9 Sequences, functions and graphs

9.4 Exploring functions

Guidance document | Key Stage 4

Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery Professional Development (PD) materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The third of the Key Stage 4 themes (the ninth of the themes in the suite of Secondary Mastery Materials) is *Sequences, functions and graphs,* which covers the following interconnected core concepts:

9.1 Exploring linear equations and inequalities

9.2 Exploring non-linear sequences

9.3 Exploring quadratic equations, inequalities and graphs

9.4 Exploring functions

9.5 Exploring trigonometric functions

This guidance document breaks down core concept *9.4 Exploring functions* into four statements of **knowledge, skills and understanding:**

9.4 Exploring functions

- 9.4.1 Understand functions and their representation
- 9.4.2 Explore other functions
- 9.4.3 Combine and transform functions

Then, for each of these statements of knowledge, skills and understanding, we offer a set of key ideas to help guide teacher planning:

- 9.4.1 Understand functions and their representation
  - 9.4.1.1 Connect a graphical representation with a real-life context (including kinematics)
  - 9.4.1.2 Understand that a function is an operation that maps a set onto another set
  - 9.4.1.3 Understand that a function maps an input in one set (the domain) to exactly one output in another (the range)
  - 9.4.1.4 Interpret the graph of a function as a representation of the mapping of the domain onto the range
  - 9.4.1.5 Represent functions algebraically using standard notation
- 9.4.2 Explore other functions
  - 9.4.2.1 Understand the nature and graphical features of an exponential relationship
  - 9.4.2.2 Understand the nature and graphical features of a reciprocal relationship

9.4.2.3 Understand the graphical features of cubic functions

9.4.2.4 Understand why strategies to approximate solutions to cubic equations are needed and efficiently use these strategies (including iterative strategies)

9.4.3 Combine and transform functions

9.4.3.1 Recognise and use the inverse function as a reverse process

9.4.3.2 Understand the graphical relationship between a function and its inverse

9.4.3.3 Appreciate that the range of a function can be a set of numbers that forms the domain for a second function

9.4.3.4 Represent composite functions algebraically using standard notation

9.4.3.5 Appreciate the connections between the graphical and the algebraic representation of translations of functions

9.4.3.6 Appreciate the connections between the graphical and the algebraic representation of reflections of functions in the axes

Overview

This core concept explores what it means to think and reason functionally. By making connections with both real-life contexts and familiar mathematical ideas, students can build an understanding of functions that goes beyond the notation introduced in Key Stage 4.

In mathematics, a function represents a specific and precise relationship between two sets, where each element of one set (the domain) is paired uniquely with an element in the other set (the range). Function notation provides a language with which to express that one quantity is a 'function' of another, and thus to examine deeply how one quantity depends on another. This notion has been touched on in the previous three core concepts of this theme, and is explored in more depth here.

It is important to recognise that no single representation can sufficiently encapsulate the entirety of what a function is, but different representations can show different aspects of a functional relationship. It is therefore helpful to be able to move fluently between different numerical, algebraic, graphical and diagrammatic representations of functions

Functions at Key Stage 4 can be daunting for teachers, particularly out-of-field teachers whose own study of mathematics may not have gone beyond Key Stages 4 or 5. This document could be used to support teachers to consolidate or extend their own subject knowledge. Students can also find functions challenging, even though they will have been unknowingly working with functions throughout their mathematical education (see 'Prior learning', below). The approach taken in these materials is to use familiar contexts, where possible, and a range of appropriate representations.

The intention throughout is to help students to see a function not simply as a mechanical rule, but as a framework that connects input and output values in a way that is predictable and systematic. Moving between representations supports students to understand that the output value is wholly determined by the input. It is crucial that students see how changes in the input directly produce coordinated changes in the output, while the relationship between the two remains consistent. Through this lens, functional thinking becomes an essential aspect of mathematical reasoning.

Once students have begun to conceptualise functional relationships, they can also begin to understand functions as objects which can be operated on themselves. This allows for the sophisticated idea of an algebra of functions – that is, functions as things that can be combined using mathematical operations. This, in turn, leads to the awareness that the graphs of functions will be transformed by such manipulations. It can be tempting to teach the cause and effect of these transformations by rote. However, building on a solid foundation of functional thinking supports students to understand the underlying meaning – why a given combination or operation results in a particular graphical transformation.

Functional thinking is required in various areas of the mathematics curriculum, and working with functions is a skill that underpins future study of mathematics beyond Key Stage 4. An appreciation of the properties of functions is fundamental for topics such as calculus, and kinematics, as well as applications beyond the mathematics classroom, including science, psychology and physical education. This broad utility highlights why understanding functions is vital and enables teachers to draw on real-life examples in their teaching.

Prior learning

Functions and functional thinking are embedded throughout the curriculum. Students begin working with functions in some of the earliest maths that they study, even if they do not begin to articulate these relationships as functions until this current stage in their mathematical education. For example:

- Multiplication tables can be thought of as functions since they map a number to its corresponding multiple.
- Transformations of shapes are functions since they map an object to its image.
- Sequences can be thought of as functions, with their domain being the natural numbers, and the range being the numbers within that sequence.
- Graphs that map every x value on to a single y value show functional relationships.
- Most of the formulae that students have encountered by this point are examples of functions.

This is not to suggest that these topics are taught as part of functions, but that they can be drawn upon when exploring functions at Key Stage 4. Many of the general properties of functions can be explored using these familiar examples. Similarly, once students are familiar with the language of functions, they could be referenced during revision of these related topics.

While students have already seen functions – and indeed may have encountered the word ‘function’ through algebraic function machines at Key Stages 2 or 3 – the particular language and notation of functions at Key Stage 4 represents new learning. Referring back to function machines, or to tables of values, may offer a familiar starting point to consider concepts such as mapping an input to an output.

An important piece of prior learning is students’ understanding of graphical representations. Their breadth of experience in this area can impact on how they begin to form a concept of what functions are. At Key Stage 3, students have considered linear graphs and the generalised equation for these. This can mean students expect all functions to have a one-to-one correspondence. Quadratic graphs, which may have been introduced late in Key Stage 3 or early in Key Stage 4, may have been their first experience of a many-to-one correspondence in a mathematical context.

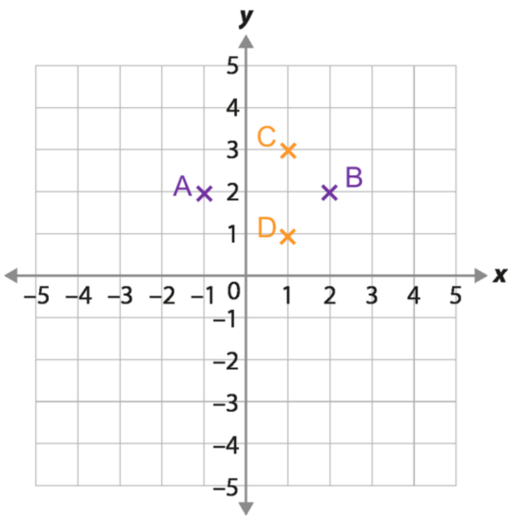
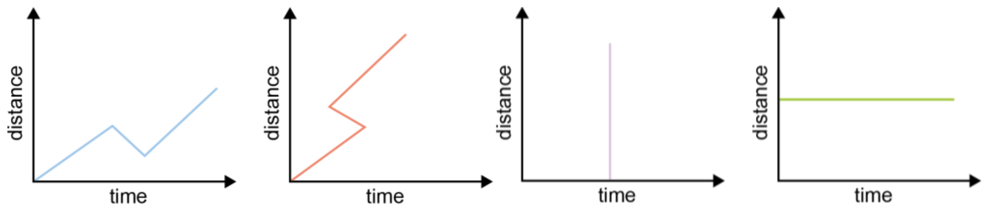
As students begin to work with functions as objects in their own right, they will apply the conventions that they have previously learned in other areas of the curriculum. For example, combining and operating on functions draws directly upon the algebraic manipulation that was taught in Key Stage 3. Similarly, an understanding of the graphical transformations that result from such manipulation builds upon knowledge that was established through transformations of 2D shapes in Key Stages 2 and 3.

Students’ understanding of functions therefore builds upon their experiences with graphs, algebra and other areas of mathematics. It is important to consider these links when planning progression through the curriculum. This is not to say that it is necessary to tackle functions in a particular order, more that it is important to consider students’ prior experience in order to ensure the most coherent approach to this new but crucial element of their learning.

The Key Stage 3 PD materials documents *1.4 Simplifying and manipulating expressions, equations and formulae, 4.2 Graphical representations* and *6.3 Transforming shapes* all explore the prior knowledge required for this core concept in more depth.

Checking prior learning

The following activities from the NCETM Secondary Assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Checkpoints '*Graphical representations of linear relationships*', Checkpoint 8 'Inbetweeners again'	 a) Think of a pair of coordinates that would be on the line between A and B. b) Think of two more pairs. c) Can you think of another one that no one else has yet? d) Think of a pair of coordinates between C and D that is not on the line between A and B. e) Can you think of a pair of coordinates that might be on the line between B and D?
Key Stage 3 PD materials document '4.2 *Graphical representations*', Key idea 4.2.3.3, Example 7	Which of these graphs show journeys that are impossible? 
Key Stage 4 PD materials document '*7.2 Using structure to transform and evaluate expressions*', Key idea 7.2.1.2, Example 11	a^n Freddie says, 'My expression has a value that is greater than 1.' Gurpreet says, 'My expression has a value that is less than 1.' Maya says, 'My expression has a value that is exactly 1.' If a is a positive integer, what do you know about the values that each person has chosen for n?

Key vocabulary

Key terms used in Key Stage 3 materials

- Cartesian coordinate system
- exponent
- gradient
- intercept

- linear
- quadratic
- sequence
- trigonometric functions (sine, cosine, tangent)
- variable

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
domain	The domain of a function, $f(x)$ is the set of arguments, x, for which that function is defined.
function	A function is a relationship between a set of inputs and a set of outputs, with the property that each input in the first set (the domain) maps to a single associated output in the second (the range). Note that the converse need not be true, for example when $f(x) = x^2$ and the domain is the real numbers. The functions that students may encounter at Key Stage 4 include: Linear function: Any function of the general form $f(x) = mx + c$, where m and c are constants. Quadratic function: Any function of the general form $f(x) = ax^2 + bx + c$, where a, b and c are constants. Cubic function: Any function of the general form $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants. Reciprocal function: Any function of the general form $f(x) = \frac{a}{x}$ where a is a constant. Exponential function: Any function of the form $f(x) = b^x$ where b is a constant. Trigonometric function: Functions of angles. A more extensive definition can be found in Core Concept document '9.5 *Trigonometry*'.
instantaneous rate of change	The rate of change at any given point is called the instantaneous rate of change and can be estimated from non-linear relationships by drawing a tangent to the curve and calculating its gradient at that point. The rate of change for a curve is not constant but constantly changes.
piece-wise linear function	A function that consists of number of straight-line functions that have discontinuities (breaks) at certain points. For example, the integer function $y = (x)$, which represents the greatest integer less than or equal to x. At each integer value of x there is a discontinuity as the function jumps to the next integer value.
range	The range of a function, $f(x)$ is the set of values that $f(x)$ can produce for the given domain.
rate of change	A rate of change describes how one quantity changes in relation to another quantity. The gradient of a line represents the rate of change and is determined by the change in y divided by the change in x.

Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible student tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

9.4.1 Understand functions and their representation

Students should be guided to develop their understanding through generating, interpreting and comparing functions using a range of different representations.

A key representation is that of the graph. Any existing misconceptions around graphical representations can undermine new learning around functions, and so it is important to establish students' knowledge in this area, including interpreting graphs in context. Students also need to be able to identify features of a graph and connect these features to the algebraic representation. Each representation of a function allows us to notice different features and students should be guided to move between representations of the function (for example, from the algebra to the graph and vice versa).

Students should be guided to build a sense of what a function is – and, crucially, what it is not – through exploration of functional relationships in context. Activities should support students' deepening functional thinking. The intention is to help them to move away from solely thinking of a function as a process (where the value of one variable is the result of an action being applied to the other variable) and towards thinking of a function as a multi-faceted object (where values vary simultaneously). Students may also begin to recognise and use a range of notations for functions, including $f(x)$, which describes the relationship between the domain and range sets (where x denotes members of the domain set and $f(x)$ symbolises the resulting members of the range set).

It can be challenging to know how deeply to explore the technicalities of functions at this stage of students' mathematical education. Consider the examples used so that students are only exploring 'true' functions, even if you do not explicitly discuss specific details such as each value from the domain being assigned to exactly one value from the range. This will help to minimise potential misconceptions for those students who go on to study mathematics at Key Stage 5 and beyond.

- 9.4.1.1 Connect a graphical representation with a real-life context (including kinematics)
- 9.4.1.2 Understand that a function is an operation that maps a set onto another set
- 9.4.1.3 Understand that a function maps an input in one set (the domain) to exactly one output in another (the range)
- 9.4.1.4 Interpret the graph of a function as a representation of the mapping of the domain onto the range
- 9.4.1.5 Represent functions algebraically using standard notation

9.4.2 Explore non-linear functions

Students' experience of functions is likely to be limited to linear and quadratic functions. A key feature of variation in task design draws upon the idea that students can only fully comprehend what something *is* after also experiencing 'what it is *not*'. Showing the non-concept and boundary cases are important to building a sense of the concept as a whole – a useful analogy is that, if a student had only heard English all their life, they would have no way of distinguishing between the meaning of the words 'English' and 'language'. It is only after experience of other languages that a student can start to see English as one language of many. Similarly, students need to experience a range of functions to fully appreciate what a function is, otherwise they might incorrectly over-generalise and infer that properties specific to linear functions apply to all functions.

Students should make connections between different representations of different functions, understanding the precision afforded by the algebraic representation of the function, and the general features of the function that may be more evident in its graphical representation. This gives a context to explore iterative strategies for approximating solutions, for example for cubic equations. Students should also, when presented with non-standard graphs, be able to infer features related to the context and use these to solve problems.

At Key Stage 4, experience of graphical representations should also include plotting and interpreting reciprocal and exponential graphs. Working with a variety of non-linear functions, such as these, gives a context to further explore functional thinking. The shift from linear relationships to a multitude of non-linear relationships provides variation for students to see and understand those necessary features that make a relationship a function. Particularly relevant when exploring other functions is the understanding of the covariation between input and output, i.e. that, if an output is a function of an input, the two variables will change simultaneously.

The language of 'reciprocal' is likely to be familiar to students from their work at Key Stage 3, and so their schema for reciprocals may be based on their experience of dividing fractions or exploring the area of a rectangle with area 1. Students' maturing mathematical understanding means that functions offer a new lens to revisit this idea. When working with reciprocal functions it is important that students understand they describe a relationship where the range is 1 divided by the elements in the domain. Students should know that division by zero is undefined, so reciprocal functions must exclude $x = 0$ from the domain for the range to be properly defined. Students' attention might also be drawn to the fact that corresponding inputs and outputs multiply to give a constant term, for example $x \times y = 1$ for the general reciprocal function $y = \frac{1}{x}$.

Students' experience of exponential functions, such as $y = b^x$ (where b is a positive whole number, x is in the domain of the function, and y is in the range of the function), will initially be quite limited. At this stage, it is expected that b will remain constant, and so more complex ideas (such as $y = e^x$ and the bivariate function $f(x, y) = x^y$) are considered beyond the scope of Key Stage 4 mathematics. However, a secure grounding in the principles of exponentiation will support all students, regardless of whether they choose to study further.

9.4.2.1 Understand the nature and graphical features of an exponential relationship

9.4.2.2 Understand the nature and graphical features of a reciprocal relationship

9.4.2.3 Understand the graphical features of cubic functions

9.4.2.4 Understand why strategies to approximate solutions to cubic equations are needed and efficiently use these strategies (including iterative strategies)

9.4.3 Combine and transform functions

As discussed above, functions can be considered in multiple different ways: as processes that map the input to the output; as the results of such processes; and as concepts or objects in their own right. Teachers should consider which construct is most relevant to the ideas of combining and transforming functions and use this to be intentional in their planning around language and representations. They should also be aware of any limitations and potential misconceptions, and how this will impact on students' understanding.

For example, when considering inverse functions, $f^{-1}(x)$, students are perhaps more likely to use their understanding of functions as a process. Function machines can be helpful here in understanding the relationship between a function and its inverse. This representation can also be used to draw attention to the connection between the range of the function and the domain of the inverse function.

Similarly, placing one function machine after another can be a helpful way to represent composition of function, $fg(x)$. However, thinking of functions as process can also unintentionally create an illusion of output coming after output, rather than a sense of the values of input and output co-varying. It is therefore important that students can think flexibly and shift their thinking to different constructs in their reasoning around functions. A deep and connected understanding of composite functions might require students to shift their perception to thinking of a function as an object. In this way, composition can be seen as both a process (i.e. $f(x)$ operating on the result of $g(x)$), *and* also as an object (i.e. the result of $f(x)$ and $g(x)$ under the operation of composition).

Students should begin to notice that composite functions have different domains and ranges than their constituent functions. This in turn should help to make links between the algebraic and graphical representations of the transformations of the function. When functions are represented graphically, the graph can be transformed as an object. This transformation of functions allows students to deepen their understanding by reasoning about the way that some combinations impact on the graph of the function.

Students' prior experiences of the language of transformation – such as translation and reflection – are rooted in work on geometry at Key Stage 3. The work at Key Stage 4 draws parallels with this to support them to transform graphs and sketch the results of such transformations. For example, before transforming, students will identify the key features of an object. At Key Stage 3, these features could have been the vertices or sides of a shape; at Key Stage 4, they might be the turning points on the curve of a function or the intercepts with the axes. After transforming, students will identify the variant and invariant features of an object when compared with its image. At Key Stage 3, this would have involved identifying which points are invariant and which lengths have been preserved; at Key Stage 4 this will involve identifying which features of the function have remained the same and which have changed.

9.4.3.1 Recognise and use the inverse function as a reverse process

9.4.3.2 Understand the graphical relationship between a function and its inverse

9.4.3.3 Appreciate that the range of a function can be a set of numbers that forms the domain for a second function

9.4.3.4 Represent composite functions algebraically using standard notation

9.4.3.5 Appreciate the connections between the graphical and the algebraic representation of translations of functions

9.4.3.6 Appreciate the connections between the graphical and the algebraic representation of reflections of functions in the axes

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

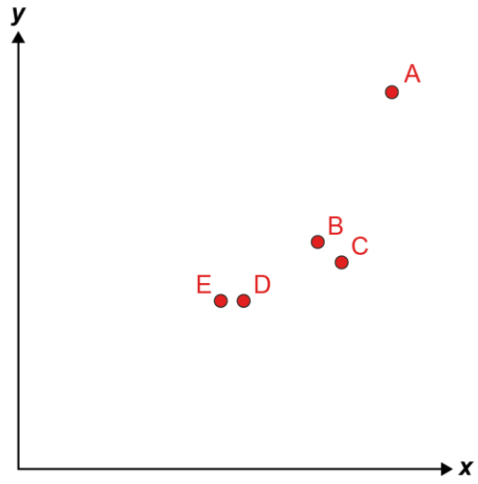
9.4.1.1 Connect a graphical representation with a real-life context (including kinematics)

Common difficulties and misconceptions
One of the key barriers to an understanding of graphs, widely identified in educational research, is that students struggle to treat graphs as abstract representations of relationships. Instead, many can incorrectly interpret them simply as pictures of the situation being studied. This is covered in some depth in the Key Stage 3 PD materials, particularly in exemplified key idea 4.2.3.3, which covers modelling real-life situations graphically. To help further address this misconception about graphs, the following examples explore what is represented by points on axes and by the lines connecting them. They then consider calculations that might be carried out to draw quantitative conclusions about the information offered, particularly in the context of kinematics. A secure understanding of these points will support the use of graphical representations in the context of functions. It is helpful for students to understand gradient as a rate of change, perhaps using the language of 'for every' or 'per' to interpret the meaning of the slope.

<table>
<tr><th>Students need to</th><th>Guidance, discussion points and prompts</th></tr>
<tr><td>

Connect points on graphs with given contextual information

Example 1:



Thomas is 14 years old. He has drawn a graph showing the ages and heights of members of his family. He includes his mum, his older brother Stu, and his younger twin siblings, Annie and Charlie.

a) Which axis has Thomas used to show age and which to show height? Explain how you know.

b) Is Thomas taller or shorter than his older brother?

c) Which other points can you label with certainty?

</td><td>

Example 1 gives students the opportunity to reason with particular points on an axis, to interpret the graphical **representation** of the information presented in the text.

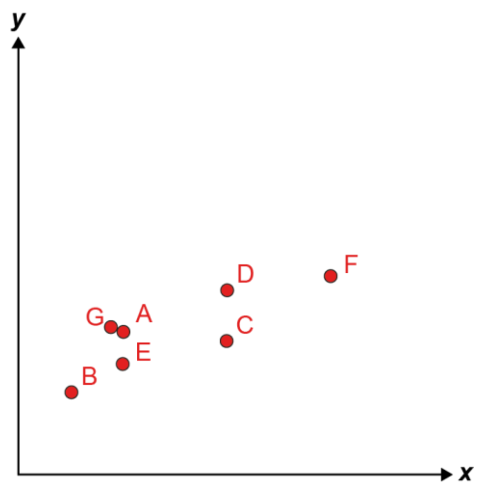
It is common for students to interpret a graph as a picture of a situation, which limits their capacity to reason with graphical representations both in and out of context. To address this, the height of the family is presented here on the x-axis of the graph, rather than the y-axis, as students might have expected.

An interesting follow-up task for **deepening** understanding might be to ask students to plot what the points might look like in ten years' time. Notice whether students recognise that Thomas's mum will have grown older but probably not taller, and so point A will move only vertically, while all of the other points are likely to shift both vertically and horizontally.

</td></tr>
<tr><td>

Example 2:

The graph shows the prices of different sized cartons of milk at a supermarket.

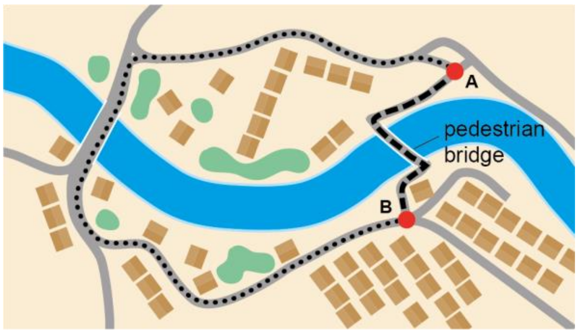
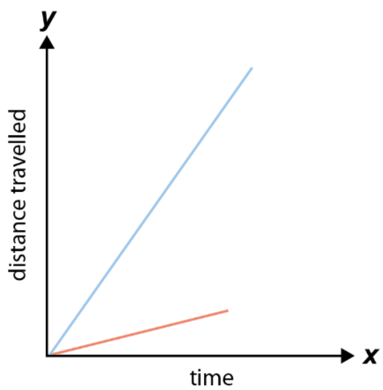


</td><td>

As in *Example 1*, the intention is to use a context to explore and reason with information represented graphically. The axes are left intentionally blank to promote discussion and so teachers will need to pay close attention to the **language** that students offer in their explanations.

The **variation** in the different points given is designed to draw out awareness of some key points that will be useful for students' initial exploration of functions at Key Stage 4. When reasoning about which axis shows which variable, students should notice that points D and C and points A and E are aligned vertically and so it is likely that these show the same size milk carton but at different price points. Noticing and interpreting similarities in points such as this is key when working with functions in a more formal way, interpreting the meaning of minimum values, or of intersections of lines.

</td></tr>
</table>

a) Explain your thinking as you decide which points represent: *(i) The most expensive carton of milk* *(ii) The best value carton of milk* *(iii) Two cartons of milk that are the same size.* *Organic milk is often more expensive than non-organic milk.* *b) Which of the points do you think show organic milk?* *c) Is there any other information you can add?* *Eilish notices that she can draw a straight line through the origin, point C and point F, and another straight line through the origin, point B and point G.* *d) What does this tell her about the cost of milk in these cartons?*	*Example 2* also starts to interpret relationships between points on lines, and part f explicitly addresses this. Students should use their prior knowledge of real-life graphs, which was previously explored in exemplified of the Key Stage 3 PD materials document *'4.2 Graphical representations'*, key idea 4.2.3.3, to make sense of the context and connect it to the graph.
Example 3: *To get from A to B a person could follow the route to walk across the pedestrian bridge, or drive over the road bridge.* 	*Example 3* offers a graph as a clearly abstracted and simplified **representation** of a situation. The intention here is to interpret what might be a slightly unintuitive representation, reasoning about the key features of the graph. You might like to comment on and explore the validity of these graphs – while the walking graph may be likely to provide a reasonably accurate representation of the journey, the driving route will be interrupted by traffic lights and other cars so is less likely to be a straight line. The teacher might like to focus on the **language** being used and how that language connects the graph and the context. For example, students might talk about the gradient, and this should be connected to the speed of travel.
The graph on the right shows how the distance travelled changes with time for each option. *a) Which line on the graph do you think represents the journey by road and which is the footbridge?* *b) How many times further does the car travel than the person walking?* *c) Which journey takes the least time?*	

Connect the shape of graphs with given contextual information

Example 4:

The sketch graphs below show how the height of a child above the ground changes as they play at the park. Which graph could show the child:

a) *On a slide*

b) *On a roundabout*

c) *Climbing up and then jumping off a climbing frame*

d) *On a swing*

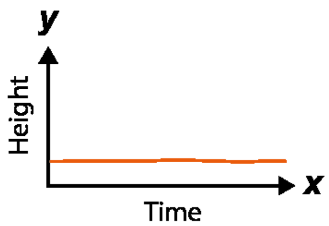
e) *On a seesaw.*

Example 4 shifts the focus of the **representation** from interpreting particular points to drawing conclusions from the shape of the graph as a whole. This is a key step when thinking about a function as a single mathematical object rather than as a collection of points or paired values. The sketched, and therefore slightly inaccurate, style of the graphs is used to draw students away from reading points to interpreting the shape of the graph as a whole.

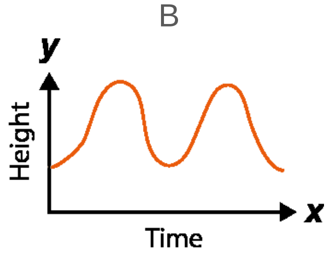
After initial thoughts are shared, the sketches might be usefully paired up and the **variation** in them discussed and interpreted. For example, the graphs showing the swing and the seesaw, and those showing the slide and the climbing frame, offer an opportunity to discuss features such as gradient or steepness and intercept.

Predict the likely mistakes and misconceptions that might arise when using this task with students. What might be the reason for those misconceptions? What experiences might confront and/or address them?

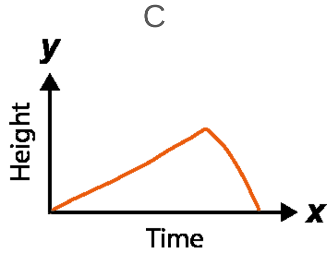
A



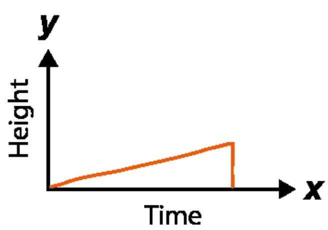
B



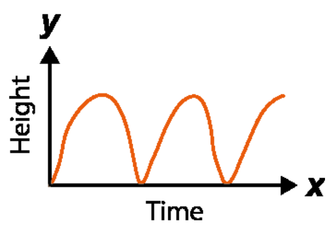
C



D



E



Example 5:

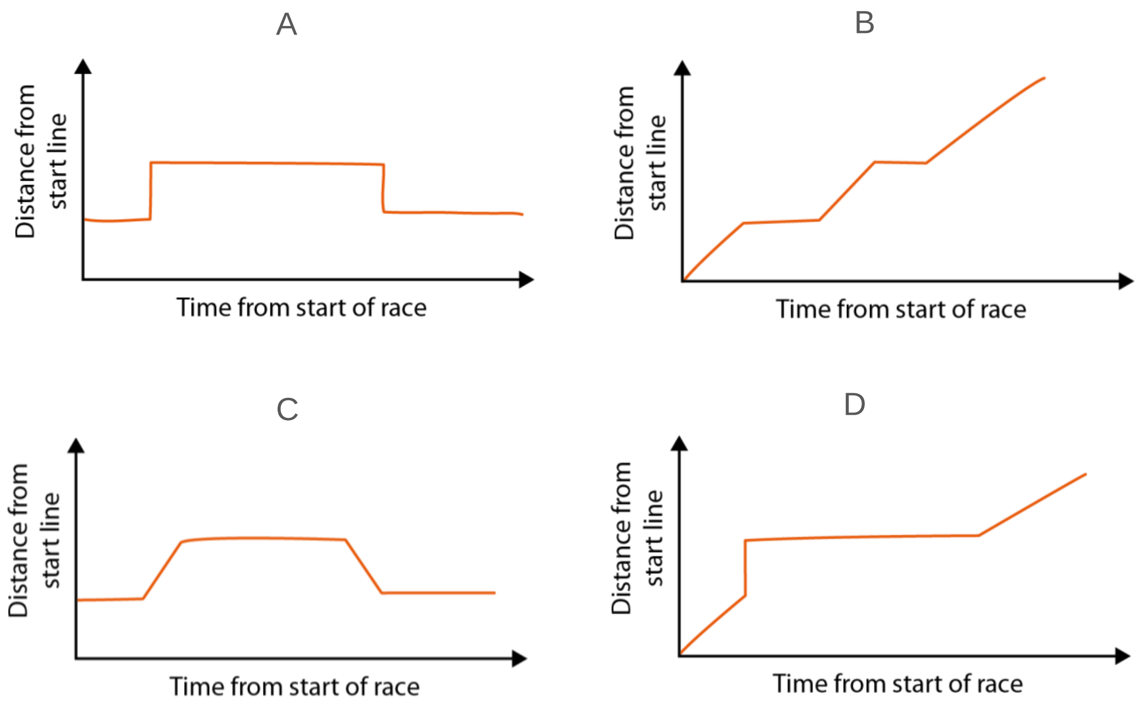
As a part of an obstacle race, runners must climb a ladder, run along a raised section of track, and then climb down a second ladder.

a) Which of the graphs below shows how a runner's distance from the start line changes in that section of the race?

b) Was the runner faster going up or coming down the ladder? How can you see this in the graph?

Example 5 explores **representations** in a similar way to *Example 1*; it challenges the notion of a graph as a picture of a situation. Students might be expected to select either graph A or graph C as representing the story given in the question, as these two graphs 'look like' the situation described.

There is an opportunity to explore mathematical **language** in this context, particularly focusing on interpreting horizontal lines on the graph and what they might mean in the context of kinematics. This is explored further in *Example* 7 below.



Use appropriate calculations to find quantitative information from a graph

Example 6:

A train travels from Exeter to London.

This information is marked on the graph (below right) and the points are joined with straight lines. The table (below left) shows the distance between stops and the time taken to travel between those stops.

Vicky says, 'I can tell that's the slowest section just by looking at the graph.'

a) How can Vicky tell? What is she looking at on the graph?

Charmayne says, 'I've calculated the average speed in miles per minute between each station – the slowest section is Tiverton to Taunton.'

b) What calculation has Charmayne carried out to find the speed between Tiverton and Taunton?

Vicky says, 'I'm going to calculate the gradient of each section of the graph.'

Charmayne says, "I've already done that!"

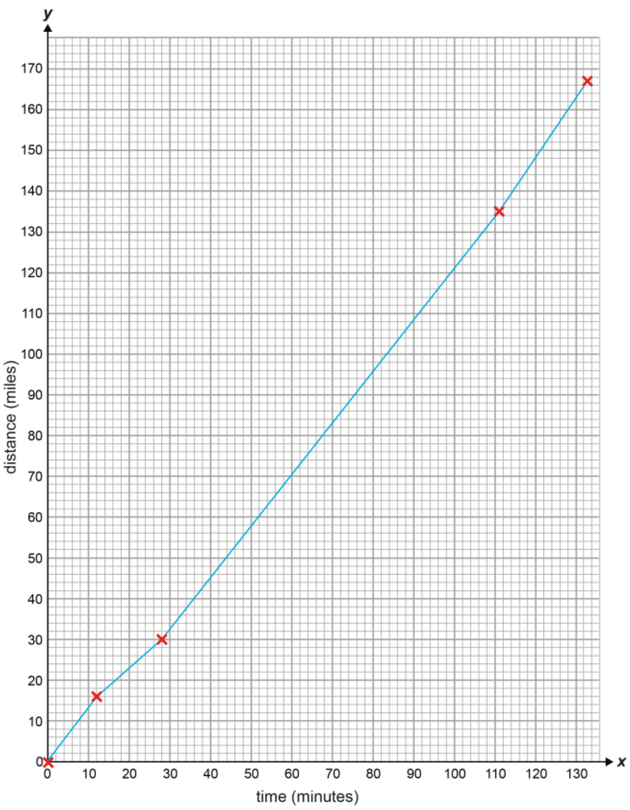
c) Explain why finding the gradient of each line segment is the same as finding the speed in miles per minute.

Station	Distance from previous stop	Time from previous stop
Exeter	-	-
Tiverton	16 miles	12 minutes
Taunton	14 miles	16 minutes
Reading	105 miles	83 minutes
London	35 miles	22 minutes

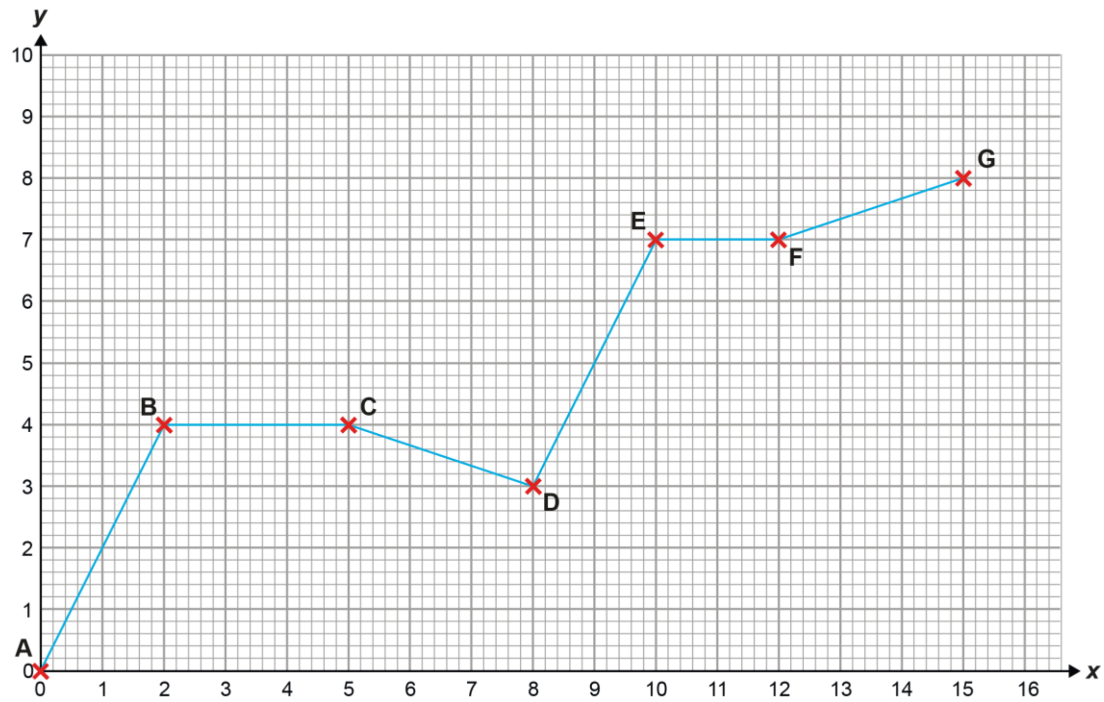
Students should be used to understanding speed as the distance travelled in a given unit of time (miles per hour can be thought of as 'how many miles do we travel for every hour we're travelling?') and similarly used to interpreting gradient as the vertical change for every horizontal unit.

The language of 'per' or 'for every' can, therefore, be helpful here in supporting students to connect the gradient of a distance-time graph and the speed of the object being represented.

As with other graphical representations you might like to explore the level of accuracy here – students should understand that the graph shows an average speed between two points. In fact, you might like to join the start and end points and interpret the gradient of that line as the average speed for the entire journey



Example 7: *The x-axis of the graph below shows the time in seconds. Any one of these labels could go on the y-axis:* • *Distance in metres* • *Speed in metres per second* • *Acceleration in metres per second* *Choose which of the labels is needed on the y-axis so that the graph shows an object that is:* *a) Travelling at a steady speed between points B and C* *b) Stationary between points B and C* *c) Slowing down between points C and D* *d) Speeding up between points C and D* *e) Speeding up between points D and E* *f) Not moving between points E and F.*	In *Example 7*, students must consider the **language** of kinematics, so any ambiguity in their understanding of key terms is likely to be exposed. Are students secure, for example, with the difference between speed and acceleration? The **variation** in this example draws attention to any gaps in understanding. By keeping the graph the same but changing the variable, students need to consider the different measures that the same shape can represent, depending upon the variable dictated by the y-axis. They may find it challenging to recognise that a horizontal line does not always represent a stationary object. *Example 7* should be paired with *Example 8*, to make explicit the interpretation of different features of a kinematics graph, and to give a context for students to reflect on the meaning of speed and acceleration and where these properties might be found in graphical **representations**.



Example 8: *a) Use the graph from Example 7 above, with the y-axis labelled as 'speed in metres per second' to calculate the object's:* *(i) speed between B and C* *(ii) speed between E and F* *(iii) total distance travelled.*	Some parts of *Example 8* will cause conflict with the responses in *Example 7*. These are intended to expose and explore misconceptions, for example that a horizontal line represents a stationary object. This provides an opportunity for **deepening** students' understanding of how kinematic graphs can be interpreted. The intention is to create a context for mathematical talk, for students to reason and justify their thinking and to understand that more than one solution might be valid.

<table>
<tr>
<td>
b) Use the graph from Example 7 above, with the y-axis labelled as ‘distance in metres’ to calculate its:
(i) speed between B and C
(ii) speed between D and E
(iii) total distance travelled.
</td>
<td>
Consider how you might use this pair of tasks in your classroom. Describe how you could introduce them and what talk structures (such as ‘Think, Pair, Share’) you would use. Do you think it’s important to gather responses from all students? If so, how might you do this?
</td>
</tr>
</table>

9.4.1.4 Interpret the graph of a function as a representation of the mapping of the domain onto the range

<table>
<tr>
<th colspan="2">Common difficulties and misconceptions</th>
</tr>
<tr>
<td colspan="2">
A function, as a mathematical object, can be represented as a graph and using algebra. It is important that students realise these are representations of the same thing. That is, that the algebraic notation and the graph of the function are two alternative ways of representing or indicating any particular function. Building a sense of this abstract concept is challenging, and students might prefer to focus on the more concrete and familiar elements of the topic such as the process of plotting graphs or generating pairs of values. However, it is important that these elements are complemented by opportunities to develop a more rounded sense of what a function is, and what it is not.

When students first encounter function notation, it can be tempting for them to categorise $f(x) =$ as just another way of writing $y =$. This superficial understanding can be hard to identify in the classroom, particularly at Key Stage 4 when students are only using function notation in a limited way. The activities in this key idea are designed to draw out the necessary features of functions, so that students have a workable understanding for Key Stage 4 and a solid foundation for future study if necessary. This includes the general awareness of functions as describing how two variables interrelate, as well as the beginnings of some of the language particular to mathematical functions, such as domain and range.
</td>
</tr>
<tr>
<th>Students need to</th>
<th>Guidance, discussion points and prompts</th>
</tr>
<tr>
<td>
Begin to use the language of ‘function’ to connect two variables

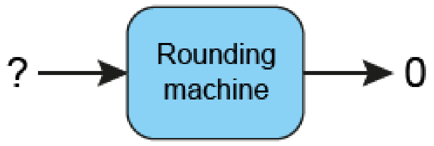
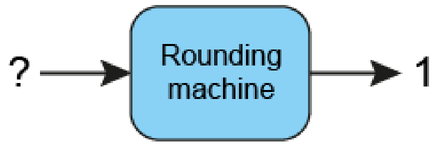
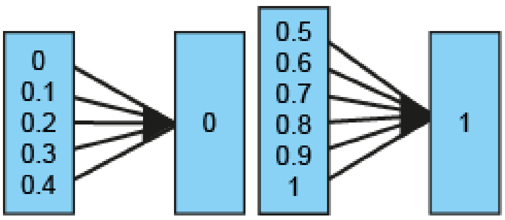
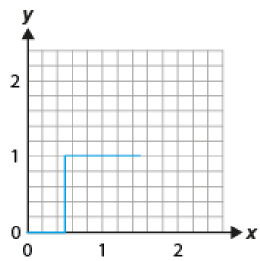
Example 1:

Jen is boiling a kettle. She says, ‘The boiling time is a function of…

a) Which of the options could complete Jen’s sentence correctly?
• … how tall the kettle is.’
• … how full of water the kettle is.’
• … how long the kettle’s cable is.’
• … how thirsty Jen is.’
• … the colour of the kettle.’
b) What other words could complete the sentence correctly?
c) What other words definitely could not complete the sentence correctly?
</td>
<td>
Example 1 begins to use the language of function. While this use of language might be a necessary part of students’ learning around functions, it is not sufficient on its own. Rather, it forms a step in their development of functional reasoning. Students’ understanding of a function at this stage should be that the boiling time is a function of a given variable if, for each value of the variable, there is a unique boiling time.

In this example, we are as interested in checking that students can identify what a function is not. So, they should be able to identify that boiling time will not be a function of Jen’s level of thirst or the kettle’s colour and suggest their own non-examples.

As with many mathematical models of real-life situations, a degree of pragmatism is required for part d. Take cooking times: students could feasibly suggest that a dish’s cooking time is a function of oven temperature. While it is technically true that, for example, there is a specific cooking time (output) for each different degree of temperature (input), in reality we do not measure cooking time with such precision. How far would you explore this with your students? Consider the balance
</td>
</tr>
</table>

d) *Can you think of some other examples of real-life functions?*	between the time that such discussions might take, and the way such discussions might enrich students' understanding.
Example 2 *Ms Tibbetts has a reward system in her classroom. She gives out gold stars for good effort and then awards one house point for every five gold stars.* *Complete the sentences in parts a and b.* a) *The number of house points is a function of…* b) *The number of gold stars is a function of…* c) *Is there more than one way to complete the sentences correctly? Why or why not?*	Much like the previous example, *Example 2* explores the **language** of function and whether students can begin to use it accurately to connect two variables. The **variation** here requires students to attend to specific features of the mathematical definition of a function, in particular its one-way nature. Support them to focus on the difference in the sentences in parts a and b. Are they able to articulate that the number of house points is a function of the number of gold stars, but not the other way around? Part c offers an opportunity for **deepening** understanding and for students to recognise that, if house points are a function of gold stars, and gold stars are a function of effort, then it must follow that house points are also a function of effort. Given the straightforward multiplicative relationship here, students may also begin to articulate how the two functions are related. Students may also have opinions about how strong the functional relationship is between effort and gold stars!
Example 3: *Aaravi inputs four different numbers into the rounding machine below. The output each time is 0.*  a) *What might her input numbers have been?* *She then inputs four new numbers. This time, the output each time is 1.*  b) *What might her input numbers have been this time?* *Aravi then draws the representation below for the rounding function.* 	*Example 3* uses a familiar context – rounding – to support students to begin to use the **language** that is commonly associated with functions. In parts a and b, a function machine is used to emphasise the connection between values when rounding; that is, that an input is 'mapped onto' an output. Part c is deliberately open so that teachers can respond to students' contributions. It should be used to highlight that there are infinite possible inputs and that these are not easily seen in this representation. The function machine may be a familiar **representation** to students and is likely to have featured at some stage in their learning around number or algebra. However, the function machines have limitations and, for students' understanding of functions to develop beyond a 'input/output', process-based model, it is not helpful for it to be the only image that they rely on. You might consider representing the rounding function as a piece-wise linear graph, as shown.  As rounding is a context that will be familiar to all students, it might be tempting to assume that this example is less suited to higher attaining students. However, it is designed for **deepening** students' understanding, so that they can expand their definition of what a mathematical function is within an accessible framework. The graphical representation above might challenge students: they will be used to using graphs to relate two variables, but perhaps not to seeing the 'chunky' shape of a many to one function. If you

<table>
<tr>
<td>

c) *Is this a helpful representation of the function? Why or why not?*

</td>
<td>

choose to explore this representation, consider possible student responses. How far will you go in your explanations around different functions at this stage?

</td>
</tr>
<tr>
<td>

Example 4:

Mel is rounding numbers between 0 and 1 to the nearest integer.

She says, 'If a number is less than 0.5, I round it down to 0. If it is equal to or more than 0.5, I round it up to 1.'

Geri says she can write this more efficiently as: $f(x) = \begin{cases} 0, x < 0.5 \\ 1, x \geq 0.5 \end{cases}$

a) *How has Geri summarised Mel's rule?*

b) *Use Geri's notation to include rounding numbers between 1 and 2.*

c) *Compare Geri's representation with the representations in Example 3.*

</td>
<td>

Example 4 is an option for continuing the context of rounding to further explore function notation and the how it can be used to capture the mapping of different inputs onto the two possible outputs. Students should be familiar both with the inequalities used and the rule that they denote. It is not a huge leap to deduce how the notation acts as a **representation** of the rounding function.

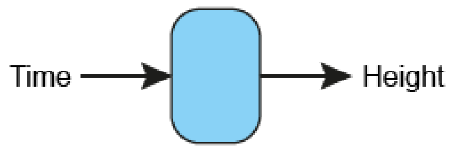
Plan ahead what **language** you will use with students. Some words will be familiar to them – input and output, for example, have colloquial meanings that approximate to their mathematical usage. However, terms such as domain and range have very particular uses in the context of functions. This needs to be explicitly taught at some stage of students' mathematical journeys, whether it is in this initial exploration of functional relationships in Key Stage 4, or in further study at Key Stage 5.

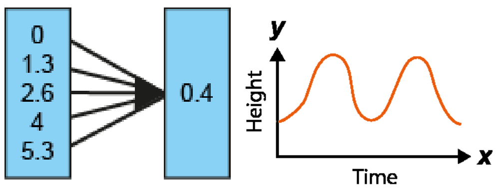
</td>
</tr>
<tr>
<td>

Example 5:

Becky is a maths teacher. She takes her children to the playground. As she pushes her daughter on the swing, she thinks about the fact that the height of the swing is a function of time.

She approximates some values for this function and creates four different representations to share with her class.





<table>
<tr><td>Time (s)</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td>Height (m)</td><td>0.4</td><td>0.9</td><td>1.2</td><td>0.8</td><td>0.4</td><td>0.6</td></tr>
</table>

a) *What is the same and what is different about each representation?*

b) *Which representation best models the relationship between height and time in this situation?*

</td>
<td>

In *Example 5*, we are again exploring different **representations** for functions and considering the strengths and limitations of each representation for drawing out different properties of functions. In this scenario, the process model that is implied by the function machine (where the function 'happens' to the input to create an output) is no longer sufficient. Similarly, the discrete coordinate pairs shown in the table of values do not fully represent the continuous movement of the swing.

This activity provides a structure for **deepening** students' functional reasoning. Students should be beginning to recognise that the values of the domain and range co-vary simultaneously. So, in this example, the height of the swing depends on the time that has elapsed, and the height and time are both continuously changing. This understanding is difficult to achieve if a student's reference point for functions is a 'function machine'-type process.

The open nature of the questions in parts a and b mean that it might not be possible to predict students' responses. Consider your classes and discuss your expectations for this activity with your colleagues. Some prompts for discussion might be:

- What understanding do you want to develop?
- What are some likely student responses?
- What additional prompts might you need to help elicit this understanding?

</td>
</tr>
</table>

Example 6:

Below are some different representations of the function $f(x) = x^2$ and some statements describing the properties of this function.

a) *For each statement, choose a representation and explain how it shows that property of the function.*

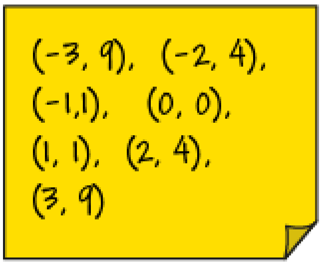
b) *Are there any representations that are of limited use in explaining the properties of this function? Why?*

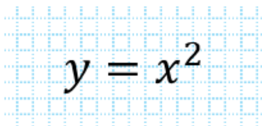
c) *Which of the properties are shared by all functions, which are shared by some functions, and which are unique to this function?*

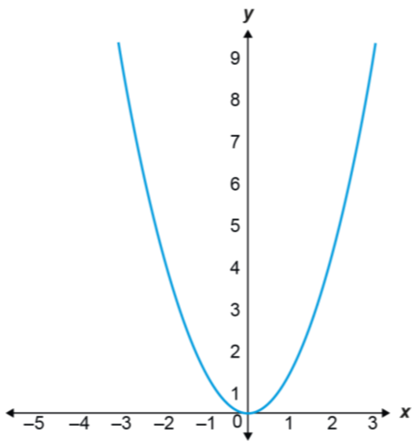
Example 6 concludes this series exploring some of the initial properties of functions. As with many of the examples, a key focus is on using different **representations** to explore the properties of functions.

Consider the **language** within each statement. The vocabulary does not go beyond Key Stage 4, but you may have chosen to use different terminology with your students. For example, you may have used the terms ‘input’ and ‘output’ rather than introducing the terms ‘domain’ and ‘range’. Conversely, you may have felt your students were ready for the term ‘many-to-one’ rather than the more descriptive definition used in statement ii).

Part c offers an opportunity for **deepening** students’ understanding by considering which of the statements would be applicable to all functions and which are unique to the particular example used here, $f(x) = x^2$. Students may be able to identify that some properties, for example statement (ii), are applicable to some, but not all, other functions. Encourage them to think of examples – if they cannot specify a function algebraically, can they sketch a graphical representation of it?







x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Properties of this function	
(i) Each value of the input maps onto a single output value.	*(ii) More than one input value maps onto each output value.*
(iii) When $x = 0$, $y = 0$.	*(iv) As the value of x increases, the change in y value depends on what the value of x was.*
(v) There are infinite sets of points that satisfy this function.	*(vi) As the value of x changes, the value of y also instantaneously changes.*
(vii) The domain includes all numbers, positive and negative.	*(viii) All of the numbers in the range are positive.*

Understand that, for the graph of any given function, a particular value of x can give only one value for y

Example 7:

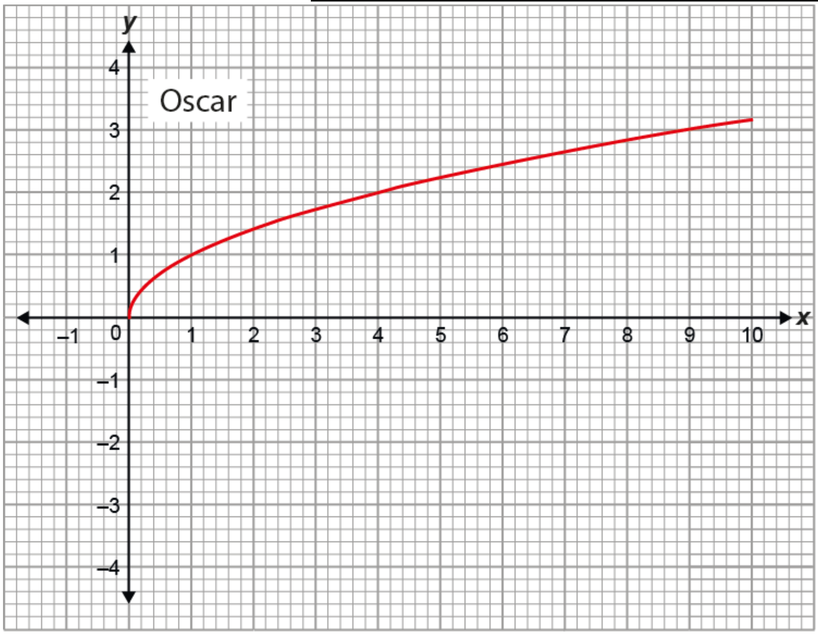
Oscar and Billy both try to plot the graph of $y = \sqrt{x}$ below.

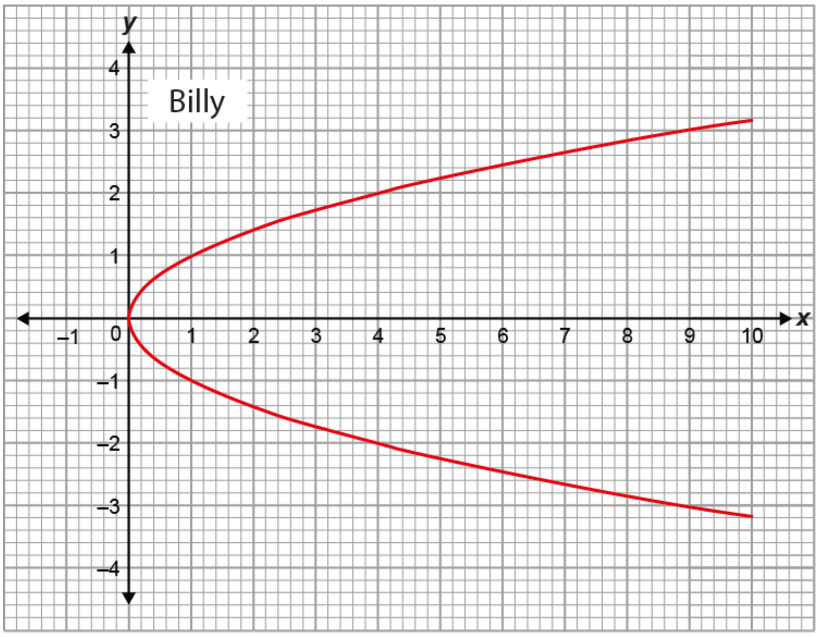
a) *What is the same and what is different about their graphs?*

Billy says that they have both drawn the function $f(x) = \sqrt{x}$.

b) *Explain why Billy is not correct*

Example 7 focuses on **deepening** students' understanding of mathematical functions. A key feature of the mathematical definition of a function is the unique nature of each y-value. *Example 7* offers a familiar context – that of square roots – to make this explicit to students. While both Oscar and Billy have plotted graphs, only Oscar has plotted a function, as each value of x corresponds to a unique value of y. This can also help to address the potential misconception that the notation $f(x) =$ is entirely interchangeable with the notation $y =$.

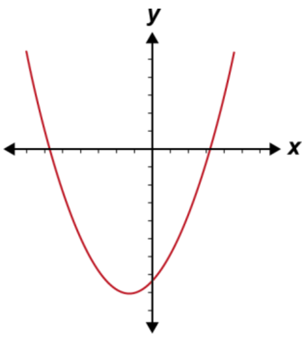




Example 8:

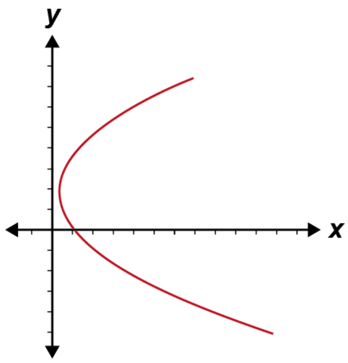
Which of these sketches might represent a function, $g(x)$*?*

A

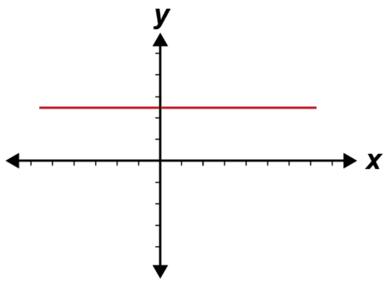


In *Example 8*, the sketches of graphs use **variation** to continue to explore properties of functions. Similarly to *Example* 7, guide students to notice that, for a graph to be a function, every line parallel to the y-axis intersects only one point. Unlike in the previous example, the graphs here are not all familiar to students. They therefore need to consider the properties of the line as a whole when deciding what is, and what is not, the graph of a function.

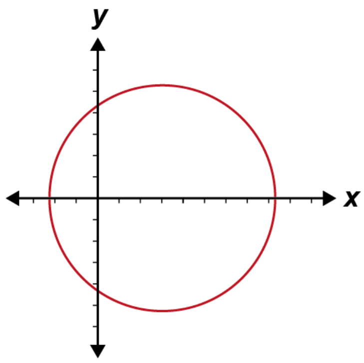
B



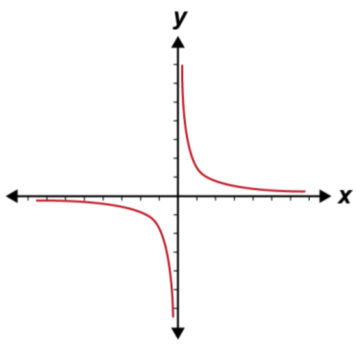
C



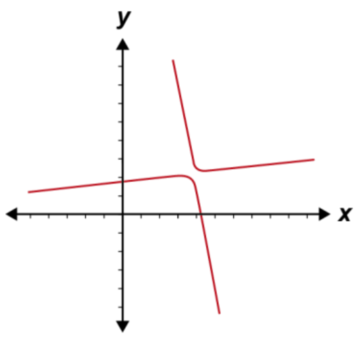
D



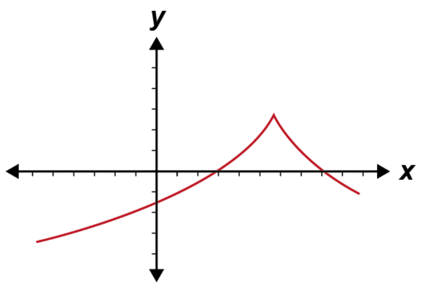
E



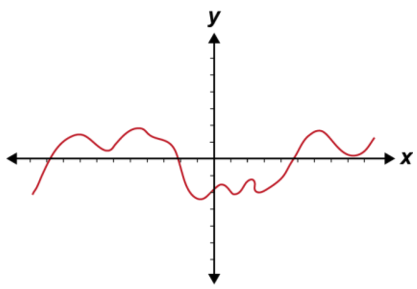
F



G

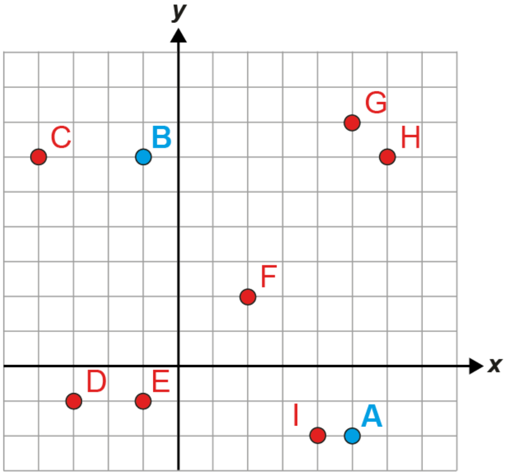


H



Example 9:

On the axes below, points A and B sit on the graph representing the function $f(x)$.



Which of the other points C to I might also be on the same graph as A and B?

The **variation** in *Example 9* consists of examples and non-examples to draw out the particular property that, for any given value of $x, f(x)$ can have just one value. So, in this case, points E and G cannot also be on the line $f(x)$. However, it **is** possible for the same value of $f(x)$ to be given by more than one x. Points C and I are examples of this. Understanding these properties is key to understanding what a function is and is not.

Pay attention to the **language** that both you and your students use, to ensure that any generalisations hold true as their understanding of functions develops. It is worth noting, for example, that all linear functions have a single value of $f(x)$ for every value of x and that, when represented on a graph, there is always both a single y value for every x, and an x value for every value of y. As students meet and work with non-linear functions, it can be challenging to understand that, although there is still a y value for every x, it is no longer true to say that there is an x value for every value of y.

Example 10:

Given that point (3, 7) satisfies the function $g(x)$, which other points might also satisfy $g(x)$?

(7, 7)	*(7, 3)*	*(3, 3)*	*(1, 7)*
(3, −3)	*(−3, 7)*	*(12, 53)*	*(0, 0)*

Example 10 again uses examples and non-examples. This time, the **representation** used is coordinate pairs plane. Connections might be made between this and the previous example since, for example, points C, B and H might lie on the line that includes (3, 3), and points A and G might lie on the line shown that includes (7, 7).

This representation allows greater focus on the equality inherent in the definition of a function. Students should notice that, for example, if (3, 7) satisfies the function, there can be no other point of the form (3, _) but there might be a point (_, 7). For the same reason, discussion could be had around the selection (7, 7) or (7, 3): students can choose one, but not both.

Appreciate that particular points on a graph offer instances of the relationship between the domain and range of a function

Example 11:

Below is the graph of $y = 0.3x^3 + x^2 - 1$.

Where possible, use the graph to estimate the value p in each case, in parts a to f.

a) $0.3 \times 1.5^3 + 1.5^2 - 1 = p$

b) $0.3 \times -2^3 + -2^2 - 1 = p$

c) $0.3p^3 + p^2 - 1 = 4$

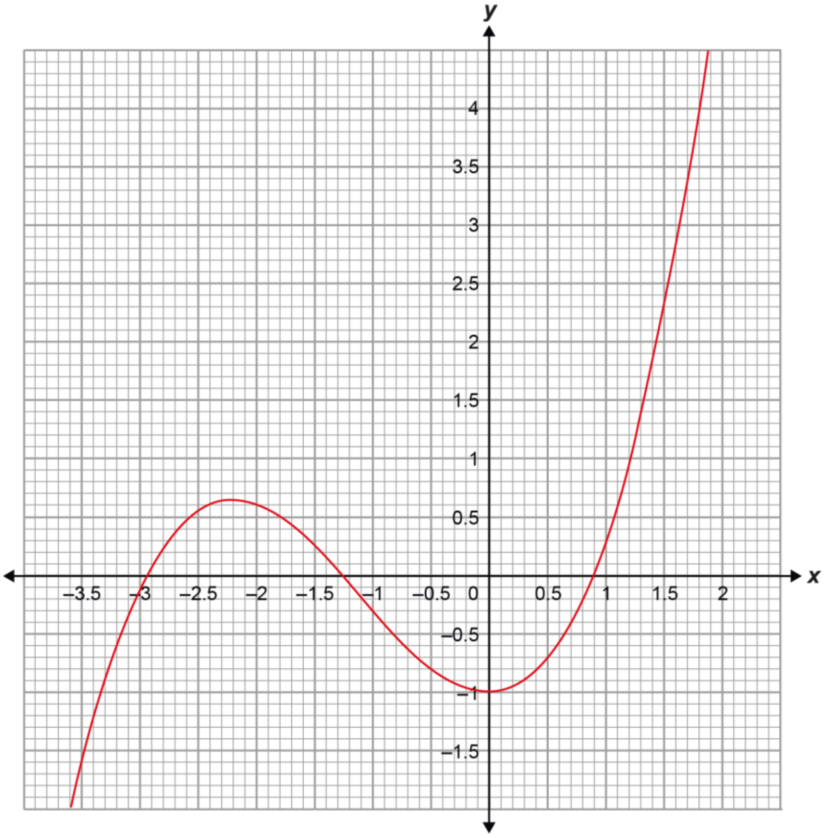
d) $0.3p^3 + p^2 - 1 = 0.5$

e) $0.3p^3 + p^2 - 1 = -1$

f) $0.3p^3 + p^2 - 1 = -1.5$

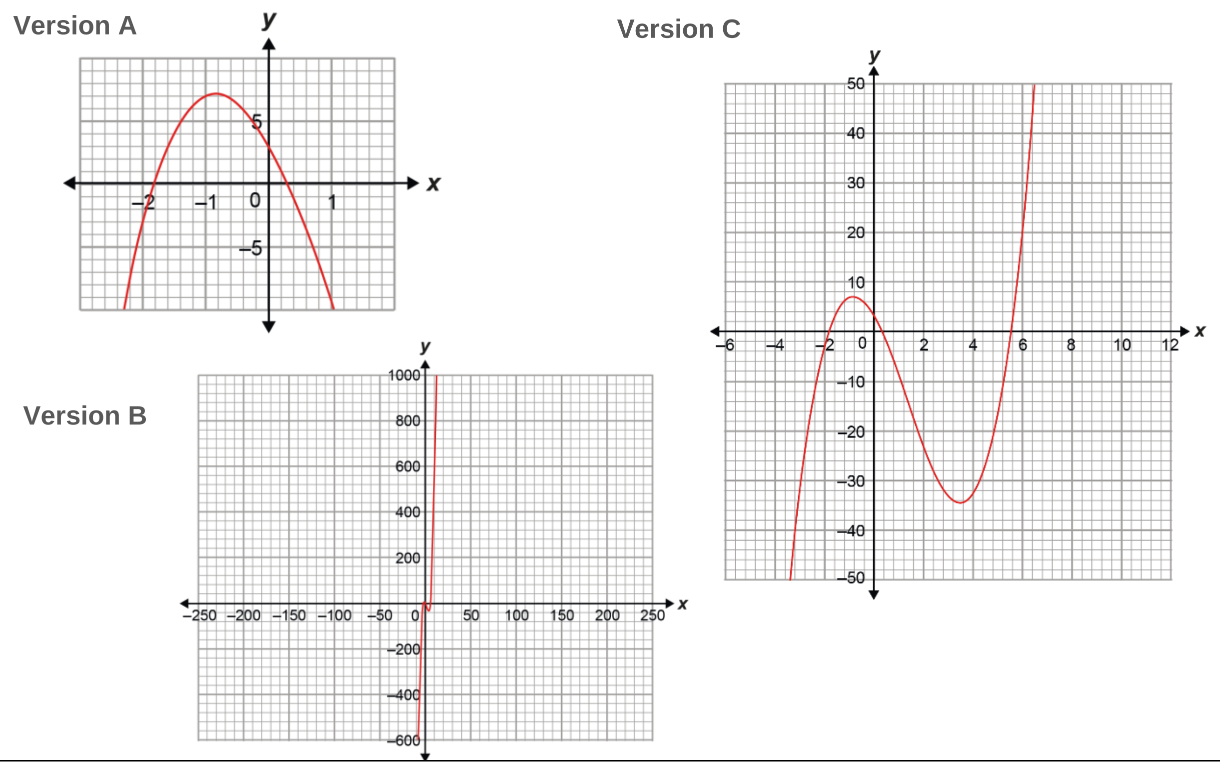
Example 11 again uses a graphical **representation** of a function, but the emphasis here is subtly different. While *Example 9* moved towards viewing the graph and function as a mathematical object, *Example 11* reverts to interpreting individual points and their relationship. This is intended to make explicit the connections between the particular cases in the functions and the values as represented on the graph. The function used here is deliberately complex to discourage calculation.

The **variation** in the questions draws attention to the invariant nature of the relationship between the domain and the range. Contrast parts a and b, where p is the value of $f(x)$, with parts c to f, where p is the value of x. This comparison should highlight that, for any given value of $x, f(x)$ can have just one value but that the reverse of this is not true. That is, that for every value of $f(x)$, there can be more than one value of x, since part d has three possible solutions and part e has two.



Example 12: *Some students are playing a game with the inputs and outputs of a function $f(x)$. When it is their turn, they have to pick an input that is between the previous two inputs.* *Josh writes $f(1) = 2$.* *Bea writes $f(3) = 6$.* *Sol writes $f(2) = 4$.* *a) It is Josh's turn again. What could he write?* *b) What could Bea and Sol write for their next turns?* *c) What do you notice?* *They play again with a different function.* *Josh writes $f(1) = 1$.* *Bea writes $f(3) = 9$.* *Sol writes $f(2) = 4$.* *Josh says, 'Although it is a different function, I will always be able to find an input that is between the previous two chosen inputs.'*	*Example 12* further develops the idea of points being particular instances of the relationship between the domain and range, with an emphasis on the infinite possible inputs (and outputs) for a continuous function. The context of the game, while maintaining the use of key **language** such as input and output, means that students can be encouraged to think of multiple possible responses to each go. Spend time on part b, asking students to contribute as many possible answers as they can. The intention is to ensure students are clear that, for any two values in the domain/range of a continuous function, there will always be a value in between them. The **variation** here is such that the same three inputs produce different outputs, so students can consider this concept in the context of both a linear and a non-linear function. They should come to understand that both share the properties inherent in continuous functions. Teachers may like to ask if students can identify functions that would fit the values given (e.g. $y = 2x$ and $y = x^2$ respectively), to consolidate prior learning, but this is not necessary to achieve the new learning point intended here. Parts d and e just ask if Josh and Bea are correct, but there is an opportunity for **deepening** understanding by asking follow-up questions. For example, '*Will this be true of all functions*?' or, '*Can you think of another function where this is true*?' In deciding how far to explore this, teachers should be guided by the time available, and the

d) *Is Josh correct? Why?* *Bea says, 'I noticed that the outputs are always between the previous two outputs as well.'* e) *Is Bea correct? Why?*	security of students' functional reasoning. It is worth noting that it would not be true for a function where there are discontinuities, unless you specify a domain that excludes these discontinuities.
Understand graphs as offering an insight into the general relationship between the domain and range of a function *Example 13:* *Below are three versions, A to C, of a graph of the same function, $f(x)$. The scales are different in each graph.* *Which version of the graph is most useful for answering the following questions? Why?* a) *Evaluate $f(3)$.* b) *Is the function quadratic or cubic?* c) *Does the point (10,70) lie on the curve?*	The **variation** here draws attention to the different insights gleaned by taking either a very close look at a small domain of the function, or by viewing the function as a whole and taking a broader domain set. Point out the way that the first graph might suggest a quadratic function, or perhaps an even power. It is not until the function is viewed at an appropriate scale that its shape is truly appreciated. While the **representations** here are limited to three static images, there are benefits to using software to explore functions dynamically. Zooming in and out allows exploration of key points as well as taking the wider view; so, when working with a class, a dynamic image should also be used wherever possible. For reference, the function used here is $y = (x-1)^3 - (x+6)^2 + 40)$. Using dynamic graphing software requires different pedagogical decisions. What strategies do you have to ensure that all students are engaged with a dynamic image being projected at the front of the room rather than a static image on their desk?



9.4.2.1 Understand the nature and graphical features of an exponential relationship

Common difficulties and misconceptions

Just as multiplication will have been first experienced as repeated addition, students' first experiences of exponentiation will be through repeated multiplication, as considered in Key Stage 4 PD materials document '*7.2 Using structure to transform and evaluate expressions*'. Students may need support with both the operation itself and the notation used to describe it. Difficulty will arise if students have not fully appreciated the impact of exponentiation on the magnitude of a number, so they should be supported to understand this.

Students should also be guided to notice that, unlike multiplication, exponentiation is not commutative: for example, $4^3 \neq 3^4$. Give them opportunities to explore the different effects that changing the exponent and changing the base have on the value of the power.

It may be useful to revisit unit fractions, so that students are clear about the relationship between the value of the denominator and the size of the fraction. Remind students of familiar phrasing from Key Stage 2, such as, 'for unit fractions, as the denominator increases, the size of the fraction decreases'. This may support students in making sense of the line where $x < 0$.

It is also important that students understand that negative indices are the same as a reciprocal of the power. To appreciate the relevance of the asymptote in the graphs of exponential functions, students will need to be familiar and confident with the fact that negative indices b^x (when $x < 0$) can also be written in the form $\frac{1}{b^x}$. This is explored further in exemplified key idea 7.2.1.2 of document 7.2 in these PD materials.

As with all functions, an appreciation of the nature of variation and covariation and how this can be interpreted through a graphical representation is key. Difficulty may arise if students don't appreciate that there is an infinite number of points between any two distinct points on a Cartesian coordinate grid, as this will limit their understanding of the continuous nature of the curve. Similarly, students need to appreciate that the set of axes is infinitely long in both directions, as this will contribute towards an understanding of tending towards infinity.

Students may need to spend some time calculating ordered pairs and plotting coordinates of exponential functions, but it is important that they also come to appreciate that there are some general features which will be present in graphs of all exponential functions. This sense of the shape of an exponential graph will support students to understand that the line/curve which represents the function is a single mathematical object which can be operated on.

Students need to	Guidance, discussion points and prompts
Relate the intersection at (0,1) to the structure of exponential relationships *Example 1* a) *Complete the table for $f(x) = b^x$ below.* b) *Then begin to plot the graphs of each function for $-1 < x < 1$.* c) *What do you notice?*	The **variation** in this example has been designed to draw students' attention to the (0, 1) ordered pair in each example. Students may initially spot patterns, questions such as "how many times larger is 1 than $\frac{1}{17}$?" will help focus students' attention on the structure. Students need a secure understanding of index laws, in particular $a^m \div a^n = a^{m-n}$, combined with a secure understanding that fractions represent a division (that is $a^m \div a^m = \frac{a^m}{a^m}$) to understand $a^0 = 1$. It may be helpful to revisit the examples from exemplified key idea 7.2.1.2 if students need to refresh their memories of this property. The task suggests that students plot just a few points of their graphs in order to focus attention on a particular feature of the function. It might be that, following this, there

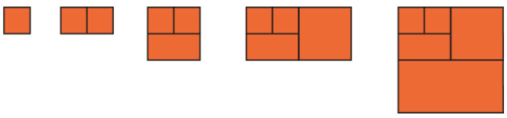
		x	
	-1	0	1
2^x	$\frac{1}{2}$		
3^x			
4^x		1	
11^x			
17^x			17
a^x			

is an opportunity for **deepening** understanding of other graphical properties using software, or by calculating and plotting further points.

What would you expect students to notice when they are completing the table of values or comparing their graphs? What would you want students to notice? What question could you ask to prompt students to notice structure?

Example 2:

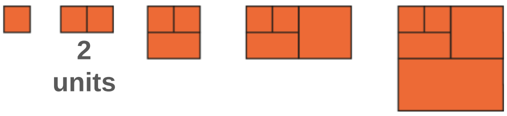
Jason and Kylie create a sequence of shapes using square and rectangular tiles, below.



a) *Describe how the areas of the shapes change from left to right.*

b) *Describe how the shapes change from right to left.*

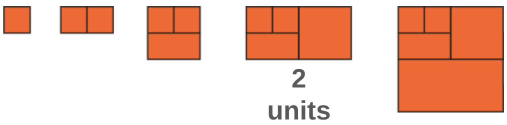
Jason says the ***second*** *shape has an area of 2 units.*



c) *What are the areas of the other shapes?*

d) *Write these as powers of 2. What do you notice?*

Kylie says the ***fourth*** *shape has an area of two units:*

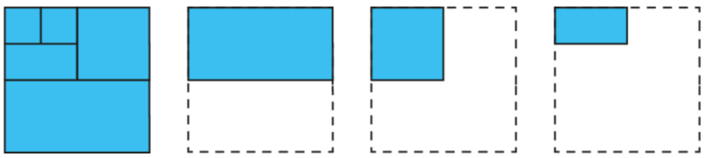


e) *What are the areas of the shapes now?*

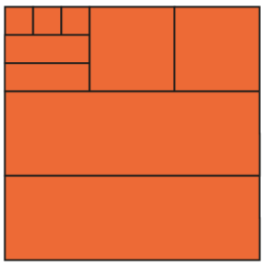
f) *Express the areas as powers of 2. What do you notice now?*

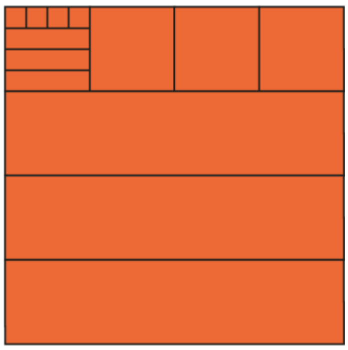
The **variation** in *Example 2* highlights the structure of the exponential relationship. By varying the 'unit' square we are drawing students' attention to the structure that is common throughout. Part g offers an opportunity for a more open-ended exploration, as the resulting areas would depend on which of the shapes students choose to consider as having an area of two units.

This **representation** allows students to explore the consistent nature of index notation and can be extended beyond the examples given here. For example, to explore negative indices, the following could be offered to students, giving a rationale for the way negative indices are used to describe fractions of amounts:

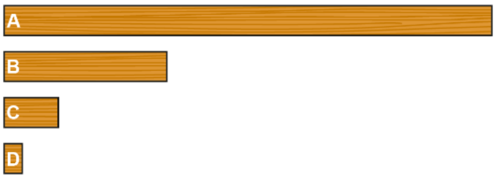


This idea is not limited to shapes that double in size. It can also be used to explore powers of three and four. Below are some possible representations for those powers:



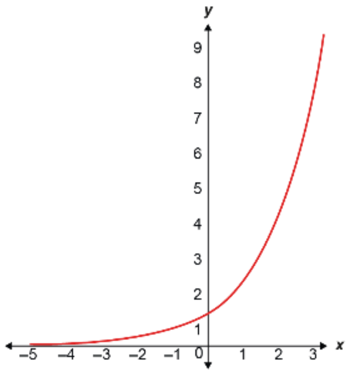


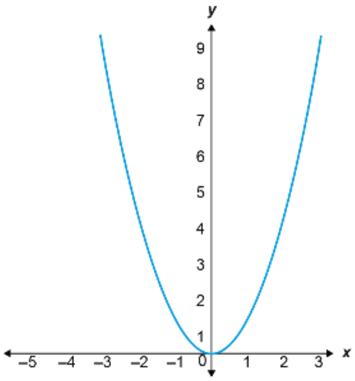
How much time do you spend exploring different representations for indices? In your experience,

g) *Which other shapes could Kylie and Jason have chosen as representing an area of 2 units? What would happen?*	which have been most successful? Are there any representations that can also link to other topics in maths?
Relate the constantly changing rate of change to the structure of exponential relationships *Example 3:* *Algae covers the surface of a pond so that the area covered doubles each day. After 30 days the surface of the pond is totally covered with algae.* *After how many days was the half of the surface of the pond covered?*	*Example 3*, and subsequent examples, give a context for students to appreciate the variable nature of the rate of change of exponential functions. *Example 3* in particular offers a task that may be familiar, but it provides a surprising result which can be used to for **deepening** students' understanding of the nature of exponential growth. Links could be made with the examples from key idea 9.2.1.3, using the with the **language** of sequences and n^{th} term to exemplify the growth of algae over consecutive days. Whilst the growth of algae may appear continuous, taking a measurement of total algae coverage on consecutive days requires a discrete perspective. Consider other real-life examples of exponential growth that you or your students may have previously encountered. For example, you could refer back to the examples from 9.2.1.3, or introduce new contexts such as the spread of infectious disease. Which of these do you find most powerful in encapsulating the nature of exponentiation? Is it the same or different for your students?
Example 4: *Here are some sticks, ordered by length. Each stick is a third of the length of the previous stick.*  a) *How many of stick B would be the same length as stick A?* b) *How many of stick D would be the same length as stick C?* c) *How many of stick C would be the same length as stick A?* d) *How many of stick D would be the same length as stick A?* e) *A fifth stick is added, stick E, which is a third of the length of stick D. How many of stick E would be the same length as stick B?*	*Example 4* gives students the opportunity to pay attention to the structure of functions with a tangible **representation**. The sticks are an unfamiliar context for functions, away from common algebraic or graphical representations, but they provide a clear visual reference. Recreating this image with physical or virtual objects (such as strips of paper or images on interactive software) may further help students to appreciate the relationships. **Language** such as 'per' or 'for every' can draw students' attention to the unit of their comparison in *Example 4*. Some students will focus on the additive relationship between neighbouring sticks, so it is important that the focus is shifted to a multiplicative comparison when relating the first stick to the second. If students find this challenging, then model the different mathematical structures and ask students to notice what is the same and what is different between different descriptions. For example: • *'I need two more of stick B to make stick A.'* (additive) • *'For every one stick A, there would be three of stick B.'* (multiplicative) *'There are three times as many stick Bs per length of stick A.'* (multiplicative)

Example 5:

Below are the graphs of $f(x) = 2^x$ and $g(x) = x^2$.





a) *What is the same and what is different about the two graphs?*

Below are the tables of values for the same functions.

b) *What do you notice? What do you wonder?*

c) *Continue the table for values of x between −10 and 10.*

d) *What do you notice? Is it the same as what you wondered about in part b?*

The algebraic **representations** of these two functions might be thought to look similar, but the graphical and tabular representations are strikingly different. Students should readily recognise these familiar functions in both forms and understand that both functions feature a non-constant rate of change, which results in a curve. They should be also able to articulate why the graph of $g(x)$ is symmetrical about the line $x = 0$.

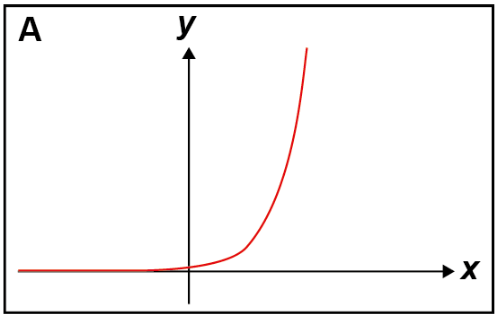
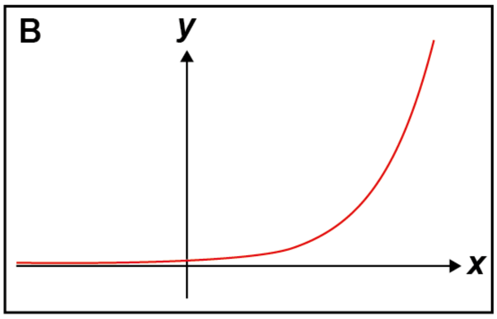
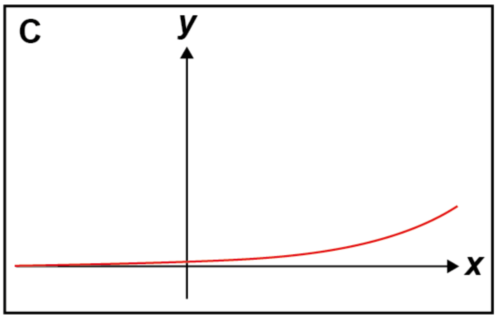
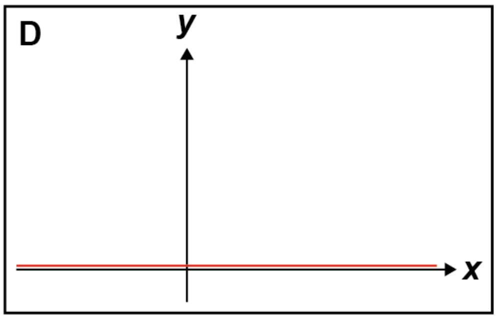
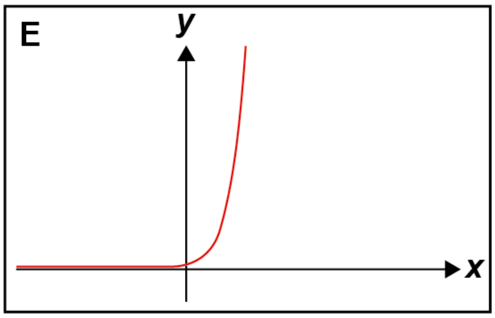
The **variation** here has been chosen to prompt students to think about the difference between functions when the variable is the base compared with when the variable is the exponent. There is another opportunity for comparison when looking at the table of values. Students should see a more striking difference in the rate of change as they continue the table of values: in one direction $f(x)$ gets much smaller than $g(x)$, and in the other much larger.

The open-ended prompts offer many opportunities for **deepening** understanding about the essential and non-essential features of exponential functions. There is a lot for students to notice, and some of the similarities, for example $f(4) = g(4)$, might need unpicking. Providing students with a similar set of questions about x^3 and 3^x may help to ascertain whether students have effectively generalised about the structures of exponential functions. Providing students with similar questions about $\boldsymbol{x^3}$ and $\boldsymbol{3^x}$ might be helpful to deepen students understanding and allow them to categorise the features common to non-linear and particular to exponential functions.

How often do you use the prompt to 'wonder' in the mathematics classroom? Students may not have often been given the opportunity to take a mathematical starting point and follow their own curiosity. How comfortable do you feel in inviting students to respond to such an open-ended question? If this is new to you as a teacher, it might be interesting to explore this question with your department and try to anticipate likely responses. Compare your colleagues' predictions with your students' actual ideas: were they the same or different?

$\boldsymbol{x}$	−2	−1	0	1	2	3	4	5
$\boldsymbol{f(x)}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32
$\boldsymbol{g(x)}$	4	1	0	1	4	9	16	25

Relate the asymptote to the structure of exponential relationships *Example 6:* *On the following page are five graphs, all drawn to the same scale. Graph A is the graph of $y = f(x)$ where $f(x) = 10^x$.* a) *Match the other exponential functions with the graphs.* b) *One graph does not have a function given. What could it be?*	*Example 6* encourages students to apply what they know about the exponential function and make decisions based on comparisons between graphical **representations**. Comparisons may be made more readily if students can physically move the graphs around; consider printing the page and cutting out the grid to create a card sort. The axes on the graphs are intentionally blank for **deepening** students' understanding of the role of the asymptote in an exponential graph. The task includes 1^x, which students should be able to reason and identify as the horizontal line. They should then be able to rank the other four graphs in order of steepness of the curve, recognising that this order can help to identify the equations when there are no absolute values to reference. Students will come up with different functions to associate with their spare graph. Discuss why there is a range of possible missing functions and focus on what is knowable from the graphs and what is not, specifically in relation to the other graphs. Comparative **language** structures may help students make sense of this. For example, *'The base is lower than ____ because the rate of change of the curve is ____.'* The instruction to 'match' the functions and graphs deliberately does not reference the strategy of ranking in order of steepness, to give space for students (or teachers, if using in a professional development setting) to apply their existing knowledge. This might mean there is some time spent considering how to approach the task – consider when this struggle is productive, and at what stage you would step in to prompt.

$f(x) = 1^x$	
$f(x) = 2^x$	
$f(x) = 3^x$	
$f(x) = 10^x$	
$f(x) = ?$	

9.4.3.5 Appreciate the connections between the graphical and the algebraic representation of translations of functions

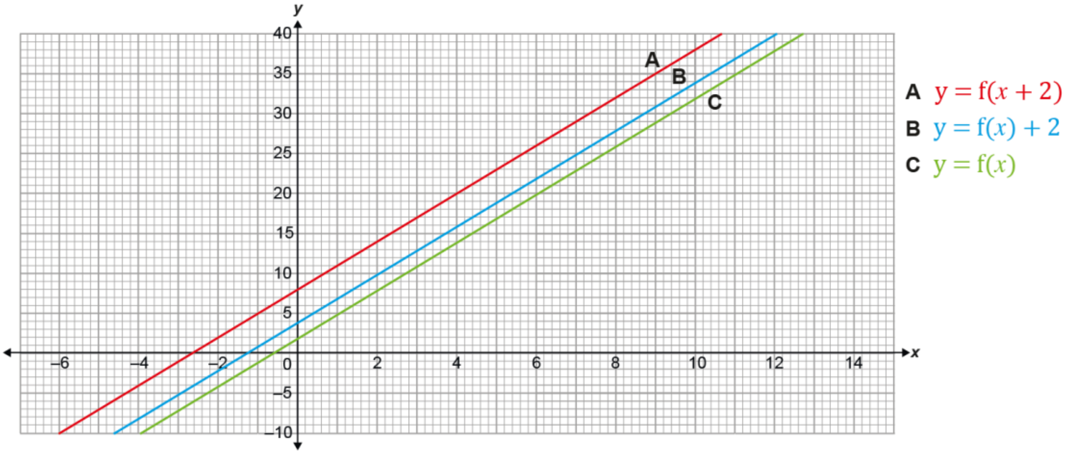
Common difficulties and misconceptions

Students should be supported to appreciate the difference between those arithmetic operations which are performed on the range set and will therefore have implications for the y-coordinates of a line/curve, and those which are performed on the domain set and will therefore have implications for the x-coordinates.

This is demonstrated in the table of values with $f(x) = x^2$ below. It can be seen that row A is equivalent to row B but shifted one space to the left, and so the same effect would be seen on the graph. Row C is equivalent to row B but with each value increased by 1 unit, which has the effect of shifting the graph upwards.

	x	−1	0	1	2	3	4	5	6
A	$f(x+1)$	0	1	4	9	16	25	36	49
B	$f(x)$	1	0	1	4	9	16	25	36
C	$f(x)+1$	2	1	2	5	10	17	26	37

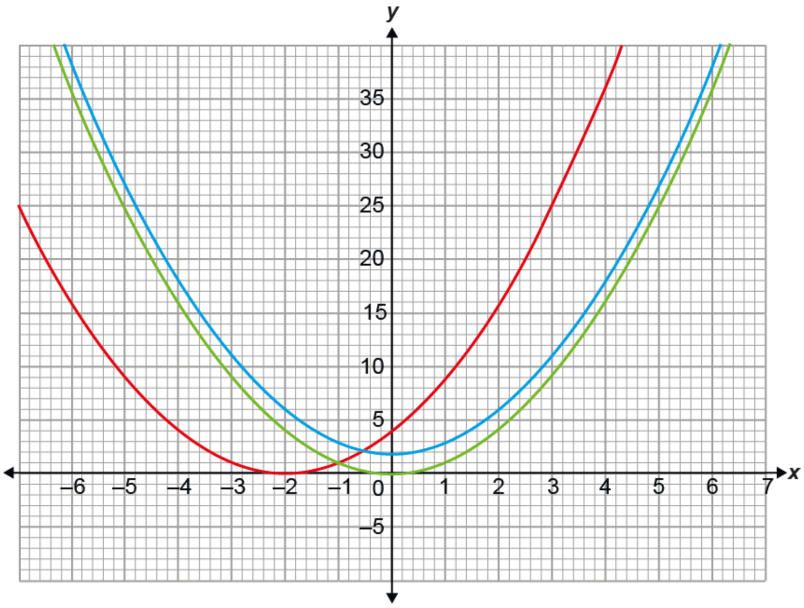
When transforming functions, it is common for students to recognise the pattern of translations for $f(x) + a$, where the object is translated a units in the positive y-direction, and $f(x) - a$, where the object is translated in the negative y-direction'. Often students will then assume that $f(x + a)$ will behave similarly and be translated a units in the positive x-direction. They should be encouraged to describe what is happening to the relationship between the domain and range sets to make sense of and connect the algebraic and graphical representations of such transformations.



When considering examples to use with students, there may be a tension between the simplicity of working with linear functions and the ambiguity in describing translations of the lines of such functions. In this example, for instance, the function $f(x) = 3x + 2$ and both of the transformations $y = f(x + 2)$ and $y = f(x) + 2$ are shown on the same axes. The linear nature of the functions means that horizontal and vertical translations are difficult to discern, and it is likely that students might describe both transformations as vertical translations. Exploring the table of values may help to avoid such ambiguity:

	x	−3	−2	−1	0	1	2	3	4
A	$f(x+2)$	−1	2	5	8	11	14	17	20
B	$f(x)$	−7	−4	−1	2	5	8	11	14
C	$f(x)+2$	−5	−2	1	4	7	10	13	16

Alternatively, the use of a non-linear graph, while it may add complication to the evaluation of the function, brings clarity around the translations that result. This is shown in the following graph and table of values using the same transformations as the previous example, but a starting function of $f(x) = x^2$:



	x	−3	−2	−1	0	1	2	3	4
A	$f(x+2)$	1	0	1	4	9	16	25	36
B	$f(x)$	9	4	1	0	1	4	9	16
C	$f(x)+2$	11	6	3	2	3	6	11	18

Support students to relate these translations of graphs to their earlier learning on translations of shapes, particularly that the image is congruent to the object but in a different position in relation to the axes. Therefore, the features of any given function will stay the same in relation to the other features of the function, but that the function, in relation to its position on the coordinate axis, will be altered.

Students need to

Interpret the notation $f(x) + a$ and $f(x + a)$, evaluating the impact of the transformation numerically

Example 1:

Below is a partially-completed table of values for the function $f(x) = 10x + 2$.

x	−1	0	1	*2*	*3*
$f(x)$	−8	2	12	*22*	*32*
$f(x) + 3$	−5				
$f(x + 3)$	22				

a) *Complete the table.*

Guidance, discussion points and prompts

In *Example 1*, the **language** of the statements and questions have been carefully considered to draw students' attention to the ways in which the function is transformed differently for $\mathrm{f(x)} + a$ and $\mathrm{f(x} + a)$. The covariation of linear functions is constant – that is, for every change in one value, the other value will change by a consistent amount. Students are therefore likely to identify easily how both the transformations $\mathrm{f(x)} + a$ and $\mathrm{f(x} + a)$ alter the value of the function at each point. However, in non-linear functions, the changing nature of the covariation means that it is harder to predict the impact of the transformation $\mathrm{f(x} + a)$ at a given point. This is unpicked further when considering the graph as a whole in *Example 4*.

Throughout the examples for this exemplified key idea, **variation** is used to draw attention to the salient features of the given transformation. This supports students to draw

Adam says, 'The value of $f(x) + 3$ is always 3 more than the value of $f(x)$.'

b) Explain why Adam is correct.

Salma says, 'The value of $f(x + 3)$ is always 30 more than the value of $f(x)$.'

c) Explain why Salma is correct.

Below is a partially completed table of values for the function $f(x) = x^2 + 2$.

x	-1	0	1	2	3
$f(x)$	3	2	3	6	11
$f(x) + 3$	6				
$f(x + 3)$	6				

d) Complete the table.

e) Is Adam's statement from part b still true? Why or why not?

f) Is Salma's statement from part c still true? Why or why not?

meaningful conclusions about the general structure of the transformations of functions, without being distracted by the details of lots of particular changes. In *Example 1*, this takes the form of transforming the same function two different ways and then applying the same two transformations to a different function. When comparing the two transformations, guide students in their comparisons. For example, by comparing pairs of values in the row $f(x)$ with their corresponding values in the row $f(x) + 3$. Or by comparing the entire row $f(x)$ with the row $f(x + 3)$

Students might begin to associate the first comparison with a vertical shift on the graph, and the second comparison with a horizontal shift. *Example 4* below further explores what this looks like graphically.

Example 2:

In the function $g(x) = 12x + 73$, the value of x changes from 176.8 to 177.8.

a) How does this change the value of $g(x)$?

The point at (19, 301) is on the line given by $y = g(x)$.

b) Write the coordinates of a point on the line given by $y = g(x) + 1$.

c) Write the coordinates of a point on the line given by $y = g(x + 1)$.

Example 2 reasons about the effect of a translation of a graph by considering particular values and points. It might be considered that parts a and c are simply asking the same question about different points. However, the changes in the **language** and context of the question are intended to shift the way that students view these two mathematically similar tasks.

To what extent do you agree that part a and c are the same question? Try the questions with colleagues, and with students. How are the responses different? Are they similar enough that it is not worth asking both questions? Or do they, as intended, draw attention to the transformation in different ways?

Interpret the notation $f(x) + a$ and $f(x + a)$, evaluating the impact of the transformation graphically

Example 3:

Below this example is a table showing the values of the function $f(x) = x^2 + 2$ for integers $-6 \leq x \leq 3$.

a) *Plot the graphs of $y = f(x)$, $y = f(x) + 3$, and $y = f(x + 3)$.*

b) *Describe the translation that maps $y = f(x)$ onto $f(x) + 3$.*

c) *Describe the translation that maps $y = f(x)$ onto $f(x + 3)$.*

d) *How would your answer to part a look different if the graphs were of $y = f(x)$, $y = f(x) - 5$ and $y = f(x - 5)$ instead?*

e) *How would your answer to part a look different if the graphs were of $y = f(x)$, $y = f(x) + 2$ and $y = f(x + 2)$ instead?*

Example 3 uses the table of values from *Example 1* above to make explicit the connection between the numerical impact of transforming a function, and the way that this transformation appears graphically. The intention is that the graph is considered as a whole. This addresses some of the challenge that students of considering the transformation of individual points in part f of *Example 1.* Moving between the tabular and graphical **representations** of the same function should support students to move from considering individual pairs of points to viewing the function as a single mathematical object. Considering even a complex function as an object means it can be efficiently operated on and translated according to a rule. The quantitative change when considering particular points is non-linear; but when viewing the function as a single mathematical object, the transformation is simple and linear.

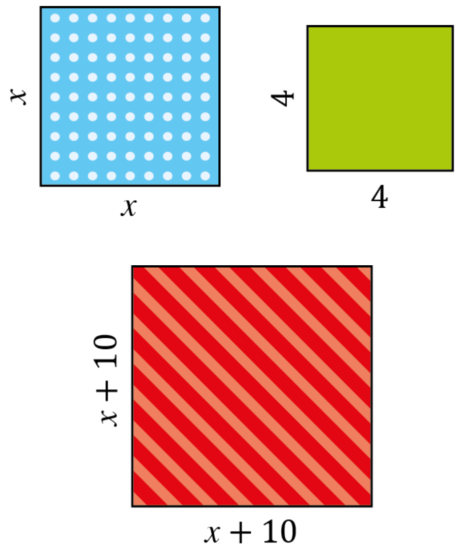
Dynamic graphing software, which enables students to visualise the transformations more readily, might be a useful tool. Teachers could use it to help check the validity of students' answers to parts d and e, or to model further transformation to consolidate the learning points. Such software also provides teachers with the opportunity to further develop the **variation** of the transformations in the example, so that they can respond flexibly to students' learning needs.

Consider how you might structure the use of *Example 1* and *Example 3* within a sequence of learning. You might, for example, ask students to connect their answers to *Example 1* part e with part b here, or *Example 1* part f with part c. Consider the spacing of these tasks to expose this comparison fully. How can you best support students to shift their perspective from considering a function as a collection of individual points to as a single mathematical object?

x	-6	-5	-4	-3	-2	−1	0	1	2	3
$f(x)$	38	27	18	11	6	3	2	3	6	11
$f(x) + 3$	41	30	21	14	9	6	5	6	9	14
$f(x + 3)$	11	6	3	2	3	6	11	18	27	38

Example 4:

The blue spotted square has sides of x cm. The red striped square has sides that are 10 cm longer than the blue spotted square. The plain green square has sides of 4 cm.



The first graph shows how the area of the blue spotted square changes as x changes. The second shows three more graphs, p, q and r, plotted on the same axes.

a) Which graph shows:

(i) The area of the red striped square as x changes?

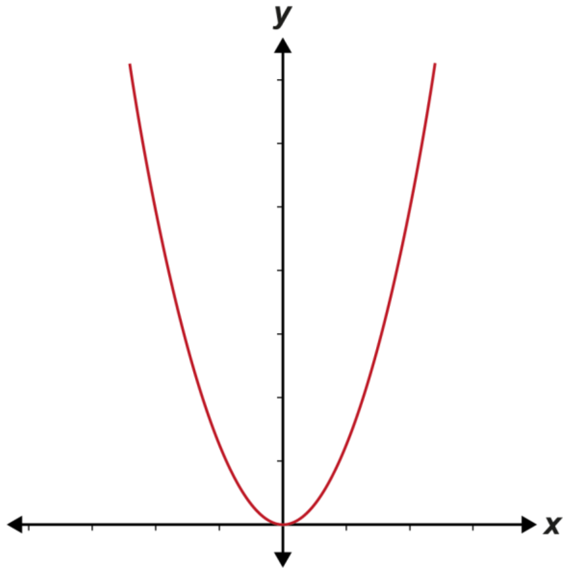
(ii) The area of the plain green square as x changes?

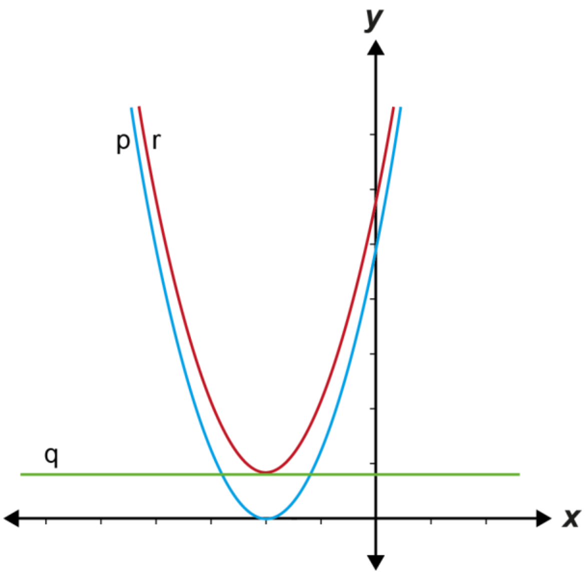
(iii) The total area of the red striped and plain green square as x changes?

b) Which points on the graph are you able to accurately give a value to?

Students may be familiar with the using an area or length to represent unknown values, as with algebra tiles. In *Example 4*, the same idea is used to look at combinations of translations, and to make sense of these translations using a physical **representation**. Students are likely to need time to reason with and connect these representations.

With time, it might be that further exploration of the limitations of these physical representations becomes a barrier. For example, the first graph given of the area of the blue square suggests that a negative length is possible. Discussing how to interpret this may support in **deepening** students' understanding of the connections between the graph and the pictorial representation.





Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a collaborative professional development activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 9 Sequences, functions and graphs* can be found here:

https://www.ncetm.org.uk/media/23eejt3r/ncetm_ks4_cc_9_solutions.pdf