Example questions, with some notes and solutions

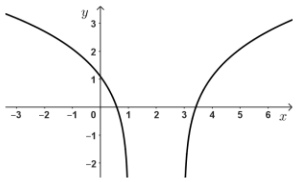
A MAT style multiple-choice question

The graph of

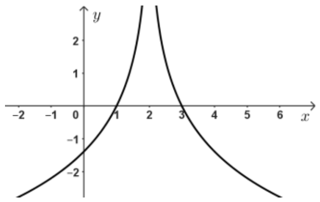
$$y = \ln\left(\frac{1}{x^2 - 4x + 3}\right)$$

is sketched in

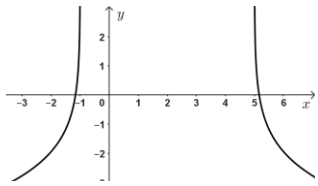
a)



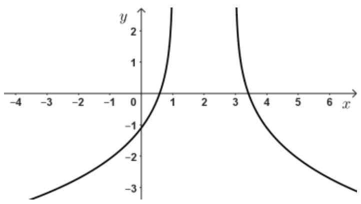
b)



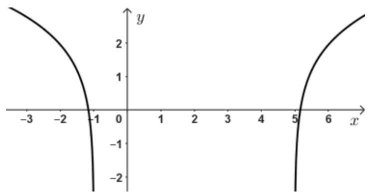
c)



d)



e)



Comments

Students may be intimidated by the complicated-looking composite function – many would want to reach for a graphing calculator, but calculators are not allowed in these tests!

With a fluent knowledge of

- the natural log function
- quadratics
- the reciprocal function

and some careful reasoning, it can be answered quite quickly.

Solution

$$\ln\left(\frac{1}{x^2 - 4x + 3}\right) = \ln\left(\frac{1}{(x-3)(x-1)}\right)$$

$(x-3)(x-1) < 0$ for $1 < x < 3$ so $\ln\left(\frac{1}{x^2-4x+3}\right)$ is undefined in this region and the graph of $y = \ln\left(\frac{1}{x^2-4x+3}\right)$ will not exist there. It will exist for $x < 1$ and $x > 3$. This narrows it down to a) and d).

As $x \to 1$, $x^2 - 4x + 3 \to 0$ from the positive direction so $\frac{1}{x^2-4x+3} \to +\infty$, $\ln\left(\frac{1}{x^2-4x+3}\right) \to +\infty$ so d) is the correct answer. This is quite nice as it links to the graph of $y = \ln x$ where, as x increases, y increases.

Another nice feature: since $y = x^2 - 4x + 3$ has reflection symmetry in $x = 2$, the graph of $y = \ln\left(\frac{1}{x^2-4x+3}\right)$ will also have reflection symmetry in $x = 2$.

A STEP style question (inspired by STEP 2 2021 Q3)

In this question, x, y and z are real numbers.

Let $\lfloor x \rfloor$ denote the largest integer that satisfies $x \leq x$ and let $\{x\}$ denote the fractional part of x, so that $x = \lfloor x \rfloor + \{x\}$ and $0 \leq \{x\} < 1$. For example, if $x = 6.8$, then $\lfloor x \rfloor = 6$ and $\{x\} = 0.8$ and if $x = -6.8$, then $\lfloor x \rfloor = -7$ and $\{x\} = 0.2$.

(i) Solve the simultaneous equations

$$\lfloor x \rfloor + \{y\} = 3.8$$
$$\{x\} + \lfloor y \rfloor = -5.3$$

(ii) Given that x and y satisfy the simultaneous equations

$$x + \lfloor y \rfloor = -7.6$$
$$\{x\} + \{y\} = 1.1$$
$$\lfloor x \rfloor + y = -7.3$$

Show that $x + y = -6.9$

(iii) Solve the simultaneous equations

$$2x + \lfloor y \rfloor + \{z\} = 8.9$$
$$2\{x\} + y + \lfloor z \rfloor = -1$$
$$2\lfloor x \rfloor + \{y\} + z = -1.3$$

Comments

This is interesting in that it doesn't require much "maths" to be thrown at it. It is mostly problem solving once the definitions of $\lfloor x \rfloor$ and $\{x\}$ are understood. Developing a clear understanding of previously unfamiliar definitions is vital for success in university-level maths.

It links nicely to simultaneous equations in that it is about what you are trying to do when solving them (find the values of x and y that 'work') rather than just applying a set method. That being said, it does respond helpfully to some of the standard techniques for solving simultaneous equations such as adding the equations together.

The last part has more than one answer, despite the equations being 'linear'. This is something A level students are unlikely to have encountered before.

Solution

(i) From $\lfloor x \rfloor + \{y\} = 3.8$, $\lfloor x \rfloor$ must be 3 and $\{y\}$ must be 0.8
From $\{x\} + \lfloor y \rfloor = -5.3$, $\lfloor y \rfloor$ must be negative ($\{x\}$ is defined as positive) and so $\lfloor y \rfloor = -6$ and $\{x\} = 0.7$

Hence $x = \lfloor x \rfloor + \{x\} = 3.7$ and $y = \lfloor y \rfloor + \{y\} = -5.2$

(ii) Adding the three equations gives $x + \lfloor y \rfloor + \{x\} + \{y\} + \lfloor x \rfloor + y = -7.6 + 1.1 - 7.3$
$2x + 2y = -13.8$ since $\lfloor x \rfloor + \{x\} = x$ and $\lfloor y \rfloor + \{y\} = y$
So $x + y = -6.9$

(iii) Adding all three equations gives $4x + 2y + 2z = 6.6$
So $2x + y + z = 3.3$ $(*)$

Adding equation (1) and (2) gives $2x + 2\{x\} + \lfloor y \rfloor + y + z = 7.9$ since $\lfloor z \rfloor + \{z\} = z$
i.e. $2x + y + z + 2\{x\} + \lfloor y \rfloor = 7.9$

Using $(*)$, $2\{x\} + \lfloor y \rfloor = 4.6$
From this $\{x\} = 0.3$ or $\{x\} = 0.8$ (since both would give rise to something point 6)
If $\{x\} = 0.3$, $\lfloor y \rfloor = 4$
If $\{x\} = 0.8$, $\lfloor y \rfloor = 3$ (so there are two possibilities for x and y)

Adding equation (1) and (3) gives $2x + 2\lfloor x \rfloor + y + \{z\} + z = 7.6$ since $\lfloor y \rfloor + \{y\} = y$
i.e. $2x + y + z + 2\lfloor x \rfloor + \{z\} = 7.6$
Using $(*)$ $2\lfloor x \rfloor + \{z\} = 4.3$
$\{z\} = 0.3$ since $2\lfloor x \rfloor$ is an integer.
This gives $\lfloor x \rfloor = 2$

Finally adding equation (2) and (3) gives $2x + y + \{y\} + \lfloor z \rfloor + z = -2.3$
since $2\{x\} + 2\lfloor x \rfloor = 2x$
i.e. $2x + y + z + \{y\} + \lfloor z \rfloor = -2.3$
Using $(*)$ $\{y\} + \lfloor z \rfloor = -5.6$
$\lfloor z \rfloor = -6$ and $\{y\} = 0.4$

There are two possible sets of solutions
$x = 2.3, y = 4.4, z = -5.7$
$x = 2.8, y = 3.4, z = -5.7$