National Research and Development Centre
for adult literacy and numeracy

Number

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Maths4Life

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For further details see ncetm.org.uk and maths4life.org. The Maths4Life website will be live and maintained until the end of March 2008 when it will transfer to ncetm.org.uk

Contents

Approaches to learning about number

This booklet is aimed at teachers working with learners from about entry level 2 to level 1. We suggest some approaches that we have found effective in enabling learners to work on developing an understanding of number, and of the relationships between operations. These are not new ideas, but rather an attempt to collate some strategies that we have seen used effectively. The ideas can be used with other teachers or alone, in order to plan lessons and to challenge your beliefs about mathematics teaching and learning.

The key emphasis here is that learners should be encouraged to develop a 'relational' understanding of number rather than simply an 'instrumental' understanding ("Relational Understanding and Instrumental Understanding", Richard Skemp, 1976, in *Mathematics Teaching* issue 77). That is, an understanding of the properties of and the relationships between numbers and between operations, so that in solving new problems they can draw on this and also on their own everyday life strategies.

An example of 'instrumental' understanding might be: learning a process off by heart, such as 'area = length x width', whereas understanding why length is multiplied by width to find area would be 'relational' understanding. Just being able to substitute numbers into the formula A = L x W does not necessarily mean understanding what 'area' is.

The aim of our approaches is to support learners in developing an intuitive 'feel' for numbers, in finding strategies to use what they already know to work out what they do not know, and in being able to determine whether or not an answer is reasonable. The emphasis is on enabling learners to see that what they learn about number is part of a meaningful and connected whole, rather than an arbitrary collection of unrelated facts, methods and rules.

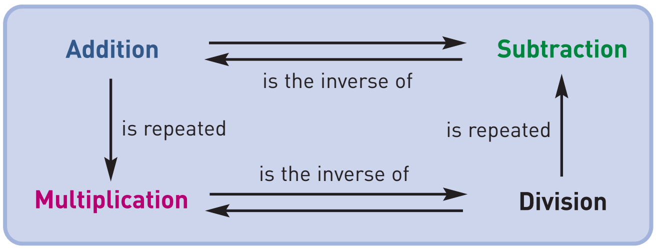
The activities outlined here lend themselves to collaborative group work because they require learners to explore ideas, to make conjectures, to reason, to look for patterns, and to compare and discuss their methods with each other. Asking a learner simply to "work out sums" can be limiting. Instead, working together to investigate numbers, to explore the mathematics in a real life context or to thrash out a problem in a vocational context may allow learners to develop a flexibility with numbers that will enable them to solve similar mathematical problems in a range of different contexts.

Many learners may feel that listening to the teacher and completing their own worksheets individually is the main way of learning. However, we believe that learners learn more if they actually enjoy the activity, have a chance to discuss what they do, explain their work and reach a shared understanding. There is now widespread support for the value of collaborative work in developing concept understanding (for example, *Collaborative Learning in Mathematics: A Challenge to Our Beliefs and Practices*, Malcolm Swan, NRDC and NIACE, 2006).

Learning is most effective if the task benefits from learners working together. The task itself could, for example, be pitched a bit higher, just outside an individual learner's comfort zone, so that it needs a second opinion; it may also involve practical equipment that needs a second pair of hands. An explanation of the benefits and ground rules is important for all learners before starting collaborative tasks, so that each group member gets a chance to express an opinion and challenge what others say.

Linking the four rules

Although being able to juggle the four rules is clearly central to developing agility with number, being numerate involves more than simply being able to add or divide in isolation. Learners need an understanding of the ways in which operations are connected, and need to be able to make sensible judgements about how to use them.



At entry level learners may need to spend time exploring addition and subtraction strategies. The activity section in this booklet provides some ideas for work with whole numbers.

Clearly, knowing multiplication facts can make life more comfortable. However, simply being able to recall them in parrot fashion is different from being able to understand and use the relationships expressed in these facts, and being able to apply this knowledge in a range of different contexts.

If learners are encouraged to think about the properties of numbers, and to look for connections and patterns, then multiplication tables no longer appear as a random assortment of numbers.

Learning multiplication tables can be approached in many different ways; some will suit some people better than others. It is good practice to encourage learners to explore a range of these strategies; for example, considering aural and visual patterns, using finger methods, comparing multiples of different numbers, using digit sums, partitioning numbers. Examples of some of these strategies are shown below.

Essentially, learners can learn to use the facts that they already know as a springboard to find out the facts that they do not know.

If a learner doesn't know 8 x 6, work it out by finding 2 x 6, double the answer to find 4 x 6, then double again to find 8 x 6.

Or work out 5 x 8 ($^1/_2$ of 10 x 8), then add another 8 to find 6 x 8.

As well as being able to access multiplication facts, learners also need to be able to identify a range of contexts where they might use these facts. Some may know that 6 x 7 = 42, but may not always realise how this fact is relevant in solving practical problems such as:

- → how many days there are in 6 weeks
- → how many centimetre squares are needed to cover a 7cm x 6cm rectangle
- → the price of half a dozen pencils @ 7p each
- → sharing equally a bill of £42.00 between 6 people
- → the total number of cakes in 3 packets with 14 cakes in each

Many people find division difficult and confusing, and may tend to get stuck with half learned algorithms. It is important that division is not taught a 'separate' topic from multiplication; we need to emphasis that each of these operations is the inverse of the other.

The wording of division questions often makes them seem more difficult: they are often written in the passive e.g. '15 biscuits are shared between 5 children', whereas people might be more likely to say '5 children share 15 biscuits '. This can cause some confusion about which number is being divided by which. A further confusion is that in some contexts learners might assume that the question 'how many' implies multiplication, so that this sort of problem seems puzzling: 'How many 10cm^2 tiles will it take to fill a space measuring 300cm by 10cm?' Many may tend to calculate 300 x 10 rather than 300 ÷ 10. They may be more used to having the parts and finding the whole, rather than working from the whole to find the parts.

Making connections

For some people number facts are simply a random assortment of "things"; it is important to show learners that the world of numbers is an interconnected web full of interesting relationship and patterns. The more a learner can see of these connections, the more at home they will feel with number, and the less they will have to rely on memory.

Work on the four rules needs to be fully integrated into other mathematics topics, namely shape, measure and handling data, rather than taught as a series of dreary sums outside of any meaningful context. Here are some examples:

Multiplying and dividing by powers of ten :

- converting pence to pounds and vice versa
- using metric units
- converting decimal numbers to percentages

Making decisions about which operations to use:

- comparing costs or measurements
- comparing 'best buys'
- using ratio and proportion
- doing scale drawings
- working on perimeter, area and volume
- using averages
- calculating probability

Interesting number work can also be developed as part of mathematical investigations of number properties and patterns.

Estimation must be a normal feature of all number work. Approximation techniques such as rounding to the nearest ten or hundred are often taught as a separate topic, so that instead of incorporating these concepts into thinking about everyday calculations, many learners see 'rounding' as something they have to do only when faced with a question that specifically mentions it.

General questions about numbers

Learners themselves often raise very interesting questions about number. Using these as a basis for discussion can generate useful opportunities for learners to reflect on their own thinking and can also help to bring out misconceptions into the open.

Here is a sample of questions we have collected. You could think about which misconceptions they might expose or what thinking processes will be revealed.

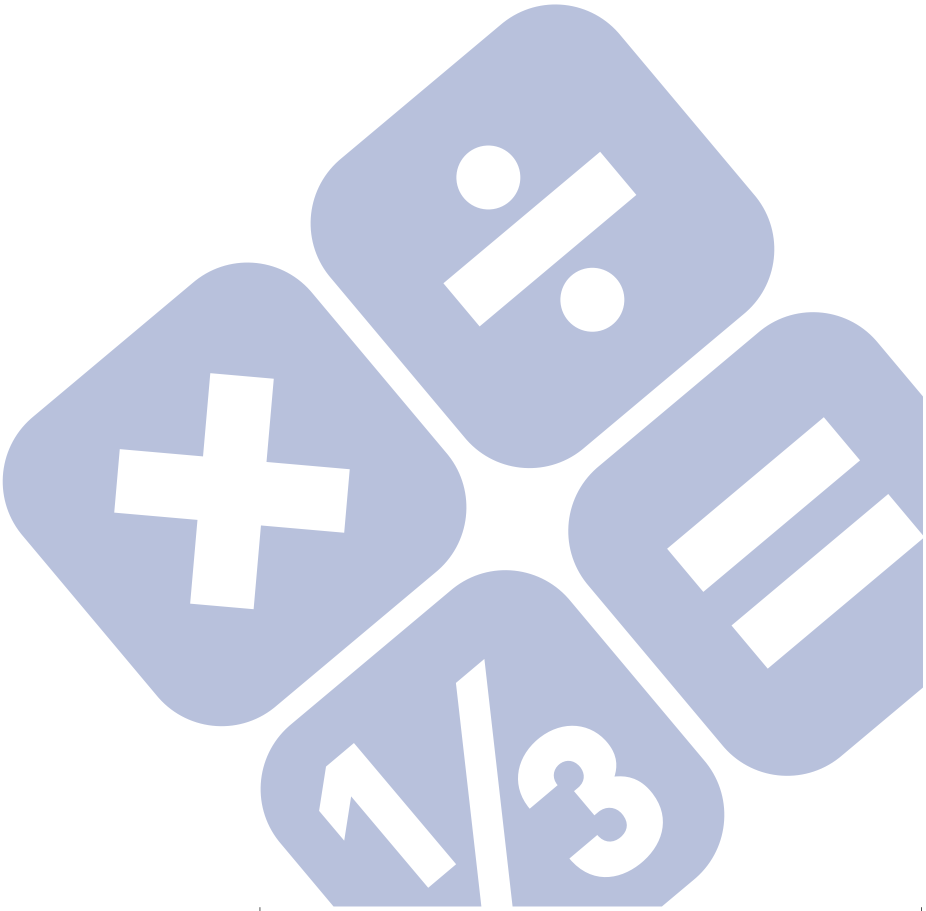
- Are there more odd numbers than even numbers?
- Are there some numbers you just cannot divide?
- Is it possible to find half of any number?
- What is the smallest number?
- If you go below zero is that the same as a fraction?
- Is it true that zeros do not really matter?
- Do calculators make you lazy?
- How can you have more than one correct answer?
- Is it best to learn number facts off by heart?
- How can you subtract a smaller number from a larger number?
- How can you calculate if you do not know your tables?
- Is there any connection between multiplying and dividing?

- If you round numbers up to make things easier, won't you get the wrong answer?
- What is the correct method for doing take-aways?
- Why is division more difficult than multiplying?
- What's the rule for doing division?
- What is the correct way to set out your working out?
- How come my children are learning completely different methods to the way I was taught at school?
- Can you divide a smaller number by a larger one?
- When you multiply two numbers doesn't the answer always gets bigger?

Developing alternative strategies

Many people have their own informal strategies for doing mental calculations, though they may feel these are not 'correct' methods. It is important to encourage learners to explore these and to see the validity of any methods that are effective.

There is a range of useful mental mathematics strategies that some adult learners working at entry level may not have previously encountered. Those who are parents are likely to find that their children are learning these in school. We should teach these strategies explicitly. We should encourage learners to discuss and compare their own strategies with each other, and to choose which ones suit them best. They can also explore whether different methods suit different numbers and different contexts.



If learners are supported in developing a mental map of concepts, they will be able to see connections and use these to solve problems.

Some of these strategies include:

Rearranging the order of numbers for addition – starting with the larger number first instead of 7 + 18 it is easier to put 18 first and then add 7. Also when adding several numbers together it is useful to group those making up tens

27 + 34 + 13 + 56 + 78 + 22

'PARTITIONING' - splitting numbers into parts which may suit our purpose better, usually into tens and units, or hundreds, tens and units

17 = 10 + 5 + 2 or 346 = 300 + 40 + 6

'BRIDGING TENS OR HUNDREDS' –jumping up to and across tens or hundreds by using partitioning, because tens are 'comfortable' numbers to land on and jump from

18 + 7 = 18 + 2 + 5
(7 is partitioned into 2 + 5 because 2 brings 18 up to 20)

USING 'NEAR DOUBLES' – adding numbers that are close to each other, so that one of them can be doubled and the answer adjusted by adding or subtracting

49 + 50 = 50 x 2 -1

'COMPENSATING' – adjusting one or more numbers to make the calculation easier and then adding or subtracting the difference

37 + 49 = 37 + 50 -1
(adjust 49 up o 50 and then subtract 1)

42 – 19 = 42 – 20 + 1
(adjust 19 up to 20 and then add 1)

SUBTRACTION BY 'COUNTING ON' – counting forwards to find the difference between one number and the other

143 – 67
Start on **67**
Add on: **3** (to reach **70**) **+30** (to reach **100**) **+ 40** (to reach **140**) **+ 3** (to reach **143**)

ADDING TENS (or hundreds) before adding units

55 + 27 = 50 + 20 + 5 + 7
or
346 + 239 = 300 + 200 + 40 + 30 + 6 + 9

LEARNING HOW TO MULTIPLY (AND DIVIDE) BY 10 AND 100

It is critical that learners understand that the effect of multiplying a digit by 10 is to move that digit one place to the left, e.g. from the units to the tens, or from the tens to the hundreds; likewise that the effect of dividing by 10 involves the inverse

'GRID' METHOD FOR MULTIPLICATION see page 17

USING KNOWLEDGE OF DOUBLES FOR LONG MULTIPLICATION see page 16

DIVISION BY 'CHUNKING'– means understanding division as repeated subtraction, and taking away 'chunks' from the number we are dividing

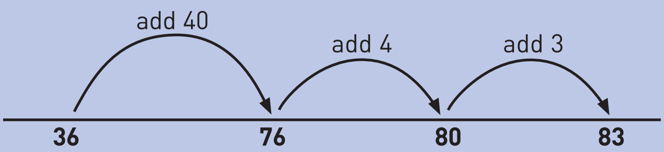
84 ÷ 6 could be worked out by taking away groups of 6 from 84, first taking away 10 groups of 6 (60) then another 4 groups of 6 (24) so that in total 14 groups of 6 are taken away.

Empty number lines

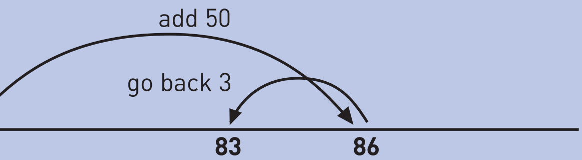
Once learners are familiar with a numbered number line, using empty number lines may help them to develop an intuitive understanding of number and to think flexibly about number operations. An empty number line has no marked intervals and does not need to be drawn neatly with a ruler; it can be quickly sketched. Learners write on it only the numbers they need. Attention is focused on writing numbers in order, rather than to scale.

The empty number line invites learners to show their thinking. It helps them to model number sequences, and think about partitioning numbers, bridging tens and hundreds, and counting forwards and backwards.

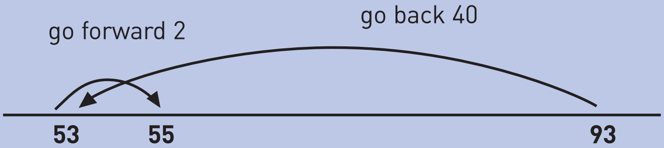
36 + 47 = 83 could be set out on an empty number line like this:



or this:



93 – 38 = 55



The number line can be a powerful tool which encourages learners to see and use relationships between numbers and between operations, e.g. between addition and subtraction, or between decimal numbers, and to see that there are several different ways of working out answers. It serves as a kind of sketchpad, a useful mental model for thinking about numbers and strategies for combining them in various ways Learners can share their thinking with each other and compare strategies to decide which might be most efficient in particular contexts.

See also Session 5 of the Thinking Through Mathematics: strategies for teaching and learning ringbinder pack for whole group activity on 'washing line' (available through www.maths4life.org)

Addition

How might you work out 9 + 26?

Traditional method

$$\begin{array}{r} 2\;6 \\ +\;\;\;9 \\ \hline 3\;{}_{1}5 \\ \hline \end{array}$$

Use a number line
(i.e. 26 + 4 + 5)

Possible mental strategies

Count on
Put the larger number first and count on in your head:
26 + 9
Use near numbers and adjust – compensation method
Add on 10 and compensate
26 + 10 - 1

How might you calculate 197 + 202?

Using near doubles
200 x 2
Then adjust
-3 + 2

Some alternative methods for 189 + 348

A combination of partition and compensating

190 + 300 = 490
490 + 50 = 540
540 - 1 = 539
539 - 2 = 537

Use a number line
i.e. 189 + 1 + 300 + 50 - 3

Partition the numbers and recombine

100 + 300 = 400
80 + 40 = 120
9 + 8 = 17
400 + 120 + 17 = 537

Use a combination of count on and compensation
Count on in hundreds:
348 + 100
+90
-1
Partition the numbers and recombine:
300 + 100
80 + 40
9 + 8 (7 + 1)

Subtraction

How might you calculate 54 – 25?

Traditional methods
decomposition ('borrowing')

$$\begin{array}{r} {}^{4}\cancel{5}\ {}^{1}4 \\ -\ 2\ \ 5 \\ \hline 2\ \ 9 \end{array}$$

$$\begin{array}{r} 5\ {}^{1}4 \\ -\ {}_{1}2\ \ 5 \\ \hline 2\ \ 9 \end{array}$$

compensation
('paying back')

Possible mental strategies

Use a number line
i.e. 54 - 20 - 4 -1

Count up:
25 + 25 = 50 (+25)
50 + 4 = 54 (+4)
Answer = 29

Count backwards from the higher number
54 - 4 = 50
50 - 20 = 30
30 - 1 = 29

Use a number line
i.e. 25 + 25 + 4

Some alternative methods for 709 – 243

Count on from the smaller number:

in 100s:	243	→	643	(+400)
in 50s	643	→	693	(+50)
in 10s:	693	→	703	(+10)
in units:	703	→	709	(+6)

Partition numbers:
700 - 200 = 500
500 - 40 = 460

460 + 9 - 3
Answer = 466

Use a number line
i.e. 243 + 7 + 50 + 400 + 9

NOTE: whenever demonstrating these strategies, it's important to make sure that we are careful to keep notation accurate. We need to make sure that learners calculating, say, 17+14 don't write:
17 + 10 = 27 + 3 = 30 + 1 = 31

Use near numbers and adjust – compensation method:
700 – 250 = 450
We only started with 700 so compensate the 9
450 + 9 = 459
And we took off too much so we need to compensate the 7
(459 + 7 = 466)

Use a number line
i.e. 709-9-200-40-3+9

Division

How might you work out 945 ÷ 8?

Layout used in many countries

```
           9 4 5 | 8
1 x 8 →   -8     |  1
          -1 4
1 x 8 →      8      1
             6 5
8 x 8 →      6 4    8
           1 1 8 r1
```

Chunking

```
             8)945
100 x 8 →   - 800
              145
 10 x 8 →   -  80
            -  65
  8 x 8 →      64 r1
              118 r1
```

Traditional layout

```
           1 1 8 r1
        8)9 4 5
1 x 8 → -8
          1 4
1 x 8 →    -8
            6 5
8 x 8 →    -6 4
```

Could also be written
118 1/8

Alternative 'chunking' method
using knowledge of doubles

8 x 2 = 16　　8 x 16 = 128
8 x 4 = 32　　8 x 32 = 256
8 x 8 = 64　　8 x 64 = 512

```
                945
8 x 64   →    - 512
                433
8 x 32   →    - 256
                177
8 x 16   →    - 128
                 49
8 x 4    →     - 32
                 17
8 x 2    →     - 16
remainder          1
```

64 + 32 + 16 + 4 + 2 = 118

Multiplication

How might you work out 27 x 46?

Traditional method

```
   4 6
x  2 7
 3 2 2
 9 2 0
1 2 4 2
```

The 2nd row could be done first

Partitioning numbers

10 x 46 = 460
10 x 46 = 460
5 x 46 = 230 (Add on ½ of 460)
2 x 46 = 92
1242

Changing numbers and adjusting

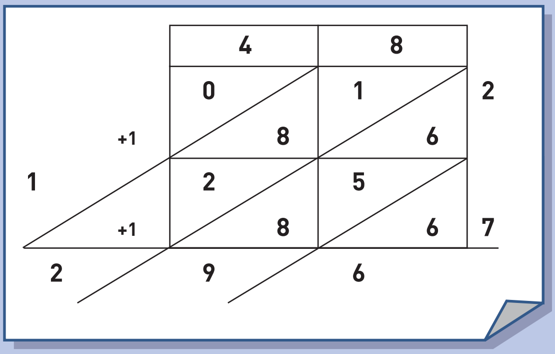
30 x 46 = 1380
3 x 46 = 138
1380 – 138 = 1242

How might you work out 27 x 48?

Grid method

x	40	8
20	800	160
7	280	56

960
336
1296



Thinking about the methods shown on pp 14-17, reflect on the advantages and disadvantages of each one. Which methods are meaningful? Why do they work? Is it helpful when place value is not retained? Try typing 'multiplication' or 'division' into YouTube to see more examples and debates on these methods.

Teaching points

- Remind learners always to make a 'guesstimate' before calculating
- Make sure learners think about whether answers are sensible or not
- Use realistic problems if you believe it will help understanding, or discrete problems if you think a context may not reveal the complexity of the mathematics
- Different methods suit different learners, as well as different contexts and numbers
- Develop activities to reinforce understanding that the order of operations in addition and multiplication is not important, whereas in subtraction and division it is critical.
- Learners should be able to recognise when there is a need to divide in a number of different contexts, using both words and symbols.
- It may take learners some time to know which way round to divide, especially if the context requires them to divide a smaller number by a larger one.
- Recognise that learners might want to use very different layouts for their work e.g. in different countries different methods for division are used
- Learners can select which methods are most appropriate for solving problems in different contexts e.g. using different layouts, number lines, factors and multiples, long division, fractions, chunking, grid multiplication, repeated subtraction etc.
- Give learners time to develop their own strategies.
- Learners need practice in developing mental agility
- Help learners develop lateral thinking by explicitly teaching problem solving strategies.
- Use learner errors and misconceptions in a constructive and imaginative way
- Effective teaching should encourage learners to think flexibly about number and to use their existing knowledge to create their own personalised solutions to problems.

The language of number

Mathematics is a language in itself and uses a very specialised vocabulary. Learners need to understand a range of symbols which express particular relationships, and they also need to be able to understand and use a specialised vocabulary. This includes some everyday words which have a very different meaning in mathematics (e.g. prime, even, odd, round, product, unit). Learning to understand and use mathematics vocabulary accurately is essential for all learners, and for ESOL learners in particular it may need a special focus.

Learners need to be familiar with a wide range of different words relating to number. However, synonyms can cause a lot of confusion, especially if taught out of context. For example, 'addition' does not always mean the same as 'increase', and 'multiplication' can refer to areas and Cartesian products, so the language of 'group' and 'share' may be unhelpful. As teachers we need to make our own judgements about which terms are appropriate and at what stage. Learners should be encouraged to talk about their thinking processes, and try and explain them in words and diagrams. Learners should try to see the meaning and structure of the operations, rather than simply learn terminology out of context.

We should however be aware that some of the language learners may use to describe what they are doing (e.g. 'borrow', 'pay back') actually clouds the picture, as it belongs to processes they have probably learned by rote without necessarily understanding. These terms may need careful unpicking, or we may use alternative terminology such as 'splitting 23 into 10 and 13' rather than 'borrowing ten'.

Activities

USING TARGET BOARDS

A target board is a grid with a set of numbers carefully chosen on to fit the purpose. It can be drawn on a whiteboard, OHP, flip chart or interactive whiteboard for use with the whole class, or printed out for small groups of learners. Learners can be asked a range of open and closed questions about the numbers.

Here is an example:

49	39	27	77
4	51	44	12
29	24	9	15
18	54	7	66
30	13	63	72

- Can you make 100 using any numbers on the board? And some more?
- Which pairs are closest in value?
- Can you find a prime number larger than 20?
- What is the biggest difference you can find between two of these numbers?
- Find the numbers which are divisible by 11
- Find the numbers which are divisible by 9
- How many numbers are in the 3 times table? Are any of these multiples of 6?
- Do any of the smaller numbers go into the larger numbers? How many times?
- Do any of the larger numbers go into the smaller ones? How many times. What is the connection between this question and the previous one?
- Can you think of a rule to get rid of half of the numbers on the board?
- Can you find three numbers which are linked in some way? How are they linked?
- Find numbers which differ by 5 or by 10 or by 15. What can you say about these number patterns?
- What is the largest sum you can make using three of the numbers?
- Make up your own target board and questions for another group of students
- Take each number in turn and try to think of a reason why it could be removed from the group, for example, 'I would remove 4 as it is the only even number less than 10'.

SORTING AND MATCHING

Learners can be shown how proportional change can help them to work out multiplication (and division) facts that they might otherwise have found difficult. Here is a simple example:

4 x **10** = **40**
Halve **4** and double **10**
2 x **20** = **40**
Halve **2** and double **20**
1 x **40** = **40**

This activity helps learners to see how they can use multiplication facts they do know to work out a series of other related facts. The cards show three different calculations in a variety of equivalent forms, using proportional change. The multiplication fact that is likely to be most familiar is green. Sorting and ordering these offers learners the opportunity to identify the relationships. Learners can produce their own sets of cards using other multiplication facts.

You should include examples where this rule does not work, in addition, for example. It is important that learners realise this rule is not generalisable for all arithmetic operations.

64 x 1½	4½ x 12	24 x 2
32 x 3	9 x 6	12 x 4
16 x 6	7 + 4	6 x 8
8 x 12	18 x 3	14 + 2
4 x 24	36 x 1½	3 x 16
2 x 48	3½ + 8	1½ x 32

CLASSIFYING NUMBERS

Asking learners to classify numbers or different representations of numbers according to different criteria encourages them to reflect on properties and relationships.

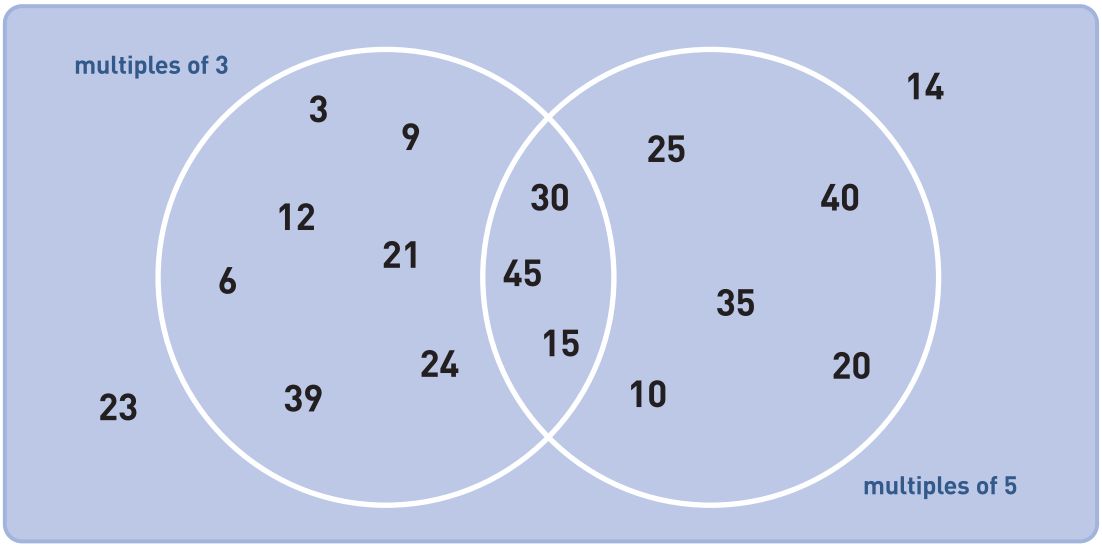
They could be asked to sort cards into groups with common characteristics. If the categories are mutually exclusive, the cards can be placed on a grid.

	multiple of 3	not multiple of 3
less than 30	6, 18, 27, 15, 24	4, 13, 23, 7, 16
more than 30	99, 72, 60, 42, 39	34, 52, 40, 74, 31, 92

Other combinations of criteria could include:

- size and whether numbers are odd / even
- whether numbers are even/odd and multiples of a given odd number
- size and whether numbers are prime / not prime
- whether numbers are odd / even and square / not square

Alternatively, where some numbers fall into more than one category, a Venn diagram with overlapping circles can be used. If the aim is to think about two sets of multiples (or possibly three sets), learners can be given a range of numbers to put into the diagram. The numbers should include some from each set, as well as some that do not fit into either set.



Problem solving

It is the everyday application of numeracy that learners sometimes find difficult. Many people can 'do' a range of isolated calculations with simple instructions e.g. 'multiply 3 by 15 then add 37', but they find it much more difficult to interpret word problems where they need to consider what sort of answer to expect, and how to make decisions about which operations to use, and in what sequence to do them.

In the real world we normally have to take into account the complexity of everyday life in problem solving. For example, in planning a bus journey, we might need to think not only about the distance and average journey time, but we also need to take into account other factors like the weather, traffic conditions at different times of day, relative costs of different types of fares, and so on. Many textbook and examination numeracy questions are oversimplified (and often contrived), so that they are not altogether realistic, and clues which might have helped the learner to identify with the context are often lost.

In order to help learners 'unpack' the mathematics involved in a problem, we need to teach problem-solving strategies explicitly.

These might include the following advice:

Understanding the problem

- → Read the problem through carefully
- → Identify key words and check their meanings
- → Explain in your own words what seems to be required
- → Estimate what sort of answer to expect
- → Look carefully at the information given – decide what is relevant

Deciding what to do

- → Think about whether you have seen a problem like this before.
- → Break the problem into steps
- → Draw a diagram or picture
- → Draw a table, chart or list
- → Use approximation techniques
- → Change the numbers to easier ones
- → Try working backwards
- → Use a process of elimination
- → Look for a pattern
- → Make a guess and then check

Looking back

- → Decide whether you have done what the question asked
- → Check whether your answer is reasonable

- Use a checking strategy to make sure your calculations are correct
- Check that you have used the correct units

Estimation and checking strategies

When faced with a calculation, many learners' instinct is to start straight away trying out one or other operation. We need to encourage them to stand back first, and make an estimate of what they think the answer might be, before they start number-crunching. They may do this by drawing on their everyday experience to judge (e.g. whether it would be sensible to pay £165, £16.50 or £1.65 for two loaves of bread), or by drawing on the knowledge they already have (e.g. 65% of £80 will be more than £40 because it is more than $^1/_2$ the total amount) or they may use rounding techniques to help them judge what sort of answer to expect.

Some learners feel uneasy about estimation, because they think it will not give them a 'right' answer; they feel that in mathematics answers are either 'right' or 'wrong' and that all answers must always be exact. They are not familiar with the idea that different degrees of accuracy may be appropriate in different situations.

Learners need to see how approximation techniques can help them judge what sort of answers to expect in any calculation. They need to be able to identify whether they are over- or under-estimating. They also need to develop confidence in making judgements about how detailed their calculations need to be in order to solve a problem, and what short cuts they can take, e.g.:

In a test question learners are be given the mass in kg of 10 people as follows:
78.4 kg, 86.5 kg, 63.2 kg, 87.3 kg, 92 kg, 78.1 kg, 60 kg, 58.7 kg, 72.5 kg, 76 kg.
The question asks whether it is possible for them all to take a lift with a maximum capacity of 900 kg.

Many learners attempt to answer this question by writing out the ten measurements and then adding them all up to give a total of 752.7 kg . They may not realise that they only need an estimate of the total mass of the ten people and that since only one person is just over 90 kg it is clear, without doing any addition, that the total mass will be less than 900 kg.

We need to encourage learners to use estimation both as their first strategy in tackling any calculation, and also to check whether their answers make sense. It can be difficult to break a culture of dependency where learners always need to ask the teacher, "Is this correct?" and expect the teacher to "mark" all their answers with ticks and crosses. However, it is essential, if learners are to develop real independence in problem-solving, that we ask them, as a matter of course, to use a range of checking strategies themselves so that they only need refer to the teacher once they have explored all other options.

These checking strategies might include:

- using estimation and approximation
- doing inverse operations
- trying an alternative method
- using a calculator
- comparing notes and discussing with another learner
- looking back at previous related work
- looking up number facts

Evaluating statements about numbers

This activity asks learners to consider some generalizations about numbers and number operations. They can be asked to discuss these in small groups and to come to a consensus decision about whether they believe these statements to be 'in more depth about relationships, properties and operations, using the headings 'always', 'sometimes' or 'never' true. In the process they will need to think of a range of examples, and perhaps counter-examples, to justify their choice.

It is best if the statements can be cut up so that each learner in turn can choose one statement to consider. Others should be encouraged to argue and challenge, and to refine their reasoning. Learners may also wish to write similar statements of their own for each other to discuss.

This activity provides an opportunity to explore common misconceptions, and also to think.

Here is a sample:

If you add even numbers the answer will be even

If you add odd numbers the answer will be odd

Most square numbers are even numbers

Any number in the 8 times table is also in the 4 times table

A number that ends in 8 will be in the 8 times table.

Multiples of odd numbers are always odd.

When you add numbers the order does not matter

When you multiply numbers you cannot change the order

There is no connection between division and subtraction

There is no connection between division and subtraction

Zeros make dividing difficult

For most numbers, when you divide them, something is left over

Some numbers you just cannot divide

You always divide the smaller number into the bigger number

You cannot take a big number away from a smaller number

You cannot take a big number away from a smaller number

Links outside the classroom

Everyone uses numbers in various contexts. We make decisions, judgements and comparisons about money and family matters, health issues, time management, work-related problems, leisure, political, social and environmental issues and much more. If we make imaginative use of these connections, this can give us plenty of choice in finding interesting and relevant contexts for work on number.

It is important to celebrate the mathematics skills that learners already use but may not value. Much of this is likely to be based on some form of 'invisible mathematics'; "the mathematics one can do, which one des not think of as mathematics – also known as common sense." (Coben, 2000) 'Invisible mathematics' might include being able to estimate accurately how long it will take to prepare and cook your dinner, knowing what range of food you will be able to buy in your local supermarket for £25, being able to adapt a recipe to feed more or fewer people, cutting something in half and half again to divide it between four people, planning a journey, sharing bills, choosing a mobile phone contract, and so on.

When learners realise that they can transfer their existing (and often untapped) mathematics strategies to problems in the numeracy classroom, they may feel more positive and confident about their work.

We also need to support learners in developing skills of making sensible judgements about which operations to use and when to solve problems in the real world. Many learners, particularly those on vocational courses, may need support to see the connections between what they do in a numeracy class and what happens in the real world outside.

They often tend to 'under-generalise', i.e. they may learn something in a specific context but not see how to transfer it to another. They may learn rules such as:

on a bricklaying course:
how to mix cement (sand: cement 3:1)

on a painting and decorating course:
how to calculate the area of the walls of a room to be painted
(height x width for each wall less area of doors and windows)

on a catering course:
how long to roast a joint of beef (15 minutes per 450g plus 15 minutes)

However, they don't necessarily see that they are using concepts they can generalise,(i.e. a ratio or formulae involving variables or a constant and a variable), and that they can apply these to other contexts, such as working out how much dye to use in a washing machine, calculating the dimensions of kitchen units to fit a given floor space, estimating the cost of car hire or of an emergency plumber with a call out fee and a rate per hour.

They may not be able to think flexibly about what they have learned. For example, they may still have difficulty in:

- working with ratios using numbers which are different to the ones they are familiar with, or dividing something three ways rather than just two
- finding out the width of a wall, given the area and the height

If they have simply learned rules off by heart without understanding them, they may also 'over-generalise' some of them, by making assumptions such as:

- 'in ratio problems the larger of the numbers is always written first', regardless of the context
- (e.g.' if there are 7 women and 10 men, the ratio of women to men is 10:7')
- 'the formula L x W applies to the area of any shape'
- (e.g. 'the area of a triangle is base x height')
- 'if you've learned a formula for cooking times, you can use exactly the same formula for roasting anything'

Learners often seem to view numeracy as a subject where answers are either 'right' or 'wrong' and they feel that if only they could remember the rules, they would manage. They often ask for rules, perhaps hoping for a magical recipe that will solve any mathematical problem.

We need to help learners make meaningful connections between the mathematics they do in the classroom and outside of it, to help them see how they can use their own reasoning, rather than trying to rely on a collection of unrelated rules.

To do this, we can draw on learners' own contexts, which may include a range of practical everyday issues, many of which will be related to money. Work on number can also be an opportunity to look at numbers in other important contexts which many learners might not otherwise feel confident enough to explore in detail on their own: we can investigate some of the mathematics behind news headlines about issues such as climate change, war, poverty, crime, education, immigration, taxation, childcare, drug addiction, health, food, sport, the music industry and much more.

Analysing misconceptions

Through looking more closely at learner errors and talking with learners, we can explore the underlying reasons for misconceptions. If we can find out why learners have made a particular mistake, then we are much better placed to be able to find an effective teaching and learning strategy. Often, just looking at their answers will not reveal what their thought processes are.

There are all kinds of reasons why a learner may make a mistake. Sometimes it may be as a result of misreading numbers, not knowing their tables, rushing their work or forgetting a rule. Other types of errors, however, may reveal a deeper misunderstanding or an over-generalisation which needs to be explicitly dealt with. In general, the teacher needs to deal with these by getting the learner to develop 'cognitive conflict' between different interpretations of the same thing, which throws doubt on their understanding. See *Thinking Through Mathematics: strategies for teaching and learning* (available at www.maths4life.org) for a discussion of this, and examples of what to do. The following list is just a selection of misconceptions you may come across.

"You can't do 3 - 6 therefore you need to do 6 - 3"

$$\begin{array}{r} 23 \\ -16 \\ \hline 13 \end{array}$$

half-learned rule for an algorithm/ not understanding that subtraction is not commutative/

Suggested strategy: Encourage learners to use a number line to visualise the calculation

"To multiply by 10 just add a zero" **10 x 15 = 150** **10 x 1.5 = 1.50** **Learning by rote without understanding / over-generalisation of a rule**	Suggested strategy: Evaluating statements 'sometimes, always, never' / explore multiplication and division by 10 and 100 / ask learners to explore the limits of rules, and develop more accurate ones such as 'to multiply any whole number, positive or negative, by 10 a quick way is to think of it as adding a zero'.
27 **+ 9** **216** *"9 plus 7 is 16 and then you put the 2"* **100 + 6 = 1006** *"one hundred and six"* **Confusion about place value / directional problems with number in columns**	Suggested strategy: Use number lines, colour coding for place value and place value arrow cards.
"The zero doesn't matter" **7 x 0 = 7** **327 ÷ 3 = 19** **Misconceptions about zero.**	Suggested strategy: Matching and sorting activities

"I've used the right layout for long multiplication" $\begin{array}{r} 15 \\ \times\ 10 \\ \hline 00 \\ 15 \\ \hline 15 \end{array}$ **Not understanding the concept / not realising that the answer makes no sense / inappropriate algorithm**	Suggested strategy: Estimating the answer, using a number line, work on multiplying and dividing by tens, explore range of strategies for long multiplication and how and when it is appropriate to use them
"that's is 6 ÷ 12" **12 ÷ 6** **Reading division the wrong way round**	Suggested strategy: Matching different representations of same calculation
"I divide £15 by 0.6 – the answer is £25.00." **If a 1 litre tin of paint costs £15, find how much a 0.6 litre tin costs.** **Not realising that this would mean the smaller tin costs more than the larger tin / not knowing how to decide when to multiply or divide or why**	Suggested strategy: ask learner to estimate what 0.5 litre tin would cost / discuss and compare how to work out costs of other tins both larger and smaller/ investigate effects of multiplying and dividing by numbers less than 1

Suggestions for resources

Readers are advised to read the background to these approaches. Full details of the work undertaken by Malcolm Swan of the University of Nottingham and Susan Wall of Wilberforce College, Hull, can be found in Thinking Through Mathematics: strategies for teaching and learning, available through the Maths4Life website www.maths4life.org.

- **Mini whiteboards and whiteboard pens and wipers**
  Use these to enable learners to easily jot down responses and work out ideas, freeing them from the worry of crossing out mistakes. They also enable tutors to assess everyone's understanding and progress.
- **Number lines**, both numbered and blank
- **1-100 grids** - for addition, subtraction, exploring number patterns, demonstrating place value, playing strategy games, making number jigsaws
- **Multiplication grids – giant ones and small individual ones**
- **Number cards - from 0-9 or 0-100.** useful for games and activities.
- **Dominoes**
  These can be custom-made, using a basic template, for matching different representations of numbers or equivalent calculations
- **Place value arrow cards and charts** – both A4 and giant ones
- **Calculators**
- **Counting sticks, metre sticks, rulers, tape measures**
- **Dice** - can be customized with stickers as necessary
- **Real or plastic coins, or coin cards** - important at Entry levels 1 & 2
- **Cuisenaire rods**
- **Diennes blocks** - for place value
- **Fraction wall** (see BBC Skillswise www.bbc.co.uk/skillswise)
- **A range of measuring equipment** e.g. scales, balances, weights, metre sticks, tape measures, measuring jugs of different capacities, calibrated in different ways
- **OHP resources** including number grids, transparent coloured fraction blocks, number lines, clock faces, calculators
- **1cm and 2 cm squared paper**
- **Coloured counters, post it notes, index cards, tracing paper, scissors, card, glue sticks, string, A3 sugar paper making posters**
- **Sets of cubes** both separate and interlocking
- **Sets of fraction cards** with fractions in numbers, area diagrams, number lines, percentages and decimals
- **Number games from BEAM 'Maths of the Month' free downloads from**
  (BEAM ref)

Do's and dont's

Do

Don't

Do
Make it clear that it is OK to make mistakes
Encourage learners to explore number patterns
Ask learners to explain and discuss their strategies
Encourage learners to select strategies appropriate to the context
Remind learners to estimate the answer before they start the calculation
Ask learners to find ways of checking answers
Avoid introducing confusing vocabulary like 'borrowing'
Use activities that are likely to encourage small group work rather than individual work
Play games which involve a range of mental mathematics strategies
Make connections between what learners already know and what is new
Always make connections between the four operations
Show numbers in a variety of representations
Provide learners with all the learning aids they need e.g. number lines, multiplication grids, place value charts, calculators for checking answers
Encourage learners to think forwards and backwards by giving an answer and asking the learners to write the question
Find a wide range of interesting contexts for number work
Suggest that learners to make up problems for each other

Don't
Let work on the 4 basic operations be a dreary diet of 'sums'
Promote rote learning of algorithms
Brush over mistakes
Teach division as something separate from the other rules.
Answer your own questions

Notes

Notes

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About the authors

Barbara Newmarch teaches numeracy/mathematics in two London colleges, and also runs numeracy teacher training courses. She has been a teacher-researcher on a NRDC project, 'The teaching and learning of common measures in adult numeracy', and a Maths4Life project on 'Formative assessment in adult numeracy'. She has written a guide 'Developing Numeracy' in the NIACE 'Lifelines in Adult Learning' series.

Tracy Part has been teaching for just over 12 years. In this time she has taught across the Skills for Life subjects: numeracy, literacy, IT and ESOL. She has worked in a further education setting for 8 years, specialising mainly in teaching numeracy, but has also taught in a number of different informal educational settings including a centre for street children and in prisons. She is presently working as a numeracy teacher trainer in the Teacher Development Unit at Lewisham College, London.

This booklet is produced by Maths4Life to provide teachers of adult numeracy with ideas about how to teach Number. The aim is to examine teaching strategies, consider where learners have difficulties, and provide some activities to help them overcome these.

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