9 Sequences, functions and graphs

Mastery Professional Development

9.1 Exploring linear equations and inequalities

Guidance document | Key Stage 4

Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The third of the Key Stage 4 themes (the ninth of the themes in the suite of Secondary Mastery Materials) is *Sequences, functions and graphs*, which covers the following interconnected core concepts:

9.1 Exploring linear equations and inequalities

9.2 Exploring non-linear sequences

9.3 Exploring quadratic equations, inequalities and graphs

9.4 Exploring functions

9.5 Exploring trigonometric functions

This guidance document breaks down core concept *9.1 Exploring linear equations and inequalities* into three statements of **knowledge, skills and understanding**:

9.1 Exploring linear equations and inequalities

- 9.1.1 Understand and interpret the graphical features of linear relationships
- 9.1.2 Use and apply the features of linear inequalities
- 9.1.3 Use and apply the features of linear simultaneous equations

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

9.1.1 Understand and interpret the graphical features of linear relationships

- 9.1.1.1 Understand the relationship between the gradients of parallel and perpendicular lines
- 9.1.1.2 Represent graphically and interpret the solution to linear simultaneous equations
- 9.1.1.3 Find and interpret the area under a straight-line graph (including in contexts such as kinematics)

9.1.2 Use and apply the features of linear inequalities

- 9.1.2.1 Represent the solution set of a linear inequality involving one variable on a number line
- 9.1.2.2 Manipulate and solve linear inequalities involving one variable algebraically
- 9.1.2.3 Represent the solution set of a linear equation involving two variables on a coordinate grid

9.1.2.4 Understand that the solution to a linear inequality in two variables has a range of values

9.1.3 Use and apply the features of linear simultaneous equations

9.1.3.1 Understand that there is either 0 or 1 solution to a set of simultaneous equations where both are linear

9.1.3.2 Understand how to maintain equality when manipulating and combining algebraic equations

9.1.3.3 Manipulate linear simultaneous equations so that they are in a format that is ready to be solved

9.1.3.4 Appreciate that linear simultaneous equations can be solved by elimination or substitution

9.1.3.5 Represent and interpret the solution to linear simultaneous equations

Overview

This core concept explores in depth the structure of linear functions: what their key features are, how graphs and symbols can be used to represent them, and how coordinates can represent particular solutions. Linear inequalities are introduced alongside linear equations to show how the two are related and to promote a connected view of these two areas of the mathematics curriculum. Simultaneous linear equations are also formalised, with the graphical representation being used to build understanding of what solutions to such equations mean and how they might be found.

In Key Stage 3, students will have had experience of solving linear equations algebraically and of drawing and interpreting linear graphs. These ideas are revisited at Key Stage 4 with a view to deepening students' understanding of how the algebraic and the graphical representations of functions are connected and, crucially, how the values of coordinates can represent solutions to linear equations, simultaneous linear equations, and inequalities.

We regard a variable as a symbol representing a mathematical object, and an unknown as a variable which can be found by solving an equation or system of equations. In practice, the two terms are often used interchangeably. We have used unknown to emphasise that a fixed value is being found, and variable to emphasise a changing value or a range of values.

It is important that students are aware that the straight-line graph representing, for example, the equation $y = 3x + 2$ contains the infinite set of points (x, y) where the y-coordinate is always equal to 2 more than the x-coordinate multiplied by 3. They should also then understand that all other points in the plane which are not on the line cannot satisfy this equality and therefore **must** satisfy either of the two inequalities $y < 3x + 2$ or $y > 3x + 2$. In addition to this, students should learn that techniques learned in Key Stage 3 for solving linear equations algebraically can also be used for solving linear inequalities, albeit with some modifications. For example, when multiplying both sides by a negative number, the inequality sign needs to be reversed; such modifications should be explored and their rationale understood.

This idea of coordinates representing a particular point or solution is also essential for students to understand the significance of a point that lies at the intersection of two line graphs. The coordinates of this point satisfy the criteria for both functions and hence form the solution for the pair of simultaneous linear equations they represent.

While exploring the two main algebraic techniques for solving simultaneous linear equations (the methods of substitution and elimination) a key aim is to support students' understanding of how the symbolic manipulation involved in these methods relates to the graphical representation. Students are likely to have seen some simultaneous equations using graphical representations in Key Stage 3, and so it is important that explicit links are made to this work and that they do not see the algebraic techniques as a set of disconnected steps.

Simultaneous equations offer a context for students to bring together elements of mathematics that they may have previously considered separate; in particular, they will be working between the symbolic and the graphical representations of functions and interpreting the results of one in the context of the other. Students often want to resort to an algorithm to solve a pair of simultaneous equations, but it is crucial that they understand the mathematical structures that underpin these approaches

An important concept to understand within this entire theme is that of a continuous function. Students are likely to have mainly experienced graphs from the starting point of pairs of values, potentially limited to integer x-coordinates. Understanding a graph, instead, as an infinite series of points represents a complex shift in their thinking. Graph-plotting software, with its capability to provide dynamic representations of linear relationships, can help students engage with and understand this concept. The mathematical definition of a function is one that is returned to and developed in subsequent core concept documents.

Prior learning

The techniques of algebraic manipulation, which form the bedrock of the understanding developed in this core concept, will have been explored extensively throughout Key Stage 3. Students should be familiar with the idea that collecting like terms, multiplying terms over a bracket, expanding binomials or taking out common factors all maintain equivalence. Students should also be able to substitute values, rearrange and simplify expressions, and solve equations.

Students will have first encountered the symbols used to denote inequality within the primary school curriculum. They will have had a good deal of experience of working with values that are described using the symbols. Students should therefore already understand the concept of inequalities, i.e. of values being unequal. However, understanding that an inequality can be written, represented and manipulated in the same way as an equation is a new step in their learning.

Working with linear equations and inequalities requires students to move between and to connect two familiar representations of a function – the symbolic expression or equation, and the Cartesian graph. Within Key Stage 3, students were introduced to graphical representations of linear relationships, the concepts of gradient and intercept, and the general equation $y = mx + c$. Students will have begun to move freely between algebraic and graphical representations, but many will need this reinforcing as they expand their understanding to include inequalities in Key Stage 4. Particularly, students need to see beyond and between pairs of coordinate values to consider the line as an infinite set of points that satisfy the equation; this is an idea that may have been introduced at Key Stage 3 but not fully embedded. Students may also be less familiar with the idea of the line dividing the plane into distinct regions.

The core concept documents *1.4 Simplifying and manipulating expressions, equations and formulae*, *2.2 Solving linear equations* and *4.2 Graphical representations* from the Key Stage 3 PD materials explore the prior knowledge required for this core concept in more depth.

Checking prior learning

The following activities from the NCETM Secondary Assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

<table>
<tr><th>Reference</th><th>Activity</th></tr>
<tr><td>Secondary Assessment materials page 19</td><td>Draw a line which is parallel to the line $2x + y = 8$ and write down the equation of that line;
… and another,
and another,
and another, …
How do you know they are parallel?</td></tr>
<tr><td>Secondary Assessment materials pages 18 and 19</td><td>What linear equations might produce the following patterns of straight lines?

Draw and write down the equations of three lines which, when drawn with the line $y = 2x + 1$ produce a tilted ‘noughts and crosses’ board similar to this:
</td></tr>
<tr><td>Secondary Assessment materials page 19</td><td>What helps you to decide whether to use an algebraic or a graphical method to solve a pair of simultaneous equation?
Is it possible for a pair of simultaneous equations to have two different pairs of solutions or to have no solution? How do you know?
How does a graphical representation help you to know more about the number of solutions?</td></tr>
<tr><td>Key Stage 3 PD materials document ‘2.2 Solving linear equations’, Key idea 2.2.1.3, Example 2</td><td>This table shows the outcome of substituting different values of p into the expressions $3p + 5$ and $5p - 1$, calculated using a spreadsheet.
<table>
<tr><th>p</th><th>$3p + 5$</th><th>$5p - 1$</th></tr>
<tr><td>−5</td><td>–10</td><td>–26</td></tr>
<tr><td>–4</td><td>–7</td><td>–21</td></tr>
<tr><td>–3</td><td>–4</td><td>–16</td></tr>
<tr><td>–2</td><td>–1</td><td>–11</td></tr>
<tr><td>–1</td><td>2</td><td>–6</td></tr>
<tr><td>0</td><td>5</td><td>–1</td></tr>
<tr><td>1</td><td>8</td><td>4</td></tr>
<tr><td>2</td><td>11</td><td>9</td></tr>
</table></td></tr>
</table>

3	14	14
4	17	19
5	20	24
6	23	29
7	26	34
8	29	39
9	32	44
10	35	49
11	38	54
12	41	59

Use the table to write down:

a) the value of $3p + 5$ when $p = 7$

b) the value of $5p - 1$ when $p = 7$

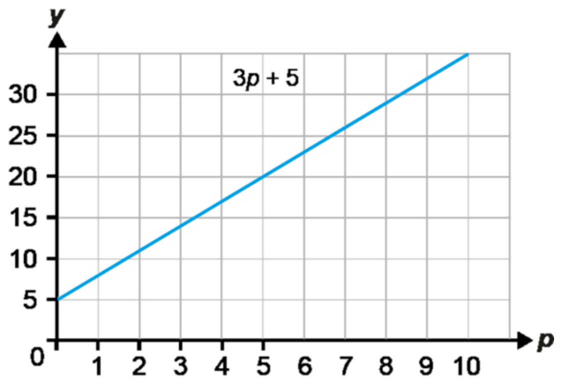
c) the value of p when $5p - 1 = 29$

d) the value of p when $3p + 5 = 29$.

Key Stage 3 PD materials document *'2.2 Solving linear equations'*, Key idea 2.2.1.3, Example 3

This line graph shows the value of $3p + 5$ for different values of p.

a) Use the line graph to write down the value of $3p + 5$ when $p = 7$.

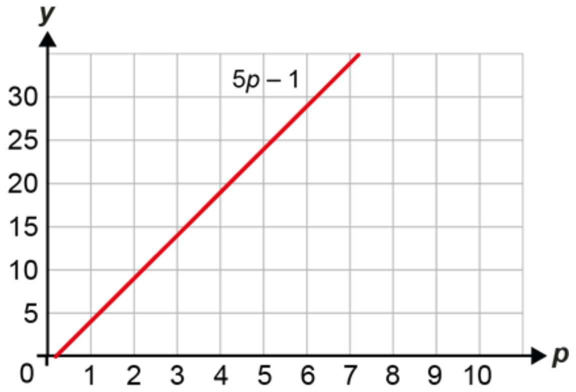


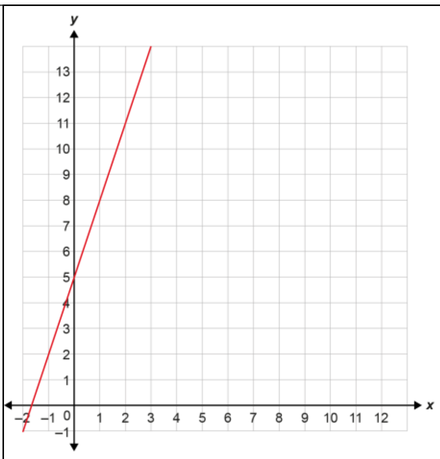
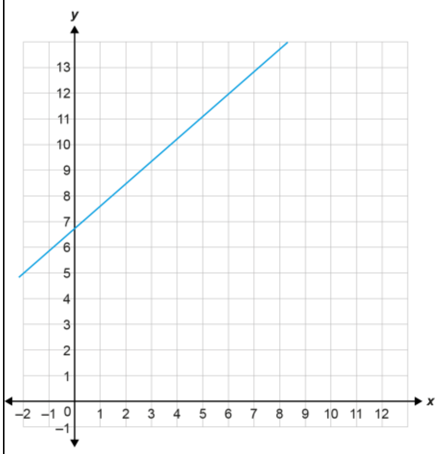
This line graph shows the value of $5p - 1$ for different values of p.

b) Use this line graph to write down the value of $5p - 1$ when $p = 7$.

c) Use the first line graph to write down the value of p when $3p + 5 = 29$.

d) Use the second line graph to write down the value of p when $5p - 1 = 29$.



Key Stage 3 PD materials document '*4.2 Graphical representations*', Key idea 4.2.3.4, Example 2	 1. This graph shows the line $y = 3x + 5$. a) Mark three points where $y > 3x + 5$. b) Mark three points where $y < 3x + 5$. c) Mark three points where $y = 3x + 5$.  2. This graph shows the line $y = x + 7$. a) Mark three points where $y > x + 7$. b) Mark three points where $y < x + 7$. c) Mark three points where $y = x + 7$.

Key vocabulary

Key terms used in Key Stage 3 materials

- inequality
- intercept
- gradient
- linear
- simultaneous equations

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
solution set	A description of all of the values that are solutions to a given equation or inequality. A single solution can be represented by a point on a graph, whereas an infinite set of solutions can be represented by a line or plane.

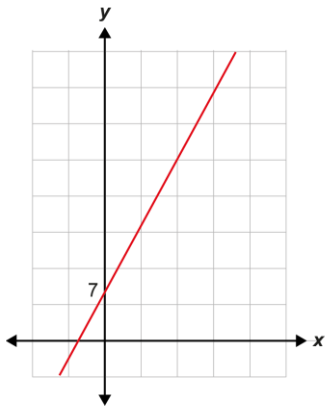
Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

9.1.1 Understand and interpret the graphical features of linear relationships

This statement emphasises that the spatial features of the graph correspond to both the function and its algebraic representation.

The linear graph below has certain spatial features that correspond to the equation $y = 2x + 7$. A linear relationship such as $y = 2x + 7$ is a function consisting of a combination of multiplication and addition. Students need to see how this 'double and add 7' arithmetic structure is mirrored in the spatial structure of the graph.



When examining the algebraic alongside the graphical representation, questions such as the following can be asked to support this understanding:

- When doubling and adding 7, how does the result change when the input is 1, compared to when the input is 2, to when the input is 3, etc.?
- When 1 is added to the input, how does the output change? What about when 2 is added? What about 2.5?
- Why is it that the $+7$ part of the equation means that the line cuts the *y*-axis at 7?
- Where does the graph cut the *x*-axis? How does this value relate to other features of the graph?
- Choose two points on the graph where it passes through the intersection of two grid lines. What might the input and output be for these points? How about points in between these points? And between them?

Students should learn to recognise the graphical representation of a function as a single mathematical object which captures its features and then be able to sketch it without recourse to the plotting of points.

9.1.1.1 Understand the relationship between the gradients of parallel and perpendicular lines

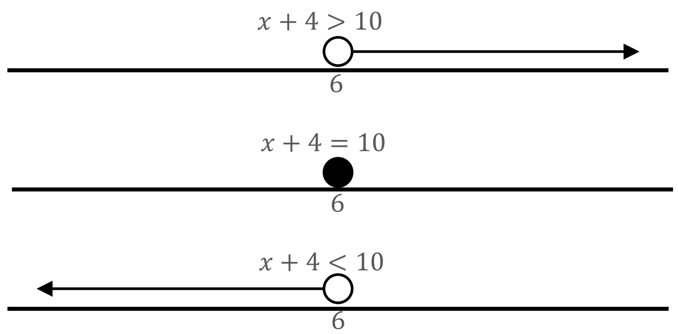
9.1.1.2 Represent graphically and interpret the solution to linear simultaneous equations

9.1.1.3 Find and interpret the area under a straight-line graph (including in contexts such as kinematics)

9.1.2 Use and apply the features of linear inequalities

Building on students' experiences with simple linear equations in Key Stage 3, this section explores inequalities, how their solutions can be determined and how their graphical representations are connected to the symbolic.

While the equation $x + 4 = 10$ has only one solution, the inequalities $x + 4 < 10$ and $x + 4 > 10$ each have an infinite number. It is important for students to appreciate why and how the solution to the equation is directly related to the solution set for the inequalities. This might be supported using a number line as shown by the three related examples below.



The understanding established through exploration of inequalities on the number line can then be developed further by exploration of regions on graphs.

Students need to understand that the transformation such as 'subtract 4 from both sides' maintains the relationship established within a given inequality. They should be familiar with this in the context of linear equations, so this should be extended to inequalities through discussion and exploration of both algebraic and graphical representations. Careful use of variation can be employed to explore similar transformations and to reason which ones also hold true for inequalities (for example, adding and subtracting any quantities from both sides and multiplying and dividing both sides by a positive quantity) and which ones do not (for example, multiplying and dividing by a negative quantity).

- 9.1.2.1 Represent the solution set of a linear inequality involving one variable on a number line
- 9.1.2.2 Manipulate and solve linear inequalities involving one variable algebraically
- 9.1.2.3 Represent the solution set of a linear equation involving two variables as a region on a coordinate grid
- 9.1.2.4 Understand that the solution to a linear inequality in two variables has a range of values

9.1.3 Use and apply the features of linear simultaneous equations

Students should already know that a linear equation such as $y = 5x - 3$ can be represented by a straight line and that every point on that line fits the given relationship; this is fundamental to students being able to understand linear simultaneous equations. Students should also know that this line is continuous and not limited to integers (see exemplified Key Idea 9.1.3.2 below).

Building on the understanding that every point on the line obeys the given relationship, students should consider a second line with a different relationship between the variables. Plotting both graphs on the same axes should lead to a realisation that there is a single point whose coordinates satisfy both of the given relationships, and is therefore the solution to the given simultaneous equations.

This awareness should then be extended to consider special cases where there is no solution (i.e. the lines are parallel), and where there are infinite solutions (i.e. the lines are concurrent).

Careful selection of examples involving non-integer solutions will help to draw students' attention to the fact that, while graphical representations like this are useful, they cannot guarantee an exact answer. Different strategies are needed for finding the pair of values common to both functions, and these are commonly known as the algebraic techniques of substitution and elimination.

9.1.3.1 Understand that there is either 0 or 1 solution to a set of simultaneous equations where both are linear

9.1.3.2 Understand how to maintain equality when manipulating and combining multiple algebraic equations

9.1.3.3 Manipulate linear simultaneous equations so that they are in a format that is ready to be solved

9.1.3.4 Appreciate that linear simultaneous equations can be solved by elimination or substitution

9.1.3.5 Represent and interpret the solution to linear simultaneous equations

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

9.1.2.4 Understand that the solution to a linear inequality in two variables has a range of values

Common difficulties and misconceptions

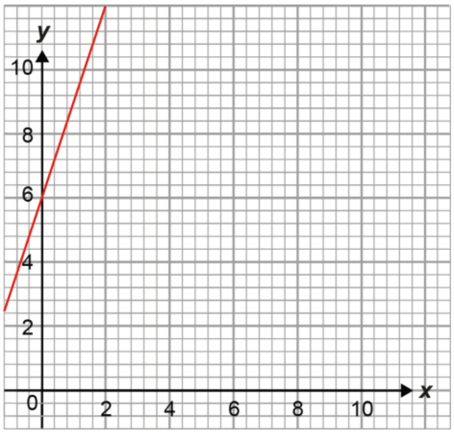
In Key Stage 3, students drew the graphs of linear equations (for example, $y = 2x - 7$) by plotting a small number of (probably integer) points and joining them with the straight line representing the equation $y = 2x - 7$. Some students may believe that the set of points is restricted to integer values or that the line 'stops' at the edge of the drawn graph. They may also not realise that there are no other points that fit the relationship. Students need to understand that there are an infinite number of points on the line and that these represent the complete set of points which satisfy the given relationship.

An awareness of the complete and infinite nature of the solution set is essential, as it will lead to the realisation that the relationship does not hold true for any points that are not on the line. Further, this awareness can build to the appreciation that one side of the line is the region where $y < 2x - 7$ and the other side is where $y > 2x - 7$. Students may guess which side of the line is which without employing reasoning and logic based on the relationship between the x- and y-coordinates, but they should be encouraged to verify their guess by checking the values of pairs of coordinates.

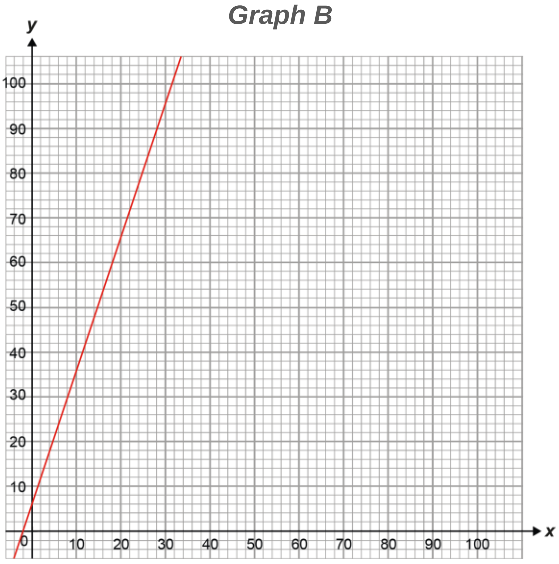
Students need to be able to distinguish between inequalities where the limiting values are part of the solution set (i.e. where the symbols $\leq$ and $\geq$ are used) and those where they are not (i.e. where the symbols < and > are used). This includes the use of solid and dashed lines in graphical representations. Ensuring that distinctions have been made clear when working with representations of the solutions of linear inequalities will help students recognise and use these conventions.

Students need to	Guidance, discussion points and prompts
Understand that a line on a graph is smooth, continuous and infinitely long *Example 1:* *Graphs A, B and C all show the same equation but on different axes.* *a) Where does the line in Graph C meet the y-axis?* *b) Where does the line in Graph A meet the x-axis?*	The **variation** in *Example 1* gives a context to discuss the infinite and continuous nature of a function as represented here by a line. It is intentional that students are asked to find features that are not immediately obvious or visible on the graphs that they are directed to. By 'zooming' in and out of the same line, and using these different reference points, students must shift their attention from 'big picture' to 'small detail'. By realising that these are different images of the same relationship, they should understand which image is helpful to draw conclusions about the value and features of the function at different points.

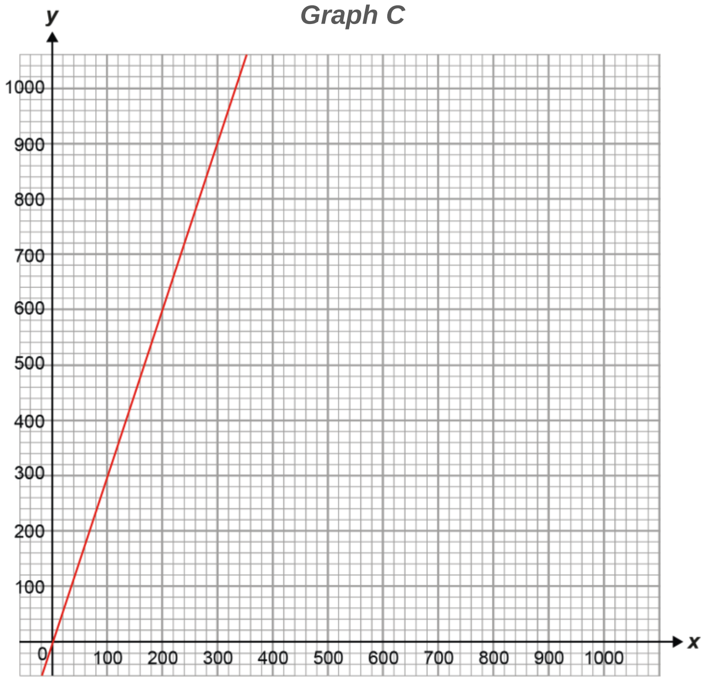
Graph A

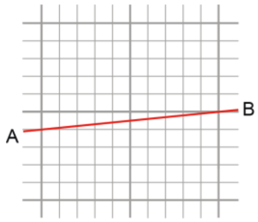
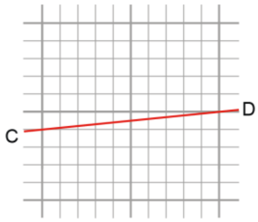
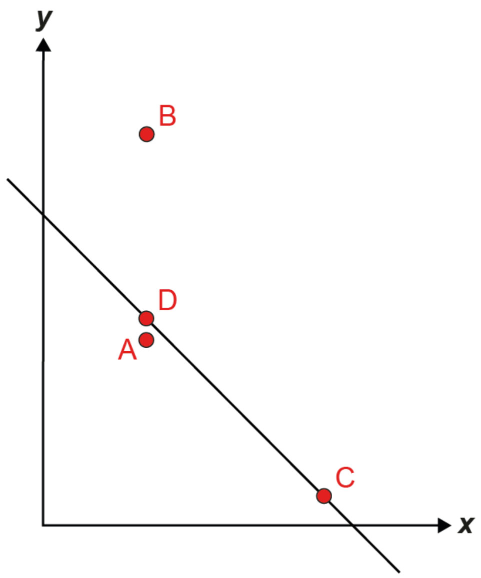


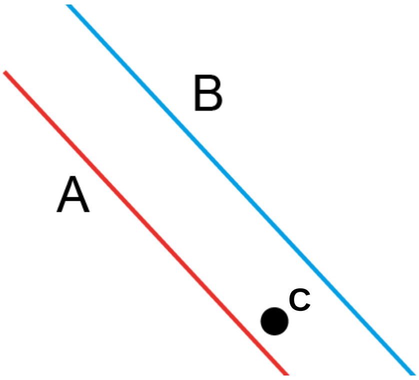
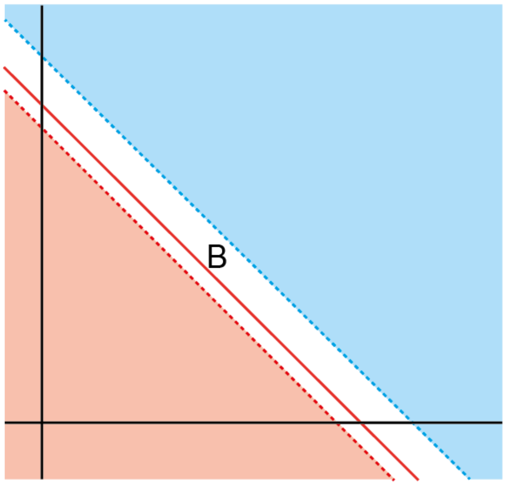
Graph B



Graph C



Example 2: *Here's a section from a graph:*  *The coordinates of the line at A are (2, 3) and at B are (4, 7).* *Here's a section from further along the same graph and using the same scale:*  *The coordinates of C are (500, 999).* a) *Write the coordinates of D.* b) *Write the coordinates of three more points on the same line.*	*Example 2* also uses **variation** to show the continuous nature of the function as represented by a line. The intention of shifting the values of the coordinates in the two-line segments by a significant amount is that students use this 'peculiar' case of segment CD to step towards generalisation. Consider how the coordinates given for C can help draw attention to the relationship between x- and y-values. Does it help students to notice the fact that the y-coordinate is one less than double the x-coordinate? Would this relationship be as obvious if the coordinates were (1002, 2003) or (505, 1009)?
Appreciate that a line on a graph delineates three distinct regions *Example 3:* *This graph shows the line* $x + y = 5$  *What might the coordinates of the labelled points be?*	*Example 3* uses the Cartesian graph as a **representation** to allow students to reason with a function and the possible values that fit with that function. Students should be able to choose values that fit with the line. They are likely to choose integer values, which suggests a limited understanding of possible values. Asking them to identify all possible values for x and y may result in just the set of possible integer values, they can then be prompted to think about 'What else might it be?' The labelling of the points A, B, C and D does not suggest an order to go 'through' them; a useful prompt might be, '*Which of these points is easiest to estimate and why?*' Alphabetical order is not a simple answer. Reasoning around the precision of the answers and noticing that, for example, the *x* value of points A, B and D will be very close is a part of the intended learning here. Discuss the responses students might give what that might tell you about their understanding of the regions identified by the given line.

Example 4: *Two lines, labelled A and B, are shown below.* *One is part of the graph $\boldsymbol{x+y=10}$ and the other is part of the graph $\boldsymbol{x+y=11}$.*  a) *Which line is which equation? How do you know?* b) *What could the coordinates of point c be?*	*Example 4* uses **representations** in the same way as *Example 3* and, together with *Example 5*, forms a useful sequence of tasks. The intention of *Example 4* is for students to work and reason generally with lines as a representation of a function, and to move attention on from particular values or points towards working with lines and regions themselves. To support this generalisation, there are no values included on the image. Students might notice and use the informal language that the area enclosed by the two lines satisfies the conditions $x+y$ is 'more than 10 and less than 11'. This language can be formalised in the inequalities $x+y>10$ and $x+y<11$.
Example 5: *The two regions shaded below are* $\boldsymbol{x+y<16}$ *and* $\boldsymbol{x+y>20}$.  a) *Which region is which?* b) *What might be the equation of the solid line B? How do you know?*	*Example 5* continues the sequence of tasks with students using a graphical **representation** to reason about the possible values of points, lines and regions defined by given functions. There are no labels on the image, so students have to use the functions given to identify values. Encouraging them to think about key points is a useful strategy to enable them to access the task.

Understand that the solution to a linear inequality in two variables has a range of values

Example 6:

a) *Categorise the following coordinates* $(x,\ y)$ *into the table below, depending on whether* $y < x + 10$, $y = x + 10$ *or* $y > x + 10$.

(4, 4) *(17, 7)*

(16.3, 3.7) *(8.9, 0.9)*

(-1.2, 10) *(14.3, -2.9)*

$y < x + 10$	$y = x + 10$	$y > x + 10$

b) *Continue the table with some examples of your own.*

The given coordinates prompt students to try non-integer as well as integer coordinates and encourage the exploration of both positive and negative values for x and y. This **variation** gives the opportunity to draw students' attention to the continuous nature of the functions. Further prompts might make this more explicit. For example, asking students to give a coordinate pair in each column that:

- *has three decimal places, or four*
- *is negative with two decimal places*
- *includes fractions with denominators of 17*
- *includes a value greater than 100.*

Encourage students to reason about where each set of coordinates corresponding to the three columns in the table lie when **represented** graphically.

Students' thinking can be extended through questions such as:

- *'Is it possible to describe the position of all of the points where* $y > x + 10$*? How about* $y = x + 10$*?'*
- *'What is the same and different about your descriptions for the positions of points in each column?'*

Encourage students to describe the graphical representation of the points and, if appropriate, to test their descriptions by plotting the given coordinates.

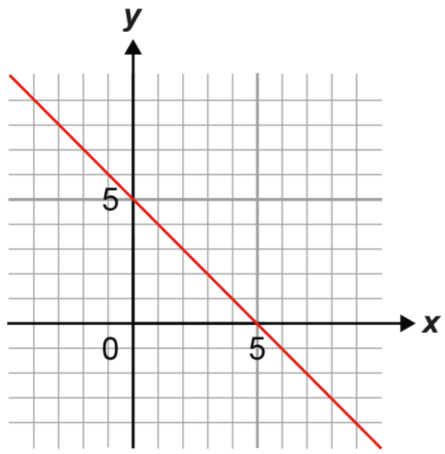
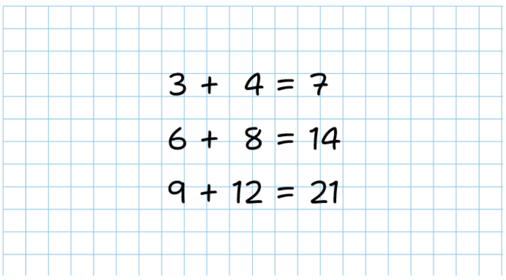
Consider the prompts and questions given above. What mathematical understanding are they intended to elicit? What other features might need to be considered? Create some more prompts and questions to draw attention to these features.

9.1.3.2 Understand how to maintain equality when manipulating and combining algebraic equations

Common difficulties and misconceptions

Students should understand from Key Stage 3 that equality is maintained within an equation when multiplying or dividing all elements of a single equation by the same value, or when adding or subtracting an amount from both sides. They now need to be able to work with equations as 'objects' in their own right, to see an equation such as $2x + 3y = 10$ as a whole rather than as a set of instructions. Further to this, they need to learn that entire equations can be combined, given certain constraints. This often proves challenging and confusing, and the use of representations and contextualised problems will greatly help students to gain insights.

Understanding that $2x + 3y = 10$ and $4x + 6y = 20$ are equations representing the same set of points and hence the same line, is a key concept that is often not well understood by students. Exemplification of the ‘equivalence’ of pairs of equations, through tables of values as well as graphical representations, can help students to better appreciate this concept.

Students need to	Guidance, discussion points and prompts
Understand that equivalence is maintained when multiplying all elements of an equation by the same amount *Example 1:* *Here is the graph of $2x + 2y = 10$:*  a) *What do you notice?* b) *How might you rewrite the equation of the graph?* c) *What does this tell you about the graph of $4x + 4y = 20$?* d) *What about $2x + 2y = 30$?* e) *What about $3x + 3y = 30$?*	*Example 1* is about making sure students realise that different equations can be **representations** of the same relationship. Students’ attention should be drawn to the fact that multiplying all elements of the equation by the same amount results in no change in the underlying relationship, so has no effect on this representation of the relationship. It is important for students to also understand ‘what it is not’; in this example, attention should be drawn to the non-equivalence of $2x + 2y = 30$ and $3x + 3y = 30$.
Example 2: *Mary, Jay and Satsuki are working together on some algebra.* *Mary writes the following equations:*  a) *What has Mary done? What do you notice? What might Mary do next?*	In *Example 2*, we are exploring a structure we will later rely on for algebraic manipulation. **Variation** is used to further reinforce the idea that algebra is an extension of number. Students should be encouraged to compare both within and between the three groups of equations. Students’ attention could be drawn to the fact that multiplying all elements of the equation by the same non-zero value results in no change in the underlying relationship. Mary might multiply all elements if the initial equation by 4 next to arrive at 12 + 16 = 28, for example. How does your department usually introduce new algebraic structures? What are the advantages and disadvantages of beginning numerically?

Jay writes:

$$10 - 3 = 7$$
$$20 - 6 = 14$$

b) *What is the same and what is different about what Mary and Jay have done?*

Satsuki writes:

$$3 + 4x = 19$$
$$6 + 8x = 38$$
$$9 + 12x = 57$$

c) *What do you notice about the value of x in each equation? Why is that the case?*

Understand that equations can be combined to create further valid equations

Example 3:

Mary writes:

$$15 + 12 = 27$$
$$3 + 5 = 8$$

She adds each term in the second equation to each term in the first and says the following must be true:

$$18 + 17 = 35$$

a) *Is she correct? Is it permissible to do this?*

She subtracts each term in the second equation from each term in the first and says the following must be true:

$$12 + 7 = 19$$

b) *Is she correct? Is it permissible to do this?*

Example 3 is an opportunity to establish that entire equations can be added in a way that maintains equality. Students could be encouraged to devise equations of their own and check their validity. This type of question can be used for **deepening** students' understanding of the structure of the arithmetic of combining equations. Understanding that the expression on the left side can be treated as an object in its own right is fundamental to understanding how to solve simultaneous equations.

In part b, students extend their understanding to subtracting values within an equation and understanding that this is likely to be problematic. It may be helpful to use **representations** such as counters to recognise that an equation remains balanced when the same amount is subtracted from each side. Crucially, it is important students understand that the amount can be subtracted partially from one term and partially from another – as long as they are both on the same side of the equation. In the case below, 8 has been subtracted from both sides, but on the left-hand side the 8 consists of a 3 and a 5.

$$(15 - 3) + (12 - 5) = (27 - 8)$$

Example 4:

Malcolm is creating equations for his friend Rita to solve. He starts with:

$$x = 5$$

a) What equations might he write next?

He writes the following:

$$2x = 10$$
$$2x + 3 = 13$$
$$3x + 3 = 13 + x$$

b) Explain what Malcolm has done.

Malcolm adds his starting equation onto his latest equation and writes:

$$4x + 3 = 18 + x$$

Rita says that Malcolm is not correct, because he has added x to the left-hand side and 5 to the right-hand side.

c) Why is Malcolm correct?

Rita now writes some equations for Malcolm to solve:

$x = 7$ and $y = 4$

$2x = 14$ and $y + 8 = 12$

So:

$2x + y + 8 = 26$

d) Can you explain what she has done?

Rita then combines $2x = 14$ and $y + 8 = 12$ differently and writes:

$$2x + 12 = y + 22$$

e) Is she correct?

Example 4 sets up a situation which students should be comfortable with before **deepening** their thinking to prepare them for work on simultaneous equations. This approach helps students see equations as objects in their own right. It demonstrates how equations that look complex are composed of simpler parts.

It is important to pay close attention to the **language** in students' responses. Phrases such as 'change the side, change the sign' are often unhelpful. This activity should better enable students to understand a 'balancing' approach where the same amount is added or subtracted from each side.

The **variation** in parts c to e supports teachers to draw attention to particular features of equivalence. This establishes some of the fundamental concepts that will be built upon when manipulating simultaneous equations. It also offers an opportunity to challenge any potential assumptions on the students' part. For example:

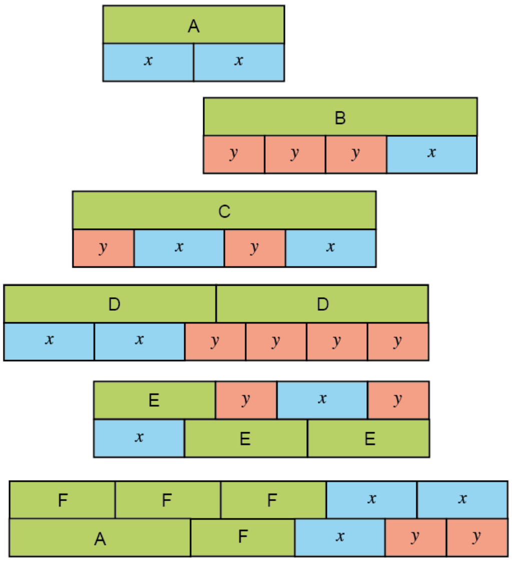
- In part c, students should be encouraged to notice that, since the starting point of $x = 5$, equations have been combined by adding terms that are different but of equal value to both sides.
- In part d, the basic equations are manipulated in two different ways. They are then combined by adding the two left-hand sides and adding the two right-hand sides to produce a further equation where the two sides are equivalent. It is important that students understand why this is a valid and permissible action and how it maintains equivalence.
- In part e, the left-hand side of the first equation has been combined with the right-hand side of the second. Students are likely to find this more challenging to accept as a permissible combination. Drawing attention to the 'equals' symbol meaning that the two sides are equivalent, and reminding them that $y + 8 = 12$ can be written as $12 = y + 8$ will help with **deepening** their understanding.

The equation used is only a suggestion, and it might be helpful for students to experience building their own equations using the structure of this example. What are the benefits and challenges to asking students to write their own equations? What 'teacher moves' might ensure that the intended focus of learning is maintained when students create their own examples?

Become proficient at combining processes to manipulate equations

Example 5:

Blue bars have a length of x and red bars have a length of y.



a) *Write the lengths of the green bars A, B, C, D, E and F.*

b) *Which green bar is longer, A or C? By how much?*

c) *Which green bar is longer, E or D? By how much?*

d) *Bars A and E are stuck together. Are they longer than C? If so, by how much?*

e) *Which green bar is twice as long as F? How do you know?*

In *Example 5,* the bar model is used as a familiar **representation** for an algebraic relationship. The representation may support students in making sense of the manipulations that are permissible, such as combining (as in part c) or doubling expressions (part d).

A challenge with the bar model is that it can be interpreted as a static, fixed relationship. It is intended to represent a dynamic relationship in which the component parts of x and y can take on any value.

For students who are less familiar with this dynamic interpretation, using comparative **language** structures can help to establish the 'rules' of the representation in which the lengths change. For example, offer question prompts such as:

- *'If x is 5 and y is 2, how long are bars A, B and C?'*
- *'If x is 2 and y is 5, how long are bars A, B and C?'*

In the second prompt, the values do not relate well to the image, since it is clear that in the image x is greater than y. This might shift students towards a more abstract understanding of the bars.

At what point in your curriculum do you introduce bar models? If students are unfamiliar with them, or have not routinely used them since primary school, is it worth introducing this task or is it better to continue to work in the abstract?

Understand that equations can be manipulated, combined and compared to create further valid equations

Example 6:

a) *If $e = 3$ and $f = 5$, find $e + f$*

b) *If $g + h = 3$ and $j + k = 5$, find:*

   (i) $(g + h) + (j + k)$

   (ii) $(j + g) + (k + h)$

c) *If $s + t + u + v = 3$ and $4mn = 5$, find $s + t + u + v + 4mn$*

d) *If $2x + 3y = 3$ and $x + y = 5$, write an expression that is equal to 8.*

The **variation** in *Example 6* draws students' attention to the basic structure, that if:

$A = B$ and $C = D$

then

$A + C$ must equal $B + D$.

To focus on this basic structure, keep some elements, like the 3 and the 5, the same throughout. Pause at certain points and discuss what students are noticing. A particularly useful prompt after part d might be, *'Is there another way to write the expression which is equal to 8?'*

Example 7: a) *If* $e = 3$ *and* $f = 5$*, find:* (i) $e - f$ (ii) $f - e$ b) *If* $g + h = 3$ *and* $j + k = 5$*, find:* (i) $(g + h) - (j + k)$ (ii) $(j + k) - (g + h)$ c) *If* $s + t + u + v = 3$ *and* $4mn = 5$*, find* $(s + t + u + v) - 4mn$ d) *If* $2x + 3y = 3$ *and* $x + y = 5$*, write* (i) *An expression that is equal to 2* (ii) *An expression that is equal to* -2*.*	In *Example 7*, **variation** is again being used to draw students' attention to the related structure: If $A = B$ and $C = D$ then $A - C$ must equal $B - D$. We are also developing the idea that knowing the value of an expression allows you to work with that expression, even when the values of the individual terms are unknown. Further questions go on to **deepen** that understanding and work with subtraction. By changing the order of the terms, students' attention could be drawn to the differences between addition and subtraction.
Example 8: a) *If* $4m + 5y = 33$*,* $4m + 3y = 31$ *and* $5m - 3y = 32$ *, what is:* (i) $(4m + 5y) + (4m + 3y)$*?* (ii) $(4m + 5y) - (4m + 3y)$*?* (iii) $(5m - 3y) + (4m + 3y)$*?* b) *What do you notice about your answers from part a?* c) *Which of your answers helps you to find the value of* y*? Why?* d) *Which of your answers helps you to find the value of* m*? Why?*	*Example 8* is important as it offers a context in which one of the variables is eliminated. This idea has been met using a bar model **representation** in *Example 5*, but here is presented algebraically. Use the correct **language** to discuss the fact that elimination occurs when the coefficients of a term are equal. Compare the outcomes of sub-question of part a, and how useful they are in moving students towards a solution. This draws attention to the different ways in which elements can be eliminated, by subtracting when there are terms with identical coefficients or adding when there are terms with equal and opposite coefficients. Students might approach *Example 8* in different ways. For example, they might write the expression and equate it to the value and then simplify (and solve if possible). Alternatively, they might substitute the values given and so rewrite the equations numerically. Parts b to d support students to notice the features of elimination, regardless of which route they take. Consider your own students. Which approach are they most likely to take? Will they need additional prompting to help elicit the key learning points?

9.1.3.4 Appreciate that linear simultaneous equations can be solved by elimination or substitution

Common difficulties and misconceptions

Students may know how to solve simultaneous equations without learning why their method works. Following algorithms such as ‘Make the coefficient of x equal and then subtract the equations’ or using mnemonics to remember a set of instructions, allows students to find the value of x and y but does not support them in understanding why such manipulations are appropriate or helpful.

This approach may have some short-term success, but it is limited to standard equations and situations, and may not be a particularly efficient method for a given problem. It is preferable for students to develop a conceptual understanding of the algebraic structures that sit behind their methods, allowing them to make appropriate choices to work towards a solution to a given problem.

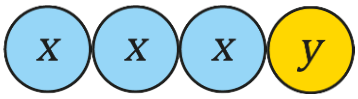
This key idea is particularly concerned with the algebraic manipulation used when solving a pair of linear simultaneous equations, but this manipulation should not be isolated from the graphical interpretation; connections should be made between the two representations.

Students need to

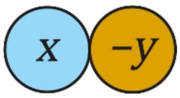
Understand how adding or subtracting two equations can result in one of the variables being eliminated

Example 1

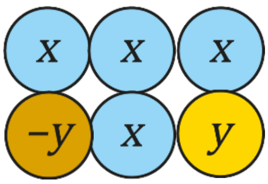
The value of the following is 19:



The value of the following is 3:



a) *What is the value of the collection below?*



b) *Can you write an algebraic expression for the collection of discs?*

c) *What is the value of one of the x discs?*

d) *Can you work out the value of y?*

Guidance, discussion points and prompts

In *Example 1*, algebra discs are used as a **representation** to highlight the manipulation of the two variables. This elimination requires a familiarity with zero pairs, which were explored extensively through double-sided counters in core concept 2.1 of the Key Stage 3 PD materials. This structure lays the foundation for adding and subtracting equations using formal algebra.

Attention can be drawn to the links between this example and the *Example 8* from the previous exemplified key idea. In this example, one unknown has been eliminated by adding expressions that have equal but opposite coefficients.

Students might need to be directed to think about the expressions, and how they are combined, linking $3x + y$, $x - y$, 19 and 3 explicitly. At what stage do you plan to intervene to ensure that these connections have been made? Is there value in giving students time for ‘productive struggle’ first?

Example 2

Ella finds two very old till receipts from a café.

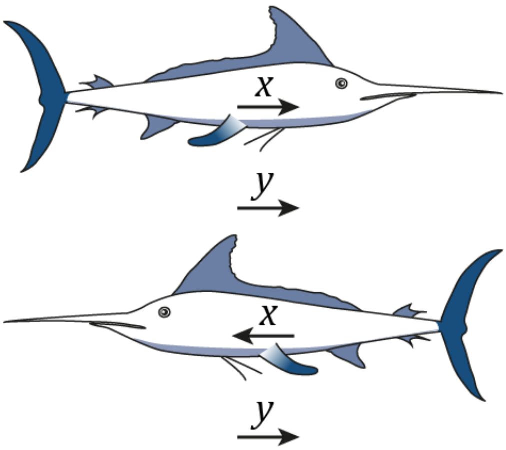
```
2 teas
1 coffee
Total:  71p
```

```
1 tea
1 coffee
Total:  49p
```

She says, 'Wow! A cup of tea was 22 p!'

a) Is Ella correct? How do you think she worked this out?

When a swordfish swims as hard as it can, it can move at 70 km/h with the current and 58 km/h against the current.



b) How could this situation be represented mathematically?

c) Find the speed of the swordfish and the speed of the current.

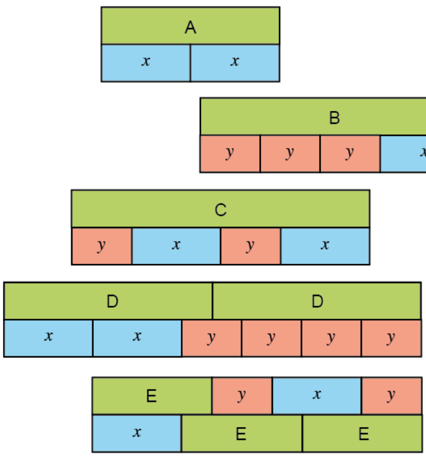
Real-life contexts are used in *Example 2* for **deepening** students' understanding of the structures behind the process of eliminating variables to solving simultaneous equations. The comparison of information, particularly in the café receipt context, may be instinctive. Students may not realise that they have, in effect, solved simultaneous equations by subtracting one entire equation from another.

In the context of the swimming swordfish, students are effectively adding one equation to another. They may need support in recognising that the current is added when the fish swims with the current and subtracted when it swims against the current. Students might be encouraged to try different **representations** until the context makes sense, before exploring the algebraic representation of the situation.

What are some common representations or contexts that you use to teach simultaneous equations? Where do particular representations have advantages over others?

Example 3:

Look at the six bar models for A-F below:



This is a continuation of *Example 5* from exemplified key idea 9.1.3.2 above. Students will have found there that the lengths of the green bars are:

A: $2x$

B: $x + 3y$

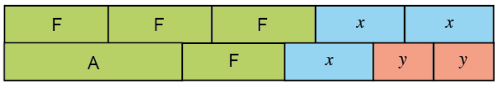
C: $2x + 2y$

D: $x + 2y$

E: $2y$

F: $\frac{1}{2}x + y$

Example 3 uses both algebraic and bar model **representations** to give context to working with the given lengths to make deductions. Students should work intuitively in the first instance, maybe by comparing the two bar models, for example, in part b. This informal discussion



a) How long is A if x is 5 cm?

b) How long is A if x is 3.5 cm?

What other lengths can you work out if you know:

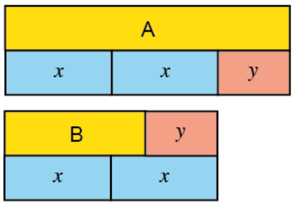
c) That A is 6 cm and C is 8 cm long?

d) That E is 10 cm and F is 9 cm long?

can then be formalised and represented using mathematical notation. The focus is for students to understand that calculating the value of x or y in one situation allows them to find other values in other situations.

Example 4:

Look at the bar model below.

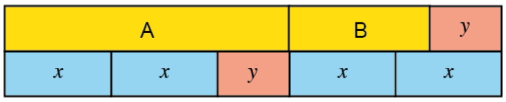


Explain how the bar model shows that the length of:

a) A is $2x + y$

b) B is $2x - y$.

The bar model below shows A and B pushed together.



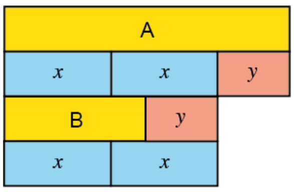
Explain how the:

c) Bar model shows that $A + B$ is $4x$.

d) Equation $(2x + y) + (2x - y) = 4x$ shows that $A + B$ is $4x$.

e) Equation $(2x + y) - (2x - y) = 2y$ shows that $A - B$ is $2y$.

The bar model below shows the difference between the length of A and B.

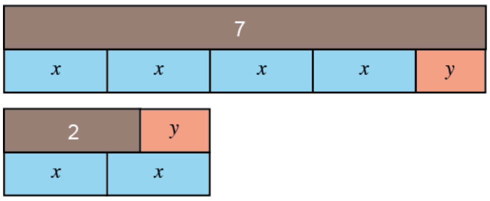
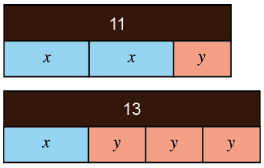


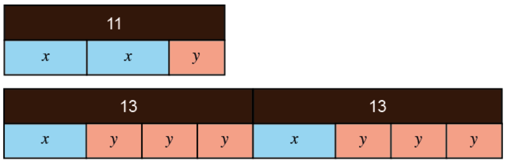
f) Explain how the bar model shows that $A - B$ is $2y$.

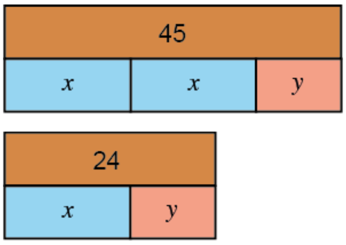
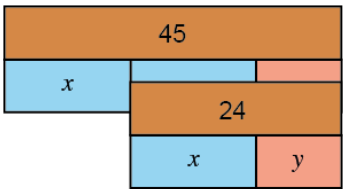
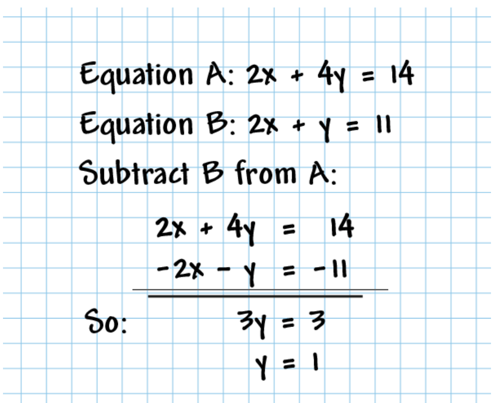
Examples 4 to *6* offer a set of tasks and **representations** exploring the different manipulations needed to solve a pair of linear simultaneous equations. Only one example is given, and each set of prompts follow a similar pattern. The first prompt in each case aims to connect the symbolic representation with the bar model, giving students an opportunity to work between these two given images.

In *Example 4*, the prompts and values allow students to note that equations can be combined additively, either by adding the pair of equations, or by subtracting one from the other. At this point, none of the lengths of the bars is fixed and so no further manipulation is needed. It is intended to make explicit some of the manipulations that may have been carried out informally in *Example 3*, further **deepening** students' understanding of the mathematical structures involved.

While each of these examples use a bar model as a representation to enable access to the concepts, it is not intended that this should be the only approach used within a sequence of teaching. What other approaches might complement this? Are there any approaches that should be avoided, to prevent confusion?

Example 5: *Look at the bar model below.*  *Explain how:* a) *The bar model shows that* $4x + y = 7$ *and* $2x - y = 2$. b) *The two bar models can be combined to show that* $6x = 9$. c) *It is now possible to find out the lengths of* x *and* y.	*Example 5* begins by connecting the symbolic **representation** with the bar model, giving students an opportunity to visualise and make sense of this abstract situation. This task makes explicit that two equations can be added to make a third, and equally valid, equation. The values and equations in this example are chosen so that adding the expressions will eliminate one of the variables. As you introduce different equations and methods, consider which **variation** to use and when, so that students meet situations where adding does not eliminate variables at the right time in their learning. Reflect on whether your students will already be familiar with the bar model, remembering that they a likely to have experienced it at primary school, even if it is not explicitly taught in your secondary curriculum. If your students are not, is it worth spending time at this point in the curriculum to introduce it? What might be the challenges and barriers? Discuss with your team how you might approach introducing bar models in Key Stage 4, and what might be different about this timing, rather than earlier in Key Stage 3.
Example 6: *Look at the bar model below.*  a) *Explain how:* (i) *The bar models show that* $2x + y = 11$ *and* $x + 3y = 13$. (ii) *The two bar models can be combined to show that* $(2x + y) + (x + 3y) = 24$. (iii) *The answer to part (ii) also shows that* $3x + 4y = 24$. (iv) *The two bar models can be combined to show that* $(2x + y) - (x + 3y) = -2$. (v) *The answer to part (iv) also shows that* $x - 2y = -2$. (vi) *Neither* $3x + 4y = 24$ *or* $x - 2y = -2$ *help to find the value of* x *or* y.	*Example 6* begins by connecting the symbolic **representation** with the bar model, giving students an opportunity to work between these two given images. The task is intended to make explicit to students that two equations can be added or subtracted to make a third, equally valid equation. The **variation** is designed to draw students' attention to situations where manipulation is required before variables can be eliminated. In part a of *Example 6*, students will find that, although the actions create other valid equations, neither adding or subtracting the two equations eliminates a variable, so they are no closer to finding a value of x or y. This is a key learning point and time should be taken to ensure that it is understood fully. Part b of this example extends one of the bar models so that the coefficients of the x are equal and so makes the equations solvable by subtracting one from the other. In this example both adding and subtracting the equations has been considered. While students may notice that different operations are successful in eliminating a variable in different situations, the choice of operation is not explicitly addressed here. The intention, instead, is to focus on **deepening** students' understanding of the algebraic manipulation that comes before elimination.

Look at the bar model below.  *b) Explain how:* *(i) The bar model shows that* $2x + y = 11$ *and* $2x + 6y = 26$. *(ii) The two bar models can be combined to show that* $(2x + y) + (2x + 6y) = 37$. *(iii) The answer to part (ii) also shows that* $4x + 7y = 37$. *(iv) The two bar models can be combined to show that* $(2x + y) - (2x + 6y) = -15$. *(v) The answer to part (iv) also shows that* $-5y = -15$. *c) Use your answers to either part ii or part v to find the length of* x *and of* y. *Which answer helped? Explain why.*	
Understand how substituting one expression for another can be used to solve simultaneous equations *Example 7* *In an arcade, a games machine takes either a token or a £2 coin for 1 game.* *On Monday, the machine indicates that 30 games have been played.* *a) If the machine collected £52, how many tokens were used?* *The arcade also has a change machine. This takes either £5 notes or £2 coins to convert to £1 coins. On Tuesday, it converted money 45 times in total.* *Ryu writes* $y = 45 - x$. *b) What could* x *and* y *represent?* *This machine has taken a total of £135.* *Ken writes* $5x + 2y = 135$. *c) Is* x *or* y *the number of £5 notes?* *Ryu says, "you can use my equation and yours together to find out how many coins were used."*	In *Example 7*, students should be encouraged to write equations as **representations** of the information presented. In the context of the games machine, information is given about coins only, and this information can then be used to ascertain how many tokens are used. The intention here is to simplify the process. Later in the example, it is necessary to manipulate equations in order to get to a position of knowing one of the values. In the context of the change machine, the numbers chosen and the way the equation is presented means that it is less conducive to using an 'elimination' method. Part d is the point which students may find most challenging. Students are likely to be comfortable with substituting a variable for a value but may need support to be sure that substituting an expression is also a valid 'move'. In everyday **language**, 'substitution' is used when something is replaced by something similar, but probably of lesser quality (for example, substitutions in grocery shopping, sport or cooking), so it is important to emphasise that the 'substitution' being made here is for something of identical value. This maintains the equality while allowing the equation to be rewritten in a single variable.

d) *How could you combine Ryu and Ken's equations?* e) *What are the total number of coins and notes used?*	
Example 8: *Look at the bar model below.*  a) *Explain how the bar model shows that $2x + y = 45$ and $x + y = 24$.* *Below, the same bar model is aligned differently.*  b) *Explain how the bar model now shows that $x + 24 = 45$.* c) *Use your answer to 2 to find the length of x and of y.*	*Example 8* begins by connecting the symbolic **representation** with the bar model, giving students an opportunity to work between these two given images. While *Examples 4* to *6* model solving simultaneous equations by elimination, *Example 8* builds on *Example 7* to explore substitution as an alternative strategy. The intention is to model that these strategies are possible, **deepening** students' understanding of the structures that enable them. This example does not go so far as to support students in making strategic choices about which method is most effective and efficient in a given situation. Practice in method selection will also be necessary, and some suggestions for this are offered in the next two examples.
Understand that elimination and substitution are equally valid methods for solving simultaneous equations *Example 9:* *Eli and Sarah are solving equations.* $2x + 4y = 14$ $2x + y = 11$ *Eli writes:* 	In *Example 9*, students are offered two different sets of workings to continue, so that their focus is on **deepening** understanding of the validity of both methods. It is important that students know both elimination and substitution will lead to the same solution. Using the same example for both methods acts as a form of **variation** as the differences between the two approaches are made more obvious against the background of the same pair of equations. Comparing and contrasting should enable students to appreciate that, while both approaches lead to the solution, there are cases where one approach is more efficient than the other.

Sarah writes: $y = 11 - 2x$ So: $2x + 44 - 8x = 14$ $-6x = -30$ $x = 5$ *a) Use Eli's value for y to find x.* *b) Use Sarah's value for x to find y.* *c) What do you notice?* *d) When do you think Eli's method will be easier than Sarah's? When will Sarah's method have the advantage?*	
Example 10: *Sarah and Eli are going to solve the following sets of equations.* ***Without solving,** decide for each pair:* *(i) whether elimination or substitution would be more efficient* *(ii) and why this is the case.* *a)* $2x + 3y = 167$ *and* $2x + y = 107$ *b)* $y = x + 3$ *and* $2x + 5y = 75$ *c)* $x + y = 5$ *and* $5x = y + 4$ *d)* $2p + q = 13$ *and* $p - 4 = -4$	Using *Example 10*, students should be given opportunities to evaluate both substitution and elimination so that they can determine which approach will be more efficient for a given example, and why. Encourage students to inspect the equations without solving them to make their decisions. Ask if they have enough information, or if they need to write the first step of the solution to help understand which is going to be the most appropriate method. The intention is to support students in coming to their own conclusions about the properties that make one or other process more efficient, rather than learning an algorithm for decision making, thus **deepening** their understanding. For some of these pairs of equations, such as in part c, there might not be a clear method that stands out as most efficient and the choice might be the personal preference of the solver. Teachers may want to develop their own sets of questions like these. There are many examples of simultaneous equations questions to be found, and it can be an interesting exercise to look for edge cases in order to explore the key features that determine the choice of method. How might this then inform your teaching of simultaneous equations?

9.1.3.5 Represent and interpret the solution to linear simultaneous equations

Common difficulties and misconceptions

Linear equations with the unknown on both sides of the equals sign were covered in the Key Stage 3 PD materials ('*2.2.1.3 Understand that a solution is a value that makes the two sides of an equation balance*'). One interpretation of the solution to a linear equation was the value of x that makes two expressions have the same value. For example, the solution to $3x + 5 = x - 1$ is the value of x when the line $y = 3x + 5$ and $y = x - 1$ intersect. The two expressions were considered dynamically by plotting values in a table and by representing these as pairs of values as graphs. In Key Stage 4, students come to recognise that these are simultaneous equations and develop algebraic techniques for solving them. It is important that students are reminded of these key underpinning concepts, so that they are building a conceptual understanding rather than just replicating process.

Central to later understanding of methods for solving simultaneous equations is the idea that, once one unknown has been found, the other one can also be found. At the heart of this is the understanding that:

- A solution (an *x*-value and a *y*-value) must satisfy **both** equations.
- The coordinate which represents the solution must lie on **both** straight lines.

Students can become immersed in applying the techniques – calculating values, plotting points, drawing graphs and manipulating algebraic expressions. They can then lose sight of these two basic principles and that the aim is to end up with a solution which satisfies both equations **simultaneously**. Activities and questions which draw attention to the 'simultaneous' aspect of this work will help students to understand this and prevent techniques from becoming meaningless algorithms.

Students need to

Understand that a solution to a pair of simultaneous equations must satisfy both at the same time

Example 1:

Gaelen and Jennika are playing a game on a hundred grid. They each spin a 0 to 10 spinner and then roll a normal die.

- *The spinner determines which square they will start on.*
- *The die determines how many squares they move each turn.*

Gaelen starts on square 1 and moves forward 3 squares per turn. Jennika starts on square 5 and moves forward 1 square per turn.

a) *Will they ever be on the same square at the same time?*

b) *What would change if Jennika started on square 7 and moved forward 2 squares per turn?*

Guidance, discussion points and prompts

Examples 1 and *2* are not directly related to the method of solving pairs of simultaneous equations. Instead, they work on **deepening** students' understanding of the ideas that underpin simultaneous equations, encouraging them to reason and visualise with functions. This should support later work on manipulating and finding a solution to a given problem.

Using a tabular **representation** for these values gives an insight into the change as the sequences increase, for example:

	G	J
Starting square	1	
First number	4	6
Second number	7	7
Third number	10	8
…	…	…

Ask students to consider how the starting number and the increment affect the behaviour of the resulting sequence of numbers.

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Example 2:

Chloe is listing solutions to $x = 2y$.

She writes:

x = 1, y = 0.5
x = 2, y = 1
… …

Jamie is listing solutions to $x + y = 30$.

He writes:

x = 1, y = 29
x = 2, y = 28
… …

Do any sets of solutions appear in both lists?

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In *Example 2*, we are **deepening** students' understanding of what a solution is, and how many solutions are in a solution set in different circumstances. This example is intended to highlight the idea that an individual linear equation might have multiple solutions, but that solutions for simultaneous equations need to be true for both equations. Teachers may wish to explore different equations that exemplify the case where there are no solutions, or multiple solutions to two equations.

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Example 3:

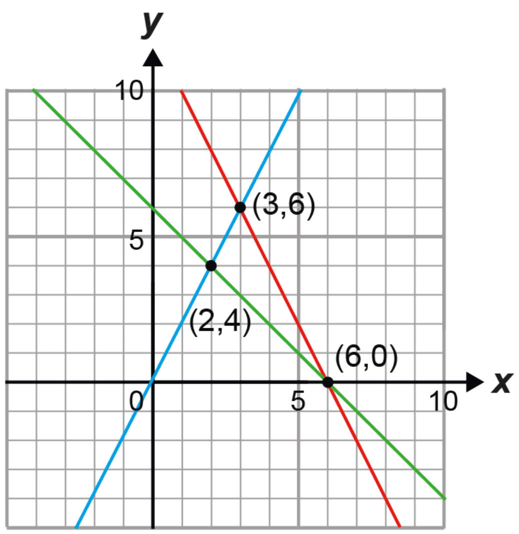
Here are three equations:

- $y + 2x = 12$
- $y - 2x = 0$
- $x + y = 6$

a) *Choose a pair of equations and solve them together.*

b) *Can you find a solution that works for all three equations simultaneously?*

Below is a graph showing all three equations on the same axes:



c) *Use this graph to explain your answer to part b.*

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Example 3 gives students further experience of treating equations as objects, as well as making the link between the equations and their graphical **representations**

This example offers opportunities for **deepening** students' understanding of what it means for a pair of values to be a solution for a pair of equations. You may find it helpful to ask questions such as:

- *'Can you find another set of three equations that has positive, integer solutions for each pair of equations?'*
- *'What if negative solutions are allowed?'*
- *'What if non-integer solutions are allowed?'*
- *'Can you find a set of three equations where there is a single pair of values that is a solution for all three?'*

Contrasting the last question with the previous ones will help to highlight that a solution to a pair of equations does not automatically mean that this will be the solution for an additional equation.

Try to come up with similar sets of equations that highlight different aspects of simultaneous equations. It can be a useful exercise to explore systems of equations and the features that determine the solution sets.

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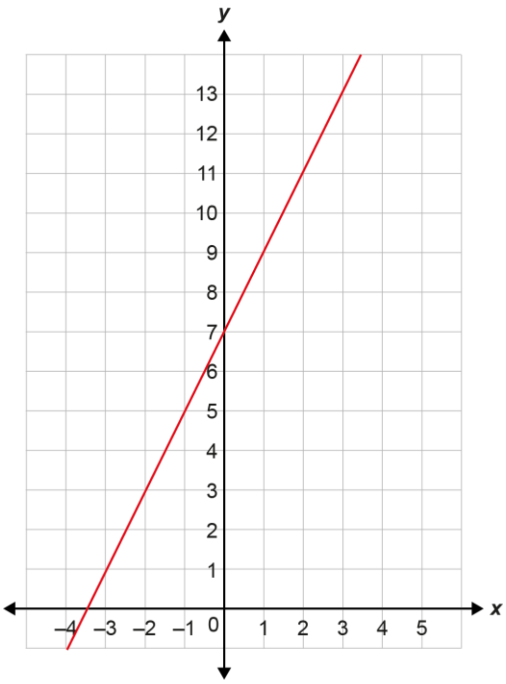
Example 4: *Look at the graph below.*  a) *How does it relate to Example 1?* b) *On the same axes, draw graphs that relate to the other scenarios in Example 1.* c) *How do your graphs help to explain your answers from Example 1?*	The equation $y = 2x + 7$ was the algebraic **representation** of the relationship in part b of *Example 1*. In *Example 4*, it is presented graphically so that students can explore the similarities and differences between counting, which is discrete, and the continuous nature of the function. It is important that students understand this continuity, so they can interpret the single point of intersection between the two lines when working with simultaneous linear equations drawn on a Cartesian graph. Connect this representation to the **language** in the description in *Example 1* for further exploration. The intention is to draw out the understanding that *Example 1* offers a subset (the integer values in the first quadrant of the graph) rather than the whole set of values for this function. Consider which of the terms in this paragraph are necessary to help students fully comprehend this idea.

Interpret graphically whether a pair of simultaneous linear equations will have one, none or an infinite number of solutions

Example 5:

The graph below shows two lines.

One line represents the equation $y = 17x + 30$.

a) *Is this equation for line A or line B? How do you know?*

b) *What might be the equation of the other line?*

Marijke is told that the equation of line B is $y = 18x + 2$. *She says, 'The two lines look parallel, so there's no way that I can find the point of intersection as they'll never meet.'*

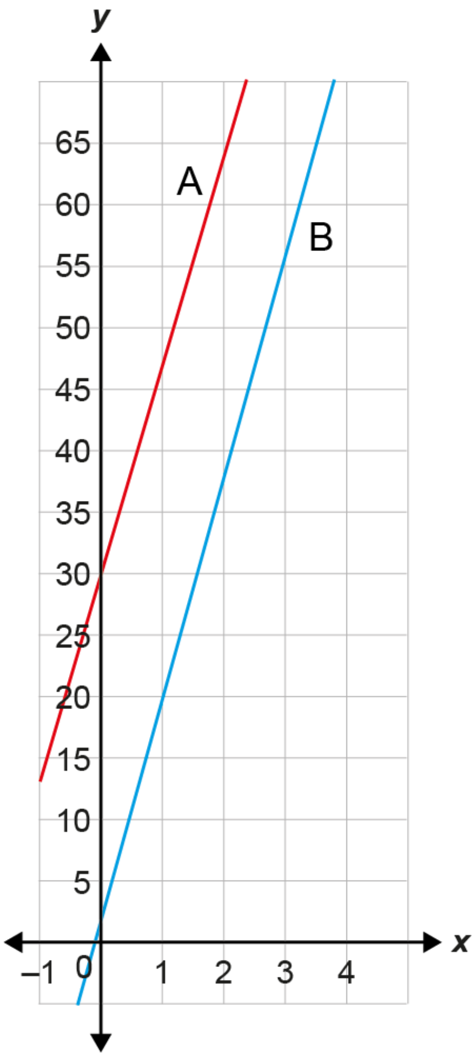
c) *Explain why Marijke is wrong.*

d) *Where do you think the lines will meet? Will it be further up or down the axes?*

e) *Write the equation of a line that is parallel to* $y = 17x + 30$ *and so will not have an intersection.*

f) *What happens when you try to solve the simultaneous equations algebraically?*

Example 5 offers a context in which students can understand why a pair of simultaneous equations might not have a solution (because the lines that represent them are parallel and so have no intersection) and use the given equations to identify this situation. In working between the two **representations** (the symbolic and the algebraic), students can choose which representation is best used to explain each part of the question. Teachers should discuss these choices and connections to make them explicit for the students.



Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 9 Sequences, functions and graphs* can be found here:

https://www.ncetm.org.uk/media/23eejt3r/ncetm_ks4_cc_9_solutions.pdf