformation of the object to image B can be desribed as a single translation, they may choose to count across from a point on the object to a corresponding point on image B. Encourage students to think numerically/algebraically as well as spatially/geometrically. Establishing that adding the two column vectors will give the resultant vector of the combined translation is key to **deepening** students' understanding of the different ways of describing and thinking about transformations.



Example 6:

Below is a new set of transformations, where the L-shaped object from Example 5 has been slightly modified. The object has been translated $\binom{6}{-7}$ *to give image A', which is then reflected in the line* $y = -x$ *to give image D'.*

a) *What is the same and what is different about image D from Example 5, and image D' here?*

This time, the transformation of the object to image D' looks like a rotation of 180° about a centre of rotation at $\left(\frac{1}{2}, 0\right)$.

b) *Why is this the case? Have each of the vertices really undergone this transformation?*

Image B' is not shown but, like image B, it is created by translating image A' along the vector $\binom{4}{7}$.

c) *Describe the single transformation that maps the original object onto image B'.*

Image C' is also not shown. Like image C, it is created by rotating image A' 90° anticlockwise about $(1, 0)$.

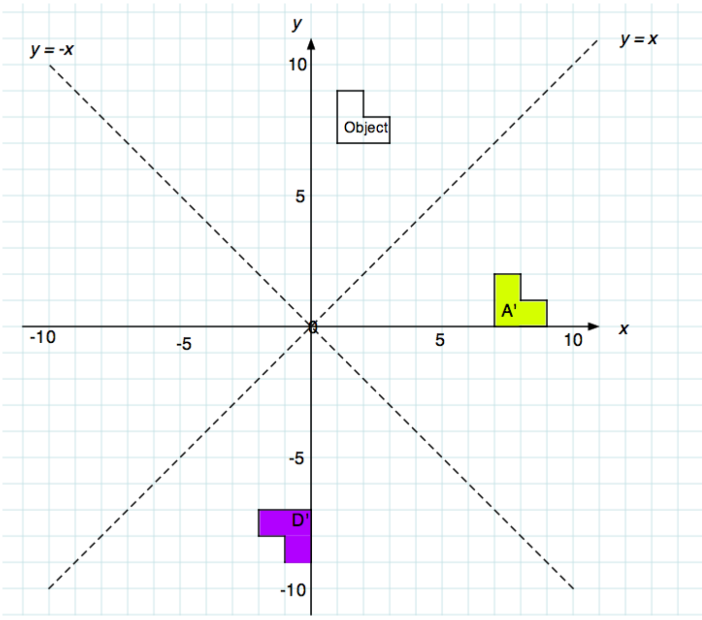
d) *Describe the single transformation that maps the original object onto image C'.*

e) *Why might there appear to be more than one answer to part d?*

Example 6 takes the L-shaped object from the previous examples and modifies it so that it now has one line of symmetry. The transformations explored here are the same as those in the previous example, so that direct comparisons can be made. This **variation** highlights the importance of identifying the way in which each vertex of the object has been transformed, rather than relying on the appearance of the image. If the object was rotated 180° about the point $\left(\frac{1}{2}, 0\right)$. the image would look the same visually, but the orientation of the shape is in fact different. When the object is not symmetrical, as in the previous few examples, the orientation is more easily identifiable.

Consider carefully the **representations** used to exemplify transformations. Shapes with line or rotational symmetry can lead to misconceptions. The same L shape has been used in all these examples, so that any differences students observe are due to the different combinations of transformations, rather than the shapes. The shape in just this example is varied to explore the effect of a symmetrical object on the appearance of transformations.

The **language** in this question has been very carefully selected – referring to what transformations might 'look like' or how they may 'appear'. This is so that students are not misled by the appearance of the shape into thinking that certain transformations are equitable when they are not. This gives the opportunity to ask students how they can check the transformation in more detail without making assumptions, leading to an investigation of the transformation of individual points. It is important that students become accustomed to considering and comparing the positions of the vertices in order to identify transformations accurately.



Example 7:

An L-shaped object undergoes some transformations, shown below this example.

The object is reflected in the line $y = x$ to give image A.

Image A is reflected in the line $y = -2$ to give image B.

a) *Determine whether a single transformation can be used to describe a transformation of the object to image B.*

Image A is translated $\begin{pmatrix}-6\\0\end{pmatrix}$ to give image C.

b) *Determine whether a single transformation can be used to describe a transformation of the object to image C.*

Image A is rotated 90° anticlockwise about the origin to give image D.

c) *Determine whether a single transformation can be used to describe a transformation of the object to image D.*

Example 7 explores scenarios where the first transformation is a reflection. Students often struggle with reflections when the line of reflection is not horizontal or vertical. It is important that they have plenty of opportunity to work with **representations** of reflections in lines other than the x- and y-axes. Attention should be drawn to the fact that every line that joins a point on an object to its image is perpendicular to the line of reflection. This is also the case with horizontal and vertical mirror lines, but in these cases the images are often intuitive and so this important feature can be overlooked.

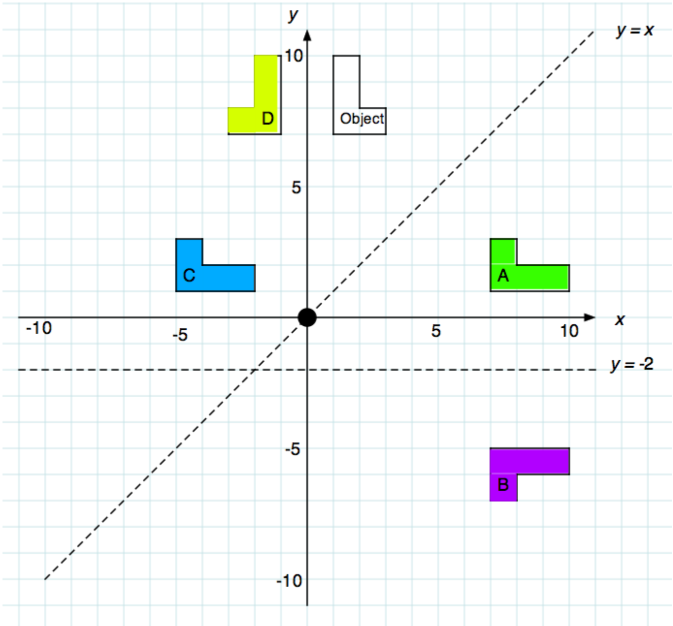
Once students have identified the transformation in part a as being a rotation, which is fairly instinctive, they then need to determine the centre of rotation, which is less so. Recognising that the centre of rotation is the intersection of the line $y = x$ and the line $y = -2$ is a key step in developing students' understanding of combining reflections. When describing two successive reflections as a single transformation, it is important to deploy **variation** in the questioning and further examples that are shown, so that students can start to generalise. It can be helpful to record their responses in a structured way, to help them spot patterns. For example, the table below organises the responses to question stem, '*What is the single transformation when there are two reflections and…*'

… the lines of reflection are perpendicular to each other?	a rotation of 180° about the point where the two lines of reflection intersect
… the lines of reflection are parallel to each other?	a translation

Having identified that two reflections can be described either as a single rotation or translation, students may benefit from by considering the reverse scenario, to help with **deepening** their understanding. They can be supported to deconstruct what has been established through question prompts such as:

- *Can a translation always be described as a combination of two reflections?'*
- *'Is it possible to describe a single rotation as the composition of two reflections?'*

In *Examples 4, 5* and *7*, the object and starting position remains the same, whilst the first of the transformations it undergoes is varied each time. The order in which they are introduced to students here is: rotation, translation and reflection. Discuss with teachers whether they consider the order in which the three different scenarios are introduced to students as being significant. How significant is the order in which the second transformations are introduced?



Use the general case to identify combinations of transformations that can be described by a single transformation

Example 8

When a general point (x, y) is reflected in the x-axis it maps to $(x, -y)$.

a) *Describe what the general point (x, y) maps to after the transformations listed below.*

(i) *A reflection in the y-axis.*

(ii) *A reflection in the line $y = x$.*

(iii) *A reflection in the line $y = -x$.*

(iv) *A rotation 90° clockwise about $(0, 0)$.*

(v) *A rotation 90° anticlockwise about $(0, 0)$.*

(vi) *A rotation of 180° about $(0, 0)$.*

b) *Describe the single transformation that will produce the same result as:*

(i) *A rotation 90° anticlockwise about $(0, 0)$, followed by a rotation of 180° about $(0, 0)$.*

(ii) *A reflection in the y-axis, followed by a reflection in the x-axis.*

(iii) *A rotation 90° clockwise about $(0, 0)$, followed by a reflection in the x-axis.*

Example 8 moves away from specific examples of transformations to a consideration of the general case. It aims to encourage students to generalise what they have noticed when transforming objects. Students often have more difficulty describing transformations than performing them, and they may be entirely unused to thinking about transformations without a graphical **representation** to refer to. They might find it helpful exemplify the transformation a number of specific points before identifying the behaviour of a general point.

The combination of two rotations in part b (i) can be easily identified as a single rotation. This provides an accessible way in to establishing combinations of transformations that can **always** be described by a single transformation, regardless of the features of the object that is undergoing the transformation. Recognising that the starting object/shape is made up of a number of connected points is key to **deepening** understanding of the relationship between the behaviour of the general point and the conclusions that can be made about the transformation of any shape. When students have successfully identified the single transformations in parts c (a rotation of 180° about the origin) and d (a reflection in the line $y = x$), ask them to use their answers for part a to identify some different combinations of transformations that can be described using a single transformation.

Discuss with teachers the transformations in part a, and how the chosen order can help to develop students' understanding in a progressive way. How would accessibility be affected if an alternative starting point was used in the rubric? For example, if the initial reflection was in the line $y = x$ rather than the x-axis? Teachers need to find a balance between careful use of specific points and an emphasis on the general case.

Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 11 Geometry* can be found here:

https://www.ncetm.org.uk/media/3xcpkpft/ncetm_ks4_cc_11_solutions.pdf