9 Sequences, functions and graphs

Mastery Professional Development

9.3 Exploring quadratic equations, inequalities and graphs

Guidance document | Key Stage 4

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Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The third of the Key Stage 4 themes (the ninth of the themes in the suite of Secondary Mastery Materials) is *Sequences, functions and graphs*, which covers the following interconnected core concepts:

9.1 Exploring linear equations and inequalities

9.2 Exploring non-linear sequences

9.3 Exploring quadratic equations, inequalities and graphs

9.4 Exploring functions

9.5 Exploring trigonometric functions

This guidance document breaks down core concept *9.3 Exploring quadratic equations, inequalities and graphs* into four statements of **knowledge, skills and understanding**:

9.3 Exploring quadratic equations, inequalities and graphs

- 9.3.1 Explore and solve quadratic equations
- 9.3.2 Use and understand the graphical features of quadratic relationships
- 9.3.3 Work with quadratic inequalities
- 9.3.4 Explore and solve simultaneous equations (where one is quadratic)

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

9.3.1 Explore and solve quadratic equations

- 9.3.1.1 Understand that all quadratics can be written in the form $a(x-h)^2+k$ (completing the square)
- 9.3.1.2 Understand that an equation written in the form $a(x-h)^2+k=0$ can be solved (where $a>0$, $h\geq 0$)
- 9.3.1.3 Understand that all quadratics can be written in the form $(ax+b)(cx+d)$
- 9.3.1.4 Understand that an equation written in the form $(ax+b)(cx+d)=0$ can be solved
- 9.3.1.5 Understand that the quadratic formula is a generalisation of the process of completing the square to solve $ax^2+bx+c=0$
- 9.3.1.6 Identify efficient methods to solve a variety of quadratic equations

9.3.2 Use and understand the graphical features of quadratic relationships

9.3.2.1 Understand that the gradient of a quadratic curve constantly changes, and there is one point where it is zero (called a turning point)

9.3.2.2 Use the completed square form of a quadratic to locate the turning point

9.3.2.3 Estimate and interpret the area under a quadratic curve (including in context)

9.3.2.4 Understand that the solutions to $ax^2 + bx + c = 0$ lie on the intersection of $y = ax^2 + bx + c$ and $y = 0$

9.3.2.5 Understand the connection between the graphical and algebraic interpretations of roots and intersections

9.3.3 Work with quadratic inequalities

9.3.3.1 Understand that a quadratic splits a plane into three regions, and be able to define them

9.3.3.2 Solve algebraically inequalities of the form $(ax + b)(cx + d) >< 0$

9.3.4 Explore and solve simultaneous equations (where one is quadratic)

9.3.4.1 Recognise that the points of intersection of linear and quadratic graphs satisfy both relationships and hence represent the solutions to both those equations

9.3.4.2 Understand that there are either zero, one or two solutions to a set of simultaneous equations where one is linear and the other is quadratic

9.3.4.3 Use substitution to find the solution to a pair of simultaneous equations where one is linear and one is quadratic

Overview

Students first met quadratic expressions in Key Stage 3, but it is not until Key Stage 4 that they will have had the opportunity to explore quadratics in depth. The work in this core concept is likely to be students' first experience of conceptualising the structure of non-linear relationships, and they should be guided to understand the key properties that are specific to quadratics. By using quadratics as a step away from linearity, students should also begin to notice those features that are general to all functions, and those that apply only to particular functions.

This core concept draws on ideas that have been explored in both '*9.1 Exploring linear equations and inequalities*' and '*9.2 Exploring non-linear sequences*', offering students an opportunity to formalise their understanding of the structure of quadratics. Students will build on their experience of using the distributive law to expand quadratic expressions in Key Stage 3, to then rewrite expressions by factorising and completing the square. However, at Key Stage 3, the emphasis was on understanding that expressions with the same value can have different appearances. At Key Stage 4, students need to go deeper and understand how different representations of the same function can reveal different features of the structure. This is explored in more detail in the key ideas of 9.3.1 Explore and solve quadratic equations below.

Two overarching ideas running through the theme of *9 Sequences, functions and graphs* are functional variation and covariation which, broadly, describe the way in which the output varies with changes in the input. The use of the word 'variation' to describe the relationship between variables should not be confused with its use to describe strategic choices in task design, in the way that it is used in the guidance for the exemplified key ideas below. While these terms may not be used explicitly with students, it is important for future learning that a Key Stage 4 teacher has an awareness of these connected concepts and is able to view sometimes apparently discrete topics through this lens.

Functional variation and covariation are particularly useful concepts when considering the difference between quadratic and linear functions. The term 'functional variation' relates to the understanding that a variable changes smoothly and continuously. This means that x can take any value and can also increase or decrease from that value by any increment. 'Functional covariation' is the awareness that, as x changes, y simultaneously changes and this change, for certain functions, is also smooth and continuous. The key difference between the quadratic functions of this core concept and the linear functions in '*9.1 Exploring linear equations and inequalities'* is the nature of this covariation. In linear functions, the covariation of linear functions is constant – that is, for every change in one value, the other value will change by a consistent amount. However, in non-linear functions such as the quadratics explored here, the nature of the covariation is continually changing (except at the turning point), lending the graph of the quadratic function its distinctive parabolic shape.

This core concept therefore represents a turning point in students' understanding – where ideas within different curriculum strands come together, and students begin to develop the functional thinking that is explored in the latter part of this theme. It is important that teachers are aware of this horizon knowledge, and how it could be developed further at Key Stage 5, so that students can secure solid foundations on which to build future learning.

Prior learning

Over Key Stage 3, students formalised the algebraic thinking that they began at primary school. Students need a secure grasp of algebraic notation and manipulation to be successful at Key Stage 4, and time securing these foundations of algebra supports students with the demands of this topic. In the NCETM Key Stage 3 PD materials, number and algebra are modelled together. This supports students and teachers to appreciate that algebraic manipulation is a generalisation of the structure of the number system, rather than a separate area of mathematics. These connections should continue to be emphasised and drawn out throughout new teaching at Key Stage 4. There are strong connections between Core Concepts 9.3 and 9.2, and progress with quadratic equations should inform students' understanding of non-linear sequences, and vice versa.

When manipulating algebraic expressions, students draw on their experience of number, including operating with negative numbers, finding factors and multiples, and operating with fractions. Fluency in these areas provides a secure basis on which to build algebraic understanding. Before starting new learning at Key Stage 4, students should be able to simplify expressions by collecting like terms. Noticing that quadratic and linear terms in the same variable are distinct but linked will be especially important when considering quadratic expressions.

Students should understand how to use the distributive law both to multiply an expression by a term and to factorise expressions where there is a common factor. Many will also have used the distributive law to find the product of two binomials, which is a key learning point in this core concept. This must be fully secure before students can consider rewriting quadratic expressions in the forms $(ax + b)(cx + d)$ or $(x + a)^2 + b$. Students should be confident in manipulating algebra to solve linear equations or rearrange formulae, essential skills to build on when manipulating and solving quadratics at Key Stage 4.

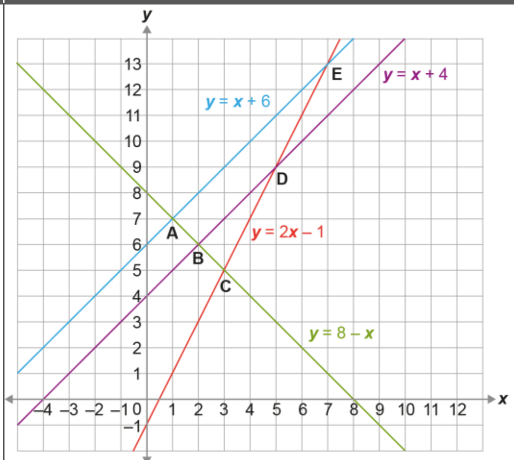
A deep understanding of graphical representations of linear equations, and the idea that the line represents the infinite points that satisfy the relationship described in an equation, should support this. Students should be clear how the elements of the graph of $y = mx + c$ relate to the equation. They should understand that the point at which a line crosses the x-axis represents the solution where $y = 0$. This key understanding will be developed further as students explore quadratics with zero, one or two solutions.

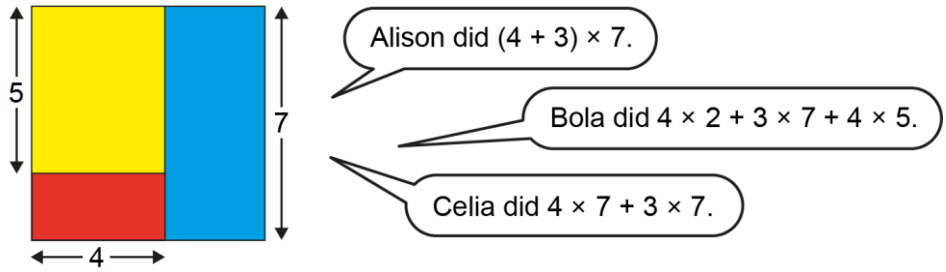
Students' work on simultaneous equations is linked to this. They should have begun to explore graphical representations of linear simultaneous equations, which extends to include one quadratic and one linear equation at Key Stage 4. They need to be aware that the intersection of two linear graphs satisfies both relationships and represents the solution to both those equations. This understanding will be further deepened through their work on inequalities and regions at Key Stage 4.

The first four themes of the Key Stage 3 PD materials all explore the prior knowledge required for this core concept in more depth. Particularly relevant are the core concept documents '*1.4 Simplifying and manipulating expressions, equations and formulae*', '*2.1 Arithmetic procedures*', '*2.2 Solving linear equations*' and '*4.2 Graphical representations*'.

Checking prior learning

The following activities from the NCETM Secondary Assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Key Stage 3 PD materials document '*4.2 Graphical representations*', Key idea 4.2.3.4, Example 4	 This graph shows the lines of these equations: $y = x + 4$ $y = x + 6$ $y = 2x - 1$ $y = 8 - x$ a) When $x = 1$, $y = 7$ is a solution to two of the equations. Which two? b) Two equations have the same value for y when $x = 5$. Which two equations? c) Write down the values of x and y that are a solution to both $y = x - 8$ and $y = 2x - 1$.
Key Stage 3 PD materials document '*1.4 Simplifying and manipulating expressions, equations and formulae*', Key idea 1.4.4.1 Example 1	Find the product of: a) $(x + 2)(x + 6)$ b) $(x + 3)(x + 4)$ c) $(x + 1)(12 + x)$
Key Stage 3 PD materials document '*1.4 Simplifying and manipulating expressions, equations and formulae*', Key idea 1.4.4.1 Example 4	Carol thinks that $(a + 3)^2 = a^2 + 9$. Is she correct?

Checkpoints *'Expressions and Equations'*, Activity G 'Same and different 2'	Look at these two expressions: $2a + b = 10$ $2a + b > 10$ a) What is the same? What is different? b) How many different values can you think of for a and b each time? Now look at these two expressions: $2a + 3 = 10$ $2a + 3 > 10$ c) How have these changed? d) How many different values can you think of for a each time? e) Is this more or less than before? Why?
Checkpoints *'Expressions and Equations'*, Checkpoint 16 'Square Numbers'	Three tiles are arranged to make a square. Alison, Bola and Celia want to find the area of the square.  Each person's method is correct. Describe how Alison, Bola and Celia each worked out the problem.
Checkpoints *'Expressions and Equations'*, Additional Activity H 'Moving twos'	Look at these three cards: a^2 $2a$ $a + 2$ a) How would you read each of the cards out loud? b) Find a value for *a* that makes the red (left-hand) card have the greatest value. c) Find a value for *a* that makes the blue (right-hand) card have the greatest value. d) Find a value for *a* that makes the green (middle) card have the lowest value.

Key vocabulary

Key terms used in Key Stage 3 materials

- associative
- binomial
- Cartesian coordinate system
- coefficient

- commutative
- distributive
- equation
- expression
- factorise
- formula
- gradient
- intercept
- linear
- quadratic
- simultaneous equations
- solution
- substitute/substitution
- unknown
- variable

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
completing the square	The process of writing a quadratic expression in the form $a(x+h)^2+k$. In this form, $a(x+h)^2$ is a perfect square and k is a constant that preserves the value of the original expression.
quadratic formula	The quadratic formula states that the solution(s) to any quadratic equation in the form $ax^2+bx+c=0$ are: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. It can be derived from the generalised form of completing the square.
region	An area of a graph where every point satisfies a certain condition. When represented graphically, the region is bounded by a solid line if it includes the points on the line, and a dashed line if it does not. For example: • The region to the left of the dashed line $y=x$ satisfies the inequality $y>x$, as, in every coordinate pair, the y-coordinate is greater than the x-coordinate. • The region to the right of the solid line $y=x$ satisfies the inequality $y\le x$, as, in every coordinate pair, the y-coordinate is less than or equal to the x-coordinate.
root	The roots of an equation are the solutions to that equation where $y=0$. When represented graphically, these are the points at which the line intersects with the x-axis. A linear equation will have one root, whilst a quadratic equation will have zero, one or two real roots.
turning point	The point at which the gradient of a curve is zero.

Knowledge, skills and understanding

Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

9.3.1 Explore and solve quadratic equations

In this set of key ideas, students will explore the various forms of quadratic expressions and consider how they can be used to find solutions to equations where the expression is made equal to zero. The focus is particularly on the solution set, and the understanding that having a single solution is a feature of linear equations, not equations more generally, and that for a quadratic equation equal to zero there may be zero, one or two solutions. While the graphical representation of quadratics is explicitly taught in the next set of key ideas, it is likely that reference to the quadratic graph, and particularly the number of intersections of the curve with the x-axis, will be helpful in making sense of the number of solutions.

Manipulating a quadratic expression, for example by completing the square or factorising, can allow students to appreciate the different properties that these different forms reveal. Students will be used to different forms of the same quantity or expression representing the same thing. However, in the case of the mathematics described here, the range of different forms is fundamental for appreciating the different aspects of the function that they reveal.

When a quadratic equation is written in a factorised form, such as $y = (x + 1)(x + 3)$, it is easy to discern the real roots of the equation, where the curve crosses the x-axis. When the same equation is written in completed square form, as $y = (x - 1)^2 - 4$, it is easy to find the turning point of the graph. The y-intercept is easiest to identify when the equation is written in perhaps its most familiar form, $y = x^2 - 2x - 3$, or when the x^2 and x terms are factorised: $y = x(x - 2) - 3$. Depending on the equation itself, each of the above forms could be considered useful for finding a value of y for a given value of x. Students may have a personal preference, but it is important for them to understand that manipulating a quadratic equation in this way is not arbitrary, and that the choice of form is driven by purpose.

- 9.3.1.1 Understand that all quadratics can be written in the form $a(x - h)^2 + k$ (completing the square)
- 9.3.1.2 Understand that an equation written in the form $a(x - h)^2 + k = 0$ can be solved (where $a > 0$, $b \geq 0$)
- 9.3.1.3 Understand that all quadratics can be written in the form $(ax + b)(cx + d)$
- 9.3.1.4 Understand that an equation written in the form $(ax + b)(cx + d) = 0$ can be solved
- 9.3.1.5 Understand that the quadratic formula is a generalisation of the process of completing the square to solve $ax^2 + bx + c = 0$
- 9.3.1.6 Identify efficient methods to solve a variety of quadratic equations

9.3.2 Use and understand the graphical features of quadratic relationships

A graph is a spatial representation of a function, so the spatial features of the graph correspond to features of the function itself. Similarly, the algebraic representation will have features that correspond to the graph. It is important that students are supported to make these links between corresponding features. They will primarily have worked with linear graphs up until this point in their mathematical education, although they may have had some experience of drawing and plotting quadratic curves. Working more deeply with graphs of quadratic functions offers students a step away from graphs of linear functions and an opportunity to develop their understanding of those aspects that are universal to all functions.

Key ideas when working with functions are that all points on the line represent solution pairs for the function, and that the line itself can be thought of as consisting of an infinite number of such points. In understanding some of the key points – the roots, the turning point – students come to understand the behaviour of the function as a whole. Ultimately, they should be able to see the graphical representation of a function as a single mathematical object which captures its features and be able to sketch it without recourse to a table of values. While it is still important that students are able to plot points using a set of coordinate pairs, they should also develop an understanding of the general features of a quadratic graph and how to determine the specific points for the graph under study.

9.3.2.1 Understand that the gradient of a quadratic curve constantly changes, and there is one point where it is zero (called a turning point)

9.3.2.2 Use the completed the square form of a quadratic to locate the turning point

9.3.2.3 Estimate and interpret the area under a quadratic curve (including in context)

9.3.2.4 Understand that the solutions to $ax^2 + bx + c = 0$ lie on the intersection of $y = ax^2 + bx + c$ and $y = 0$

9.3.2.5 Understand the connection between the graphical and algebraic interpretations of roots and intersections

9.3.3 Work with quadratic inequalities

Building on students' experiences with simple linear equations in Key Stages 3 and 4, and with linear inequalities in core concept 9.2, this set of key ideas explores quadratic inequalities. Students should understand how their solutions can be determined and how their graphical representations are connected to the symbolic. It is likely to be the first time students will have considered non-linear inequalities and their meaning, and so explicit reference will need to be made to what is the same and what is different when working with linear and non-linear inequalities. In particular, students need to be clear that the plane is, again, split into three regions where all points are either greater than, equal to, or less than the specified quadratic. By focusing on these points of similarity, students should be able to connect and apply their previous learning on graphical representations of inequalities, rather than seeing these key ideas as a particularly significant area of new learning.

9.3.3.1 Understand that a quadratic splits a plane into three regions, and be able to define them

9.3.3.2 Solve algebraically inequalities of the form $(ax + b)(cx + d) >< 0$

9.3.4 Explore and solve simultaneous equations (where one is quadratic)

Students should be aware that a quadratic equation such as $y = (x - 3)(x + 2)$ can be represented by a curve and that every point on that line fits the given relationship. Understanding that this line is continuous and not split into sections or limited to integers is a key awareness for students, as is understanding the continually changing nature of the covariation between x- and y- values.

In the statements of knowledge, skills and understanding outlined in 9.1.3, students learnt that there is only one point that satisfies both equations when working with a pair of linear simultaneous equations. However, when working with linear and quadratic functions together, they discover that there is a range of possible situations: there could be zero, one or two points that satisfy both equations. It is significant that that the correct answer could be that there is no possible answer. It might be the first time that some students have encountered an impossible question in the context of algebra. This might challenge a belief about mathematics, that there's always one single correct answer. This should be considered carefully and be made clear.

Draw students' attention to the fact that graphical representations like this are useful, but an exact answer cannot always be ascertained by only drawing the graphs. Students need different strategies for finding this value that is common to both functions.

9.3.4.1 Recognise that the points of intersection of linear and quadratic graphs satisfy both relationships and hence represent the solutions to both those equations

9.3.4.2 Understand that there are either zero, one or two solutions to a set of simultaneous equations where one is linear and the other is quadratic

9.3.4.3 Use substitution to find the solution to a pair of simultaneous equations where one is linear and one is quadratic

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

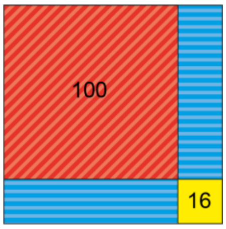
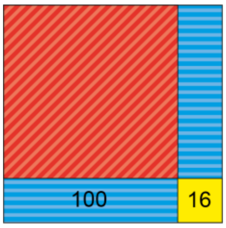
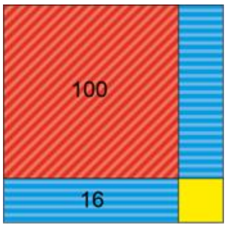
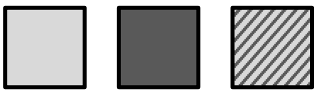
Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

9.3.1.1 Understand that all quadratics can be written in the form $a(x-h)^2+k$ (completing the square)

Common difficulties and misconceptions
It is fundamental to this key idea that mathematical notation allows for objects with the same value to be represented and symbolised differently. For example, the expressions $x+x+1+1$, $2x+2$ and $2(x+1)$ all look different but, for a given x, have the same value. This understanding (sometimes summarised as 'same value, different appearance') is fundamental to working with quadratic functions where different representations of the same relationship give different insights into that function. A secure understanding of the area model for multiplication is also helpful to enable students to fully conceptualise completing the square. Students will need to be able to relate the area model to factorising and expanding expressions, and link this to the new learning when completing the square. If students are familiar with algebra tiles, these may be used to complement the representations used here. The NCETM has produced a series of professional development guidance documents to support this, *Using mathematical representations at KS3*, which includes specific guidance on the use of algebra tiles. This can be found on the NCETM website.

Students need to	**Guidance, discussion points and prompts**
Understand that any square can be deconstructed into two smaller squares and two congruent rectangles. *Example 1:* *These squares are made up of a red diagonally-striped square, a yellow square and two congruent blue striped rectangles. Some of the areas are given in each situation.* *Find the dimensions of each of the smaller shapes:* *a)*  *b)*  *c)* 	*Example 1* is useful for students who are less familiar with the area model for multiplication. The intention is to support students to make sense of the lengths and areas and the interconnections between them. This preparatory work introduces the construction and deconstruction of a square in such a way that this **representation** can be used to access and understand the structures underpinning the algebraic manipulation when completing the square. The **language** in the rubric for this and subsequent examples has been carefully considered to ensure clarity for all students, using language that should be familiar by Key Stage 4. For example, the different shapes in *Example 1* are identified using different adjectives, so that students with visual impairments are not disadvantaged by references just to colour. Before sharing these examples with your classes, reflect on the language used and consider your own students' needs: do any of the terms need revision before new learning is introduced? Before looking at the whole set of examples within this key idea, reflect on your current model for teaching it: • Do you have a departmental approach to completing the square? • Are there any tasks or chains of reasoning that your department feels all students should experience?
Example 2: *Perran has several square tiles, which are all the same size. There are three different colours: plain pale grey, plain dark grey and dark/pale grey stripes.*  *He rearranges the tiles to make a larger square, shown below:*	This example is very similar to *Example 1* but here the **representation** focuses on the striped tiles, and how this total number of tiles must be split into two equal pieces to maintain the dimensions of the large square. Part c then raises the idea that not all tiles will necessarily be used when making a square, and that an adjustment of some sort might be necessary. Discuss when (or if) students in your school meet the area model. Is it a consistent model of multiplication used across your department, or is it introduced when necessary for a particular class? Consider the benefits and drawbacks of weaving a model like this through your curriculum.

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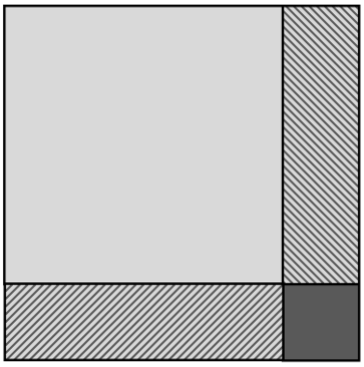

He uses 100 plain pale grey tiles and 40 of the striped tiles in total.

a) How many plain dark grey tiles does he use? How do you know?

b) He decides he wants the striped rectangles to be wider and buys a pack of 50 striped tiles. How many more plain dark grey tiles does he need to buy?
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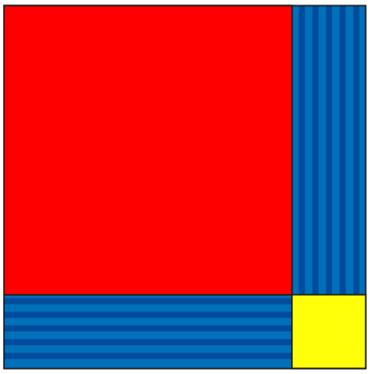
Example 3:

Yukiko and Xavier are asked to find the area of the square below.



Yukiko writes:

$$(7 \times 7) + (2 \times 3 \times 7) + (3 \times 3) = 100$$

Xavier writes:

$$(7 + 3)^2 = 100$$

a) Are they both correct?

b) Can you describe how each of them saw the problem?

c) If the sides of the larger, plain yellow square increase by one, how do Yukiko and Xavier's calculations change?
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One aspect of **variation** is considering multiple solutions to one problem. *Example 3* draws attention to calculations which have the same value but a different appearance. This is important to students' developing appreciation that completing the square allows particular features of the quadratic function to be seen and understood.

In *Example 3*, particular values are used, and students are asked to connect the symbolic **representation** and the pictorial representation. They should be supported to identify which values must correspond to which lengths – for example, noting that Yukiko has found the area of each composite shape separately, and so 7 and 3 must be the lengths of the larger and smaller squares respectfully. Part c starts to consider a more dynamic interpretation. This can be continued by asking questions about other possible changes. For example, how would the calculations change if the smaller, yellow square changed in size by one unit.
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Example 4:

Louie has 100 tiles in a square. He wants to rearrange the tiles into smaller squares.

He knows he can separate them so that he has 100 individual small squares.

a) What is the next largest number of equal-sized squares he can have? How big are they?

b) Are there any tiles left over?

c) How many 4×4 squares could he make? How many tiles would be left over?

Louie now wants to make exactly two squares, with as few tiles left over as possible.

d) How can he do this?

e) How do you know that you have found the best solution?

Throughout this task it would be useful to pay attention to the **representations** used. Are students accounting for every tile? Is a partitioned area model (like those used in the other examples) useful?

The five part-questions suggested here offer one potential route to explore possible squares, with the intention of laying the groundwork for the next stage in students' understanding (that any expression can be written as a square, with an adjustment). The number 100 was chosen as it is a familiar square with an even side length that is easy to partition. Other side lengths could be explored, including odd side lengths and very large side lengths, thus **deepening** students' understanding. This could then build to algebraic side lengths such as $x + b$.

Consider *Examples 1* to *4* as a sequence of tasks. Is this the most appropriate order for your students? What are the benefits and limitations of using *Example 4* after students have seen the area model explicitly referenced in *Examples 1* to *3*?

Notice that any value or expression can be written as a square ± an adjustment

Example 5:

Huey is making a mosaic. He has a pile of 534 square tiles. He wants to rearrang them to make a larger square.

He knows that $23^2 = 529$ and $24^2 = 576$.

*a) Can he arrange **all** 534 tiles into a solid square? Explain how you know.*

b) What is the biggest square that can be made from 534 tiles? How many tiles will be left over?

Della has a pile of 1 442 square tiles. Like Huey, she wants to arrange them to form a larger square.

*c) What would need to be true for her to be able to do this using **all** of her tiles?*

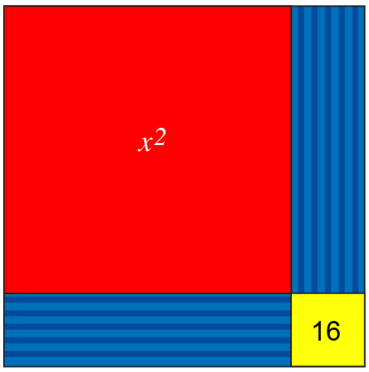
*Della than arranges **all** of her tiles to form the closest shape she can to a square. She writes $38^2 - 2 = 1442$.*

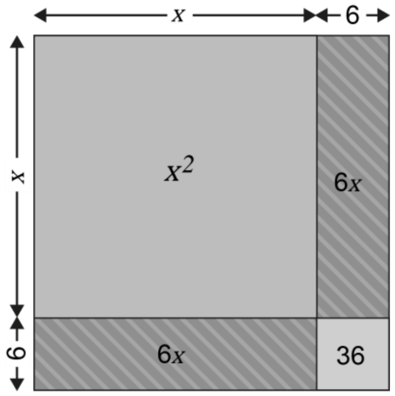
d) Describe what Della's square will look like. How do you know?

Finally, Della arranges her tiles to form the biggest square she can.

Examples 5 and *6* together move students towards the **representation** of a quadratic function written in completed square form, $a(x + h)^2 + k$. This algebraic structure should become familiar to students over the course of Key Stage 4.

A context is offered to draw students' attention to the square that is implicit in the notation, but is often missed as they work to recall a method. Mentally recalling and manipulating a square allows them to work with the structure of the algebra and manipulate the symbols accordingly. Considering Della's two different 'incomplete' squares offers students insight into the structure of the completed square form, **deepening** their understanding of how the adjustment $(+k)$ can involve either addition or subtraction.

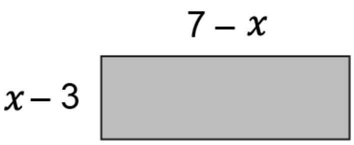
e) What would be the dimensions of her square? How many tiles will be left over? *f) Can any of Della's leftover tiles be arranged into squares? How many tiles would be left over?*	
Example 6: *Each of the expressions below represents sets of tiles, where x is a whole number and the expression as a whole is the total number of tiles:* *(i)* $x^2 + 2$ *tiles* *(ii)* $x^2 + 11$ *tiles* *(iii)* $(x + 2)^2$ *tiles* *(iv)* $(x + 2)^2 + 6$ *tiles* *For each set of tiles:* *a) Will you ever be able to arrange **all** the tiles into a solid square? How do you know?* *b) If part a was not possible, will you be able to arrange **some** of the tiles into a solid square? How big will the square be? How many will be left over?*	**Variation** is used here to give a context for students to think deeply about what a 'square' or a 'square number' can look like. It is offered first as x^2 and then as $(x + 2)^2$, to give an opportunity to connect square numbers and geometric squares. This example sets the foundations for completing the square and allows students to begin to connect algebraic manipulations and geometry, which is explored further in the next set of examples. In general, any expression of the form $ax^2 + bx + c$ can be written as a square with an adjustment, (i.e. in the form $m(x \pm p)^2 \pm q$) which is important to understand when completing the square. It may help students to fully appreciate how the **representation** of the tiles relates to this mathematical structure if they have physical tiles to manipulate. In both *Examples 5* and *6*, the **language** of 'solid square' is used. this might seem like an oxymoron, but it is intended to ensure students understand that their tiles need to be arranged to form an *area,* rather than a *perimeter*. Consider whether to address this ambiguity with students ahead of them doing this task, or whether to only explain if clarification is requested.
Connect the algebraic and pictorial representations of completed squares *Example 7:* *Four tiles (two differently-sized squares, and two congruent rectangles) are rearranged to form a larger square, as shown below with some of the areas labelled.*  *Find the dimensions of the red and yellow squares and of the two congruent stripy blue rectangles.*	The **representation** here revisits *Example 1* from this key idea, but uses a general square of size x rather than the particular dimensions used earlier. You might choose to combine this example with *Example 1* and focus on this case as a generalisation of the situation. What might inform the choice of where this task is used in a sequence? Taking this set of key ideas as a sequence, there is a pattern of mathematical experience that is intended to be coherent for students. For example, earlier examples in this key idea use 'friendly' values. *Example 4* uses particular values to build familiarity with the questions being asked, and large enough that students will pay attention to the structure rather than calculating. *Example 6* uses letter symbols to generalise, and this is continued in *Example 7*. This task structure is sometimes referred to as 'particular, peculiar, general'. Consider other times where this task structure is useful in leading students towards a generalisation.

Example 8: *Below is a square made from four tiles.*  a) *Describe how the expression $(x+6)^2$ can represent the area of this square.* b) *Describe how the expression $x^2 + 12x + 36$ can represent the area of this square.* *Look at the equation below.* $(x+6)^2 = x^2 + 12x + 36.$ c) *Do you agree that the equation, which connects both expressions, is true? How do you know?* *Look at the equation below.* $(x+6)^2 + 1 = x^2 + 12x + 37.$ d) *Do you agree that the equation is true? How do you know?*	As with *Example 3*, this makes explicit the connections between the symbolic **representation** and the pictorial. In working between these representations, students develop a deeper understanding by integrating them into a coherent mental model. Parts a and b particularly stress the connection between these two representations of the relationship. It is important that students are given time to describe this connection. In part c students may shift to a learnt method and expand the brackets to show equivalence between the expressions. It is important again to ensure that this equivalence is also recognised in the pictorial representation.
Example 9: a) *Expand the following squares:* *(i)* $(x+1)^2$ *(ii)* $(x+2)^2$ *(iii)* $(x+3)^2$ b) *What do you notice about the coefficients of x?* c) *Predict the next three expansions in the sequence and then try them. Were you right?* d) *What can you say about the expansion of $(x+a)^2$?* e) *Expand the following squares:* *(i)* $(x-1)^2$ *(ii)* $(x-2)^2$	Through the **variation** of these examples, students are guided to notice that the x coefficient in a perfect square is double the constant in the bracket. They should also notice that the numerical term is also positive. From this, expanding $\left(x+\frac{b}{2}\right)^2$ could link prior knowledge to completing the square. This relationship can be easier to access than the inverse, but early experience of expanding can make it easier for students to see the algebraic relationships necessary for completing the square. There are many different routes that this task could take next to highlight certain features of the mathematical structure. By carefully choosing the sequence of expressions to expand, teachers can emphasise different aspects of the relationship. *Example 10* offers one possible next step, which focuses on the adjustment to the numerical term, but teachers may want to choose their own next steps. For example, to next explore different coefficients of x by expanding expressions such as $(2x+1)^2$. A useful professional

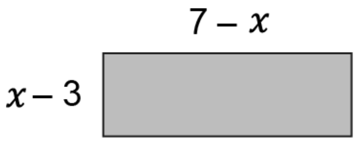
(iii) $(x-3)^2$ *f)* *What is the same and what is different about your answers to part e and your answers to part a?*	development exercise could be to create and compare different sequences of questions.
Example 10: *a)* *Expand the following:* *(i)* $(x+1)^2-1$ *(ii)* $(x-2)^2-4$ *(iii)* $(x+3)^2-9$ *(iv)* $(x-7)^2-49$ *(v)* $(x+12)^2-144$ *b)* *What do you notice about your answers to part a?* *c)* *Use your answers to part a to fill in the gaps below:* *(i)* $x^2+10x=(x+\_)^2-\_$ *(ii)* $x^2+20x=(x+\_)^2-\_$ *(iii)* $x^2+2000x=(x+\_)^2-\_$ *(iv)* $x^2+ax=(x+\_)^2-\_$	*Example 10* is designed to be used after experience of *Example 9*, with the **variation** this time drawing attention to the adjustment made to the constant term in the expression by subtracting a numerical value. In this case, all of the subtractions entirely cancel out the numerical term. This is intended to help students appreciate the structure of the two expressions, and particularly how the x term and numerical term are generated. The **language** that students are confident in using can make a huge difference to how clear and understandable teachers' explanations are. Students should be familiar with vocabulary such as 'coefficient' and 'constant' to ensure that it is clear to them which terms are being referred to.
Example 11: $(x+17)^2=x^2+34x+289$ *Which of equations A to F are also true?* *A* $(x+17)^2+10=x^2+34x+299$ *B* $(x+17)^2-100=x^2+34x+189$ *C* $(x+17)^2+a=x^2+34x+289+a$ *D* $(x+18)^2=x^2+34x+290$ *E* $(2x+17)^2=2x^2+34x+289$ *F* $2(x+17)^2+1=2x^2+68x+580$	*Example 11* uses a given anchoring point and then **variations** to explore what transformations of that equation are valid if equivalence is to be maintained. The focus here is again on understanding 'same value, different appearance', which is fundamental when working with quadratic functions. Having explored the 11 examples in this key idea, consider again the PD prompt from *Example 1:* • Do you have a departmental approach to completing the square? • Are there any tasks or chains of reasoning that your department feels all students should experience? What is the same and what is different in your views? Has anything changed in your approach?

9.3.1.4 Understand that an equation written in the form $(ax + b)(cx + d) = 0$ can be solved

Common difficulties and misconceptions
At the heart of this key idea is zero-product property: students should already be familiar with the effect of multiplying by zero and that, if the product of two numbers is zero, then one of them must be zero. They need to grasp the reasoning behind this idea and appreciate the particular property of zero (the multiplicative identity) and how it can be applied in this area of mathematics. They should not simply memorise a disconnected rule that is specific to this context, such as, 'If the product of two binomials is zero then either one of the binomials must be zero.' The reasoning supports students to develop a deep and connected understanding, where established mathematical structures are used in different contexts, rather than each context requiring a new series of paradigms.

Students need to	Guidance, discussion points and prompts
Understand that, for the product of two numbers to be zero, at least one of them must be zero *Example 1:* *The rectangle below has sides of length $x - 3$ and $7 - x$. In this case, $x = 4$.*  a) *If the rectangle starts with $x = 4$ then changes so that $x = 5$, has the area increased, decreased, or stayed the same?* b) *If the rectangle changes again so now it goes from $x = 5$ to $x = 7$, has the area increased, decreased, or stayed the same?* c) *Think about the area of the rectangle as x changes from 0 to 10. Can you describe how it changes? Are there any particular points you are certain about?*	This task helps students to make sense of a factorised quadratic function where one of the brackets has a value of zero. The **representation** also supports them to visualise the smooth and continuous changes of the two lengths as the value of x changes. Discuss with students how they would describe the shape as the value of x varies. What changes make it taller? What changes make it wider? At what value of x is it a square? Consider the precision of **language** used here, and how this impacts on students' understanding. Where would you place part c in this sequence of questions with your class? Would it benefit students to access the question by discussing it, or is it better to work on parts a and b first? To then encourage students to connect the function and the area with a focus on zero, focus on $7 - x$ and ask how long it is when $x = 4, x = 3, x = 2$ etc., and then discuss whether any of these values are problematic for the height. Students should understand that no matter what the length, if the height is zero then the area is also zero.
Example 2: *For each of statements a to g, say whether they are always true, sometimes true, or never true. Give reasons for those you have said are always or never true. Where you have said sometimes true, state under what conditions they would be true.* a) $p \times 0 = 0$	The questions in *Example 2* are best tackled through discussion. As students progress through the statements, it is important that they can identify and offer reasons for their answers. In the case of statements that are sometimes true, they should give a mathematical explanation as to when the situation is and is not true. By working on mathematical **language** in this way, they will learn through their own speaking and by listening to others.

b) $0 \times (3a + c) = 0$ *c)* $x \times y = 0$ *d)* $x^2 = 0$ *e)* $(a - 6) \times (b + 1) = 0$ *f)* $(a - 6) \times (a + 1) = 0$ *g)* $(m^2 + 1) \times (n^2 + 1) = 0$	
Understand that, for the product of two expressions to be zero, at least one of them must be zero *Example 3:* *What value of x makes the equations in parts a to f true?* *a)* $8 - x = 0$ *b)* $x - 8 = 0$ *c)* $2x - 8 = 0$ *d)* $2(x - 8) = 0$ *e)* $x(x - 8) = 0$ *f)* $(x - 2)(x - 8) = 0$	*Example 3* uses **variation** to draw attention to the differences when working additively and multiplicatively. Parts a to c require students to make expressions that sum to zero, a key skill when solving quadratics in their factorised form. Part d then builds on this and combines it with the knowledge from *Example 2*: that for the product of two numbers to be zero, one of them must be zero. For parts e and f, there are two possible correct solutions. Although they are likely to have met this in several situations, some students may find it surprising that there is more than one correct answer.
Example 4: *In the equations in parts a to d, some of the symbols have been blanked out. Is it still possible to give at least one correct value of x for these equations?* *a)* $3x \times$ [redacted] $= 0$ *b)* $3x +$ [redacted] $= 0$ *c)* $(3x + 12) +$ [redacted] $= 0$ *d)* $(3x + 12) \times$ [redacted] $= 0$	*Example 4* builds on the **variation** used in *Example 3*, with a focus on what can and cannot be deduced through reasoning when situations are additive or multiplicative, and the result of the calculation is zero. The key learning is that, although statements can be made about the blanked-out expression in parts b and c (for example, the blanked-out section in part b must have a value equal to $3x$), it is not possible to assign a particular value to x in either case. In parts a and d, it is possible to give a value for x which makes the equation true, even without knowing the full expression.
Example 5: *Melanie says, 'If $(x - 5)\ (x + 7) = 1$, then either $x - 5 = 1$ or $x + 7 = 1$.'* *a)* *Is she correct? Explain why or why not.* *Asif says, 'If $(2x + 5)\ (3x - 1) = 5$, then one of the factors must be 1 and the other must be 5, because $1 \times 5 = 5$.'* *b)* *Is he correct? Explain why or why not.* *Seema says, 'If $(1 - 2x)(4 + 3x) = 0$, then one of $1 - 2x$ or $4 + 3x$ must be 0 because $0 \times 0 = 0$.'* *c)* *Is she correct? Explain why or why not.*	*Example 5* offers common misconceptions for students to reason with and unpack. The intention is **deepening** understanding of the zero-product property in the context of solving quadratics. Consider how this task is offered to students, and how they respond. A written response might allow for more formalised reasoning, but a discussion might be more time efficient. Offering all three parts at once may address misconceptions in one go, but there may be an advantage to using each part separately when issues arise. These are all decisions for teachers to take using their knowledge of the students in front of them, and of how this concept sits within their broader scheme of work. How might you offer this task to your class? How would you expect them to work on it?

Example 6: *The rectangle below has sides of length $x-3$ and $7-x$. In this case, $x=4$.*  *If $x=4$, the area of this rectangle is 3 units.* a) *Find another value for x that gives the same area.* b) *Sue says, 'If I know the area of the rectangle is 3, there's no easy way to find the value of x. It's just trial and error.' Do you agree with Sue?* *If $x=2$, the area of the rectangle is −5 units.* c) *Find another value for x that gives the same area.* d) *Ollie says, 'If I know the area of the rectangle is −5, there's no easy way to find the value of x. It's just trial and error.' Do you agree with Ollie?* *If $x=3$, the area of the rectangle is 0 units.* e) *Find another value of x that gives the same area.* *Charlie says, 'If I know the area of the rectangle is 0 it's easy to find the value of x.'* f) *Do you agree with Charlie?*	*Example 6* revisits and extends the **representation** used in *Example 1*, this time considering the numerical value of the area of the rectangle rather than only focusing on the way that it varies. The intention is to give a context for why the product of any number and zero is zero. Using the area model for multiplication, it is clear that any length of the rectangle is irrelevant if the height is zero (and vice versa). In parts a and b, students need to consider what side lengths will result in an area of 3, and then what values of x will lead to those side lengths. Finding integer factors of 3 leads to values of 1 and 3 for the terms in brackets, so x values of 4 or 6, as above. The **language** of trial and error is used by Sue and Ollie, but this is intended as a point for students to argue against. The method of searching for solutions by inspection, as they are actually doing, is an important skill when solving quadratic equations. In both Sue's and Ollie's examples, prime numbers have deliberately been chosen to limit the number of cases that students need to consider. This enables a contrast with Charlie's example in part f; here, ensure that students understand the importance that zero takes in supporting the easy calculating of the value of x, the key focus of this task. Parts c and d use the idea of a negative value for the area of the rectangle. Students may be familiar with the idea of negative areas through use of, for example, double-sided counters or algebra tiles. While it can be valuable for **deepening** students' understanding by seeing examples such as the one here, students may find it troublesome. This strange idea, that we can use a concept that can't be seen to build an understanding of something real, is important in higher mathematics.

9.3.2.5 Understand the connection between the graphical and algebraic interpretations of roots and intersections

Common difficulties and misconceptions

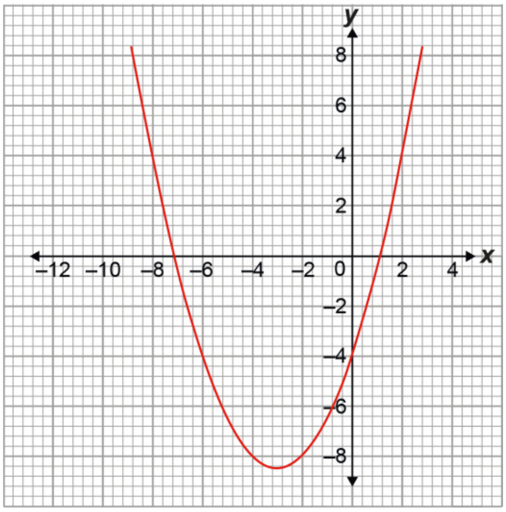
When students understand that coordinates represent solutions, they can interpret certain points on a graph and relate this to the algebra. For example, the fact that the point (3, 20) lies on the curve $y = x^2 + 3x + 2$ means that $x = 3$ is a solution to the equation $x^2 + 3x + 2 = 20$; by looking for another point where $y = 20$, students can find another solution to the equation.

Students need to

Understand that the coordinates of a point on a curve represent a solution pair to the equation that defines the curve

Example 1:

Below is the graph of $y = \frac{1}{2}x^2 + 3x - 4$.



a) *Calculate:* $\frac{1}{2} \times (-8)^2 + (3 \times -8) - 4$.

b) *How could you use the graph instead to find the answer to part a?*

c) *What other value of x has the same solution for* $\frac{1}{2}x^2 + 3x - 4$*?*

d) *Change only the **bold** numbers in the calculation below to write another calculation that gives the same answer:*

$\frac{1}{2} \times (-\mathbf{2})^2 + (3 \times -\mathbf{2}) - 4 = -8$

Guidance, discussion points and prompts

Example 1 draws attention to the relationship between the x value and the corresponding y value on a graph. Each point on the curve can be thought of as the result of a calculation. That calculation defines the invariant relationship between the x and y values. Here, rather than writing the equation in algebraic form and substituting in values, the whole calculation with relevant values is written out. This **representation** is used so that particular cases of this invariant relationship are made apparent, and students are able to generalise.

Consider extending and **deepening** this task by asking students to estimate the value of, for example, $\frac{1}{2} \times (-5)^2 + (3 \times -5) - 4$, or to write other calculations that they can calculate easily using this graph.

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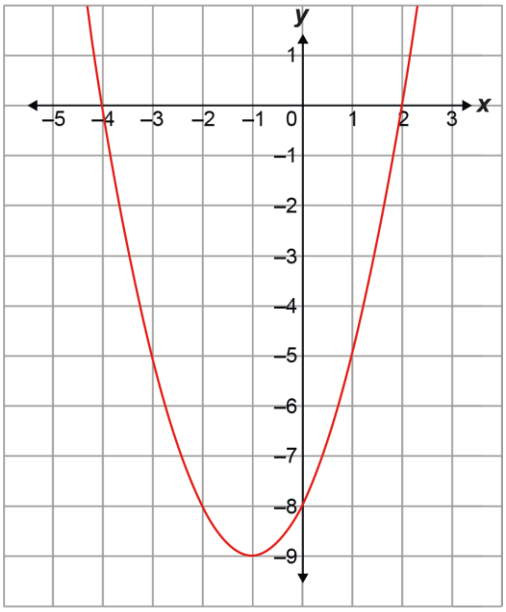
Recognise the importance of zero when working with graphs of quadratic functions

Example 2:

Below is the graph of $y = x^2 + 2x - 8$



Write the value or values of x *that make statements a to c true.*

a) $x^2 + 2x - 8 = -8$

b) $x^2 + 2x - 8 = -9$

c) $x^2 + 2x - 8 = 0$

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This task builds on *Example 1*, using a grid with simple values to raise students' awareness that the value of points on the line are solution pairs to the defining function. It is important to notice that using the graphical **representation** to read off solutions can only be an estimate. You could ask students to calculate particular values to check the accuracy of their answers. Draw attention to how the way a calculation is written can affect the ease with which it is carried out, which will help with preparation for *Example 3*.

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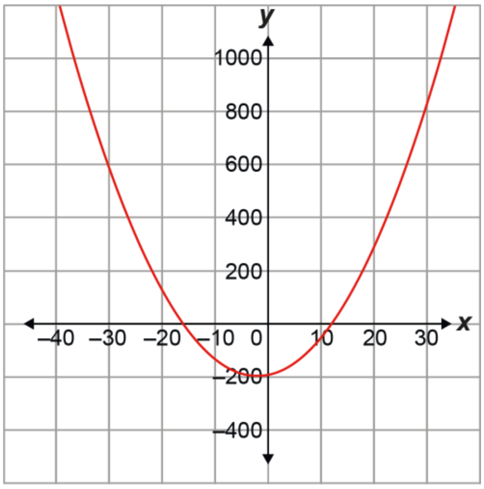
Example 3:

Below is the graph of $x^2 + 4x - 192$.



Write the value or values of x *that make the following statements true:*

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Example 3 uses the same structure as *Example 2*, but the equation has changed such that the answers to parts a to c can no longer be easily read from the grid. The **variation** between these two examples draws students' attention to the continuity of the line, and how there are infinite solution pairs. They are also likely to be more aware in this instance that their answers are an estimate. Ask them to calculate to check particular values.

Part e changes the **representation** of the function, showing it in factorised form. From their work in exemplified key idea 9.3.1.4, students should be aware of the particular property of zero, which means if $p \times q = 0$, then either $p = 0$ or $q = 0$ or both. They should be able to apply this knowledge to identify that writing the solution to $(x + 16)(x - 12) = 0$ is a relatively simple step, without any need to calculate.

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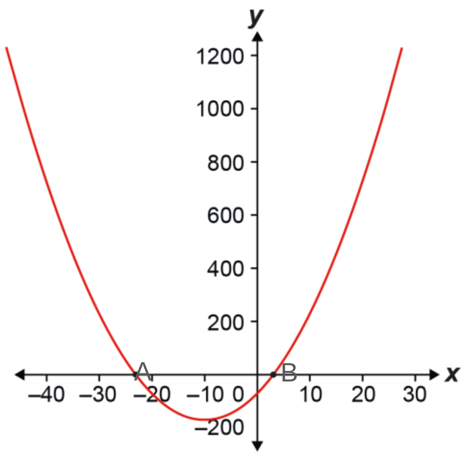
a) $x^2 + 4x - 192 = 600$ *b)* $x^2 + 4x - 192 = 20$ *c)* $x^2 + 4x - 192 = 0$ *d) How confident are you that your answers are accurate?* $(x + 16)(x - 12)$ *is the factorised form of* $x^2 + 4x - 192$. *e) Which of these calculations is it easy to write an answer for?* *(i)* $(x + 16)(x - 12) = 600$ *(ii)* $(x + 16)(x - 12) = 20$ *(iii)* $(x + 16)(x - 12) = 0$	
Example 4: *Below is the graph of* $y = (x + 23)(x - 3)$. *Explain how you can easily and accurately write down the coordinates of points A and B.* 	*Example 4* offers a simple prompt to help students to consolidate their developing understanding that the solution to an equation of the form $(x + a)(x + b) = 0$ will give the points of intersection on the x axis of the graph of the function $y = (x + a)(x - b)$. The **language** of ease and accuracy is used in the rubric, so students should identify that the graph is not necessary if the equation is presented in factorised form. Consider how students capture their explanation. What classroom structures do you use to support this sort of work? For example, is the accuracy of their answer improved if students have a paired discussion before writing? What about if students write first and then share some of the explanations before refining them?

9.3.3.1 Understand that a quadratic splits a plane into three regions, and be able to define them

Common difficulties and misconceptions

Students are likely to have experience from Key Stage 3 of drawing the graphs of quadratic equations by making a table of values and joining them with a curve, for example, $y = x^2 + 3x - 10$.

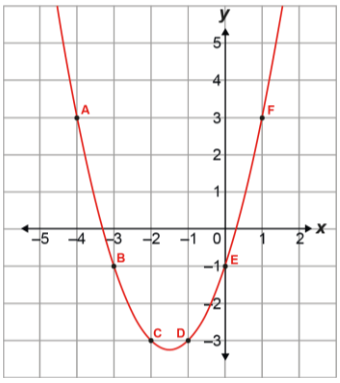
x	-5	-4	-3	-2	-1	0	1	2	3
y	0	-6	-10	-12	-12	-10	-6	0	8

When helping students to construct graphs, spend time ensuring they understand why the points are joined in a smooth and continuous curve. From working with a table of values, they might incorrectly see the curve as being composed of integer 'chunks'. Students should be made aware that it is more than just 'joining the dots' and, while the points shown in a table of values give us an impression of the shape of the curve, by joining the points we are using the fact that this relationship also holds *between* the points. Make sure students don't see the graph as simply plotted points connected by segments of a curve. Ensure that they are supported to see the curve as a whole object, valid for any point on the line, plotted or otherwise. To establish this principle, revisit ideas and techniques from Key Stage 3 through this lens.

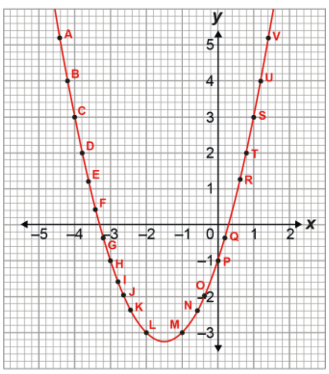
This awareness that the whole curve can be considered as points on the curve leads naturally to the understanding that the infinite points on this curve represent the solution set for the quadratic function, in this case $y = x^2 + 3x - 10$. The natural consequence is that the regions where $y < x^2 + 3x - 10$ and $y > x^2 + 3x - 10$ must be either side of this line, and students should have the opportunity to reason about this.

Students need to	Guidance, discussion points and prompts
Understand that the whole line can be considered as points on the curve *Example 1:* *Below are three different students' drawings of the graph of $y = x^2 + 3x - 1$.* *Rod says, 'I've marked **all** the points on the graph, and I've labelled them A to F.'* *Chima says, 'I've marked **all** the points on the graph, and I've labelled them A to V.'* a) *Have Rod or Chima marked all of the points on the graph? What is the same and what is different about the points they have marked?* *Freddie has used software to draw his graph. He says, 'There are no points for me to mark on my graph!'* b) *How is Freddie's graph different?* *Zippy says, 'I am certain that the coordinate (1 000, 1 002 999) is on the line in all of your graphs!'* c) *Explain how Zippy could know this. What does he mean by 'on the line'?*	*Example 1* focuses on the understanding that the graph of a function is constructed from an infinite number of points, each of which is a solution pair for that function. The **variation** here is that the function remains constant whilst the graphical representation changes. This draws attention to some possible misconceptions, particularly those arising out of the 'chunky' interpretation of the way that the function varies, by drawing attention to those points where the line meets the coordinate grid. Part c focuses on **deepening** students' understanding of the infinite nature of the function, and the invariant nature of the rule connecting the x and the y values at all points on that line. Very large numbers have been used to exemplify this here, but the point could equally have been made with other 'extremes' such as coordinates with lots of decimal places or negative values with large magnitude. The example here features a quadratic function, but the same principles could easily be modelled using any function, including linear functions (which students are already familiar with) and other non-linear functions (which they might not yet be). With your department, adapt this task to feature different types of functions. Consider how you might revisit it in these different contexts, to consolidate the learning at various points in your scheme of work.

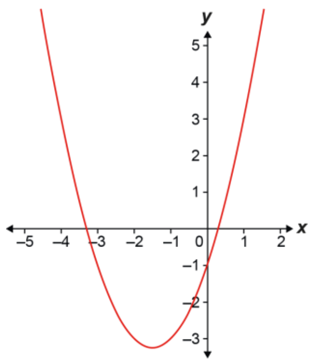
Rod's graph:



Chima's graph:



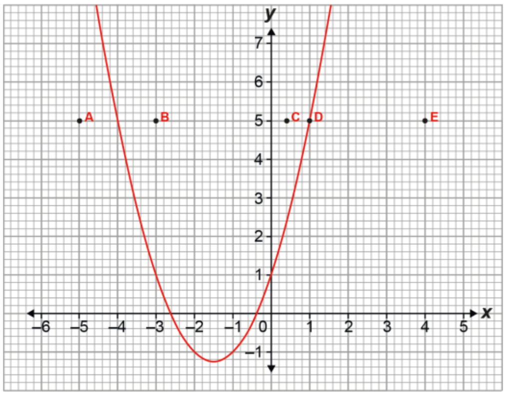
Freddie's graph:



Appreciate that a line graph identifies three distinct regions

Example 2:

The line below is the function $y = x^2 + 3x + 1$.



a) *Sort points A to E in the table below.*

b) *Where possible, write the coordinates for three more points in each category in the table.*

$y < x^2 + 3x + 1$	
$y = x^2 + 3x + 1$	
$y > x^2 + 3x + 1$	

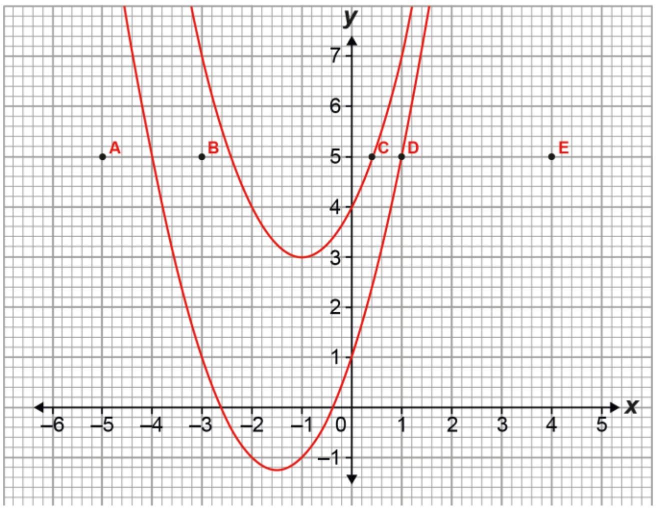
The line $y = x^2 + 2x + 4$ is drawn onto the same axes, right.

c) *Place the points A to E in the two-way table below*

d) *Where possible, write down the coordinates for three more points in each category.*

A key understanding is that the line of a function divides the plane into three distinct regions. The points in *Example 2* are all on the line $y = 5$. However, the equation is not the focus – instead, the focus is on how different points on that line are in different regions in relation to the quadratic function. That the points are the same 'height' exposes the misconception that, for example, if point A fits the inequality $y < x^2 + 3x + 1$ then point B can't be $y > x^2 + 3x + 1$ because they have the same y value. Part c then continues with **deepening** students' understanding by considering two lines, the relationship between them, and the regions they create.

In part d, the points of intersection of the two lines are not shown on this scale. Students might reason about these points and how to find them to complete the central cell in the two-way table. Prompt thinking by asking how many points there will be in that cell, and what students know about the x and y values of the coordinates of those points. Some students might reason that they can create equations to find these points (see exemplified key idea 9.3.4.2 below) but this is not the intention of this task. However, that is not to say that this same example cannot be revisited in the context of simultaneous equations! Revisiting the same example with a different learning focus can reinforce connections between different strands of the mathematics curriculum.



	$y < x^2 + 2x + 4$	$y = x^2 + 2x + 4$	$y > x^2 + 2x + 4$
$y < x^2 + 3x + 1$			
$y = x^2 + 3x + 1$			
$y > x^2 + 3x + 1$			

9.3.4.2 Understand that there are either zero, one or two solutions to a set of simultaneous equations where one is linear and the other is quadratic

Common difficulties and misconceptions

Some students say that they like or dislike maths because it's either right or wrong. Depending upon the order in which students encounter the key ideas within this document, students may not yet have encountered maths questions where the correct answer is that no answer is possible, or where there is more than one correct answer. Identifying whether or not a solution is possible is an important part of mathematics, and teachers should address that point directly with students. Take care to make sure students understand what it means when a problem has multiple (or no) solutions. It is important to relate this back to a context where the question is based on a real problem, particularly if this then means that a negative solution is not viable – such as in the context of solving a quadratic equation to find an area.

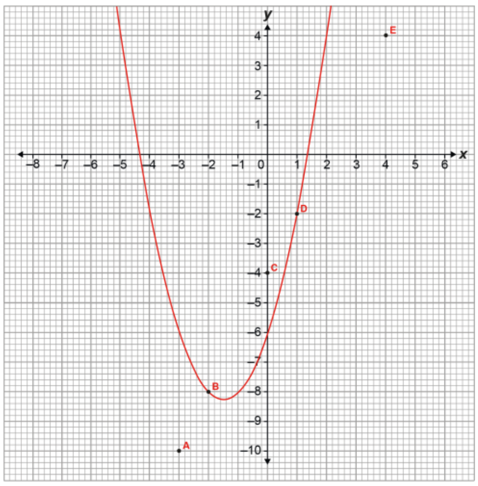
Students may find it challenging to visualise whether a quadratic and linear graph will intersect, if the scale of the graph means that the intersection is not visible in the range of the axes. Dynamic graphing software, on which students can 'zoom' in and out, can help support them to recognise where two graphs will **never** intersect and where two graphs have **not yet** intersected in the range shown.

Students need to

Appreciate that there can be up to two solutions to a pair of simultaneous equations where one is linear, and the other is quadratic

Example 1:

On the grid below is the graph of the function $y = x^2 + 3x - 6$.



Guidance, discussion points and prompts

Example 1 echoes the structure of *Example 2* in 9.3.3.1 above and this **variation** could be used to make connections. The focus in the earlier example is to understand the different regions created by the line. In *Example 1* here, students should understand that, for every value of x there is a unique value of y; and, of all the points that make up $y = 2x + 4$ and $y = x^2 + 3x - 6$, there are just two points that have the same value of x and y. Understanding that the y values are equal gives reason for the equating of the two expressions defining the line and ultimately the setting of the equation $x^2 + 3x - 6 = 2x - 4$.

It is fundamental learning in part c that the straight line can only intersect with the parabola a maximum of two times. This should be discussed and explored with students. A focus on **language** will support this and further develop understanding, as well as allowing misconceptions to be raised and challenged.

Mikey has drawn some points on the coordinate grid.

a) Sort points A to E in the table to the right.

$y < x^2 + 3x - 6$	
$y = x^2 + 3x - 6$	
$y > x^2 + 3x - 6$	

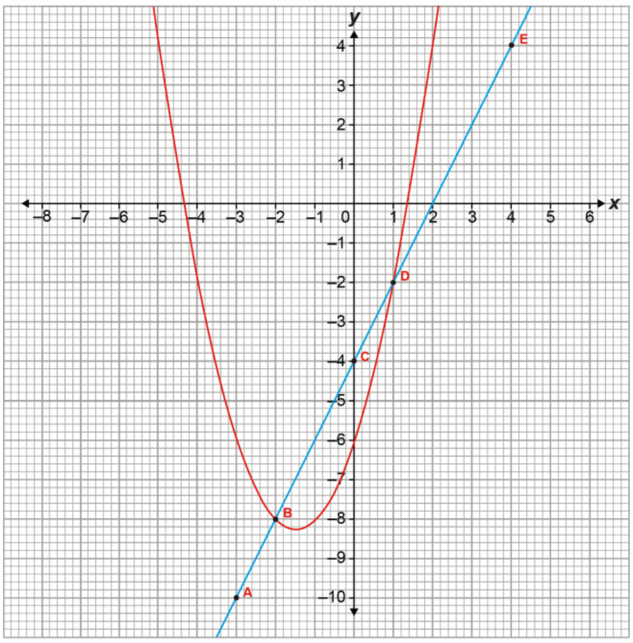
Davina notices that all of Mikey's points are on a straight line, $y = 2x - 4$.

She draws the line on the same axes, shown right.

b) Sort the points A to E into the two-way table below.

The points in the central cell of the table are ***simultaneously*** *solutions to the equations* $y = x^2 + 3x + 1$ *and* $y = 2x - 4$.

c) Explain how you know that there are no other coordinates that can be added to the central cell.



	$y < 2x - 4$	$y = 2x - 4$	$y > 2x - 4$
$y < x^2 + 3x + 1$			
$y = x^2 + 3x + 1$			
$y > x^2 + 3x + 1$			

Example 2:

On the grid below are the graphs of $y = 2x - 8$ *and* $y = x^2 - 2x - 6$.

Point B is on the line $y = 2x - 8$. *It can be seen that, when* $x = 3.4$, $y = -1.2$.

a) Write the value of $x^2 - 2x - 6$ *when* $x = 3.4$.

b) When $x = 0.6$, *what is the value of*

(i) $2x - 8$?

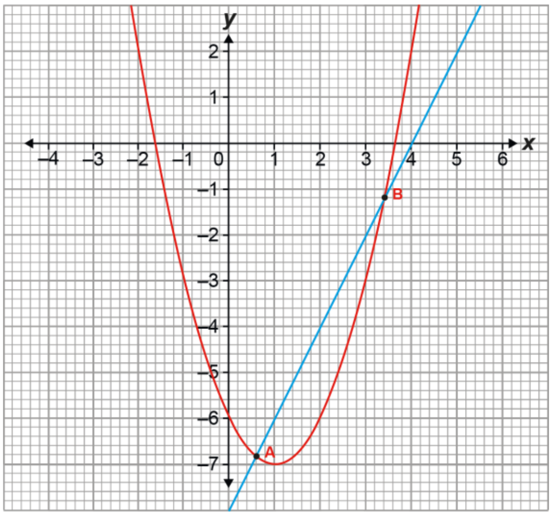
(ii) $x^2 - 2x - 6$?

Examples 2, 3 and *4* make explicit the connection between the points on intersection of the line and parabola, and the solution to the related simultaneous equations. *Example 2* focuses particularly on this connection and works with non-integer points to discourage students from substituting into the equations, and instead encourage them to use the graph. The connection between the graphical and the symbolic **representation** is explored and made explicit.

This is the first in a series of examples exploring the possible number of solutions to simultaneous equations (where one is quadratic and the other linear). Before using them with

c) *Explain how you know that there are two solutions to the simultaneous equations:* $y = 2x - 8$ *and* $y = x^2 - 2x - 6$.

students, consider the sequence with your department. Consider the thread of thinking that is being built through the gradual changes between each example. What ‘teacher moves’ might be needed to most effectively draw these ideas out within your classroom?



Example 3:

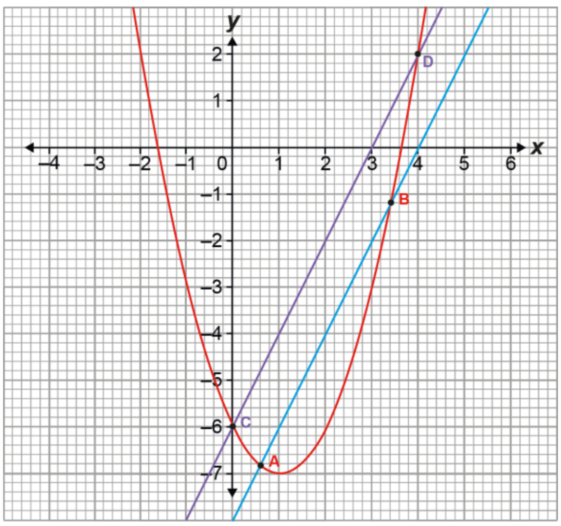
On the grid below are the graphs of $y = 2x - 8$ *and* $y = x^2 - 2x - 6$ *from Example 2. The graph of* $y = 2x - 6$ *has also been added.*

a) *Use the graph to write the solutions to the simultaneous equations* $y = 2x - 6$ *and* $y = x^2 - 2x - 6$.

Yan says, ‘When I’m solving simultaneous equations where one of the equations is linear and the other is quadratic, I can see that there will always be two pairs of solutions.’

b) *Do you agree? Why or why not?*

Example 3 part a uses the same diagram as *Example 2* but with an additional line, $y = 2x - 6$. The implicit **variation** in this series of examples draws attention to the features of simultaneous equations where one is quadratic – and, crucially, challenges students’ assumptions. Once students have encountered the parabolic curve of a quadratic graph, they may over-generalise and think that *all* quadratics have two solutions. The new line here also has two intersections with the curve, which does not challenge this view. However, as it is parallel to the original line, students may start to imagine other possible lines and realise that Yan’s assertion is not true. This is more explicitly explored in the next example.



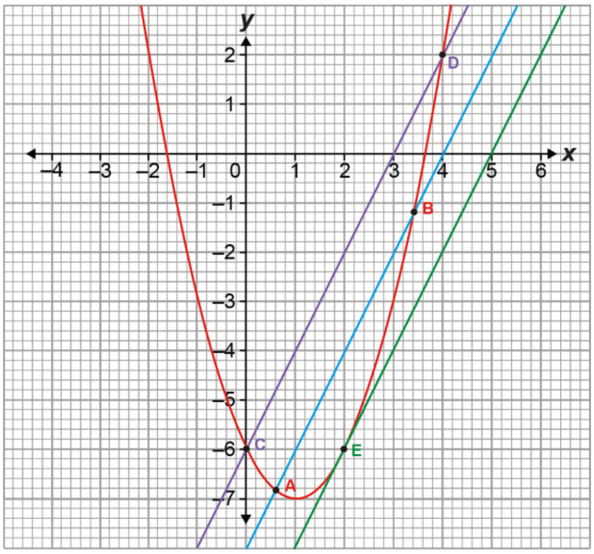
Example 4:

On the grid below are the graphs of $y = 2x - 6$, $y = 2x - 8$ and $y = x^2 - 2x - 6$ from Example 3. The graph of $y = 2x - 10$ has been added.

a) Do you agree that point E is a solution to both $y = 2x - 10$ and $y = x^2 - 2x - 6$? Why or why not?

In Example 3, Yan said, 'When I'm solving simultaneous equations where one of the equations is linear and the other is quadratic, I can see that there will always be two pairs of solutions.'

b) Use the information from part a to refine Yan's statement.



Example 4 builds on the previous two examples, this time offering evidence against Yan's assertion that simultaneous equations, where one is linear and the other quadratic, will always have two solutions. Students are unlikely to have met the tangent to a curve in a formal way but may have an intuitive sense that a straight line can 'rest upon' a curve at just one point. This is best explored through discussion and, in refining Yan's statement in part b, students will sharpen their mathematical thinking through using mathematical **language**.

Appreciate that it is possible for there be no solutions to a pair of simultaneous equations where one is linear, and the other is quadratic

Example 5:

Below is the graph from Examples 2, 3 and 4. As before, the quadratic line is $y = x^2 - 2x - 6$.

The solutions to Examples 2, 3 and 4 are still visible.

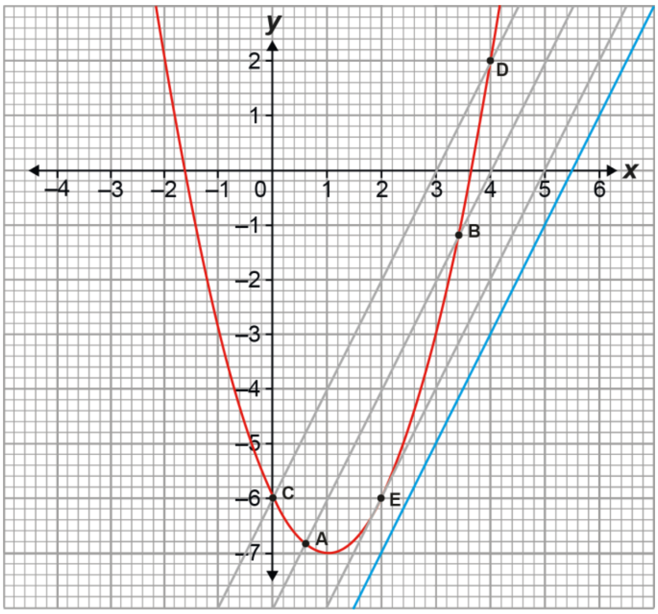
The new, blue line has equation $y = 2x - 1$.

Explain how this graph shows that there are no possible solutions to the pair of simultaneous equations $y = x^2 - 2x - 6$ and $y = 2x - 11$.

Example 5 uses the now-familiar graph from the previous three examples as an anchoring point. It now includes an additional line where it can be seen that there is no point of intersection between the linear and the quadratic equation. The **representation** allows for an understanding that the line and parabola will not meet. Discuss this further, as students come to understand that this means that the pair of simultaneous equations $y = x^2 - 2x - 6$ and $y = 2x - 11$ have no solution. Students can then manipulate the algebraic representation of the functions (as they may have done in *Example 2*).

We refer to there being 'no solution', but it would be more accurate **language** to refer to 'no real solution'. Those students who continue to study mathematics at Key Stage 5 will need to make this distinction, but it is not something that is generally considered as part of the Key Stage 4 curriculum.

The decision about whether to specify 'real' roots echoes decisions earlier in the curriculum. For examples, students may have been told that larger numbers 'cannot' be subtracted from smaller numbers, only to later encounter negative numbers. Throughout the NCETM PD materials, care has been taken to ensure accurate language is used to prevent future misconceptions such as this. However, in the case of imaginary numbers, which not all students will go on to experience, the introduction of a term that is not fully defined at Key Stage 4 may cause confusion. Discuss this with your colleagues to canvass departmental opinion.



Example 6:

The graph below shows the lines of these equations:

- $y = x^2 - 5x - 1$
- $y = -x - 1$
- $y = x - 1$
- $y = 2x - 14$
- $y = 3x - 17$

a) *When $x = 2$, $y = -7$ is a solution to two of the equations. Which two?*

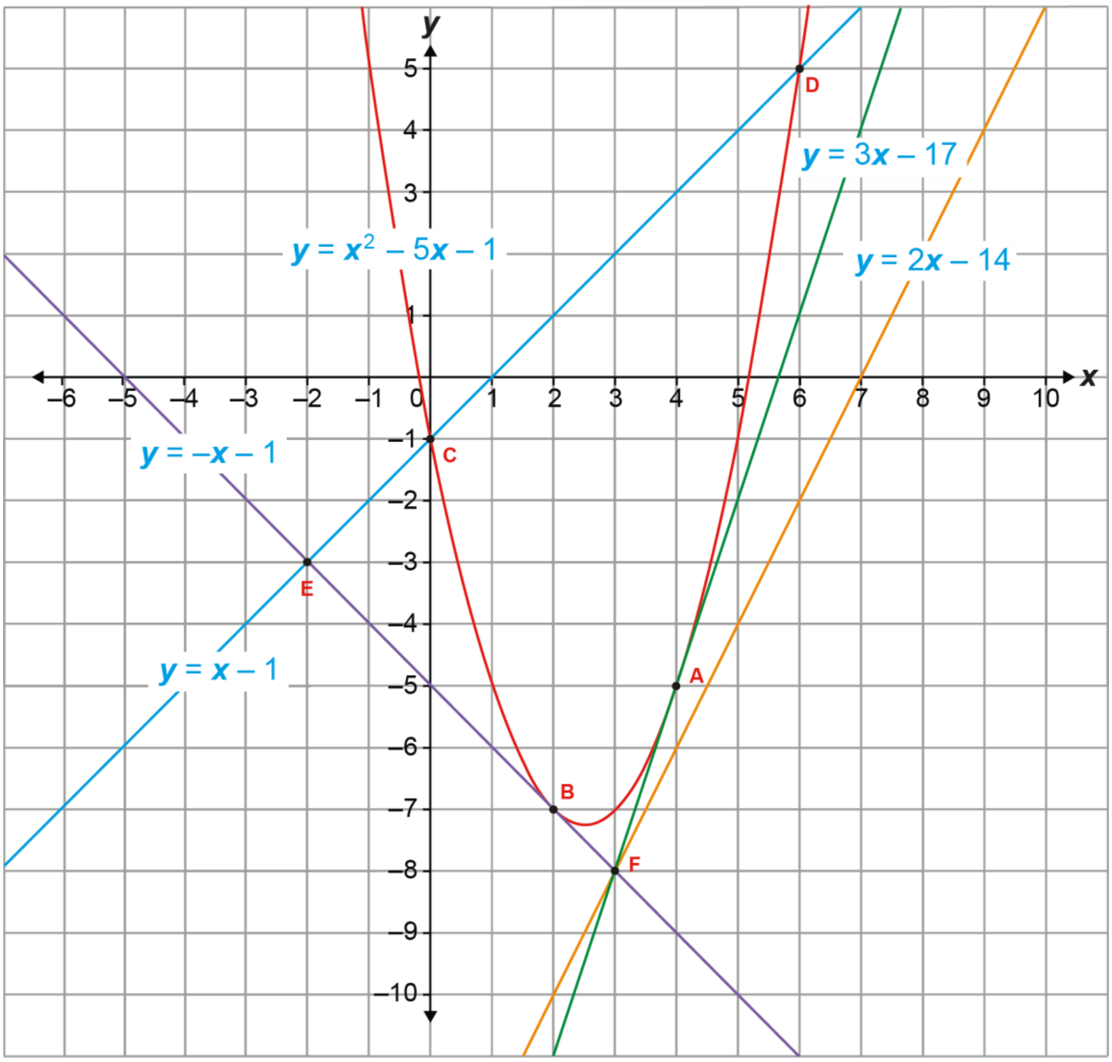
b) *Two equations have the same value for y when $x = 4$. Which two?*

c) *Write the values of x and y that are a solution to both $y = x - 1$ and $y = x^2 - 5x - 1$.*

Example 6 uses the same structure as the first example in 'Checking prior learning' above, which could be used as a precursor. The intention is to make explicit the connection between the graphical and the symbolic **representation** of the functions. By starting with the points and asking students to find the equations that match them, students will need to demonstrate a secure understanding rather than just applying a series of rote learned steps.

The questions draw attention to the intersections of the lines and parabola, **deepening** understanding of the properties of solutions to simultaneous equations. Useful discussions could be had about combinations of equations that do not intersect, and which of these will also *not* have a solution. Students should be able to articulate the difference between $y = 2x - 14$ and $y = x^2 - 5x - 1$, which will *never* meet, and $y = 2x - 14$ and $y = x - 1$, which just do not visibly meet on this graph. You could also explore other points of interest – particularly point F which is a solution to three of the linear equations simultaneously.

d) *Write the values of x and y that are a solution to both $y = 2x - 14$ and $y = x^2 - 5x - 1$.*	Reflect on the order in which you explore the examples in key ideas 9.3.2.5 and 9.3.4.2. The two ideas are entirely connected and therefore there is an argument for interleaving examples from each set of key ideas rather than tackling them separately.



Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 9 Sequences, functions and graphs* can be found here:

https://www.ncetm.org.uk/media/23eejt3r/ncetm_ks4_cc_9_solutions.pdf