MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Fractions

3.5 Working across one whole: improper fractions and mixed numbers

Teacher guide | Year 4

Teaching point 1:

Quantities made up of both wholes and parts can be expressed as mixed numbers.

Teaching point 2:

Mixed numbers can be placed on a number line.

Teaching point 3:

Understanding how to compare and order proper fractions supports the comparison and ordering of mixed numbers.

Teaching point 4:

Mixed numbers can be partitioned and combined in the same way as whole numbers.

Teaching point 5:

Mixed numbers can be written as improper fractions.

Teaching point 6:

Improper fractions can be added and subtracted in the same way as proper fractions.

Overview of learning

In this segment children will:

- be introduced to the use of fractions for quantities greater than one whole, initially as mixed numbers and then as improper fractions
- learn how to partition and combine fractional amounts greater than one whole
- continue to develop their understanding that fractions are numbers as well as operators
- solve addition and subtraction calculations involving fractional amounts greater than one whole.

This is the first segment where children are introduced to fractions for quantities that are greater than one whole. Children will initially access this concept by learning about mixed numbers, before progressing to looking at improper fractions. Throughout this segment, area models and number lines are used alongside each other, with children representing mixed numbers and improper fractions on both models. The decision to cover mixed numbers before improper fractions was taken because children will already be familiar with talking about mixed numbers in some contexts, such as *'I am eight and three-quarters'* or *'I've eaten one-and-a-half biscuits'*. They will also have a secure understanding of whole numbers (e.g. 2) and fractions less than one (e.g. $\frac{3}{4}$), so the new learning at this stage is simply the combination of these numbers to form a mixed number (e.g. $2\frac{3}{4}$).

After the introduction of mixed numbers, children will make the natural progression to improper fractions. Improper fractions are simply another way of expressing a mixed number, and so the equivalence here should be highlighted. Fluency in the times tables is essential in facilitating the conversion between mixed numbers and improper fractions (and vice versa). Converting an improper fraction to a mixed number draws very heavily on the concept of division with remainders. For example, 17 ÷ 3 gives five groups of three with two left over (five, remainder two), and $\frac{17}{3}$ is five groups of three-thirds, with two-thirds left over ($5\frac{2}{3}$). If children are not fluent in their times table facts, then much of their working memory will be occupied with performing the calculation, meaning they may struggle to focus on the concept of an improper fraction.

Having established a grounding in the concepts of mixed numbers and improper fractions, children will then apply their prior learning around addition and subtraction of whole numbers and proper fractions, to addition and subtraction of mixed numbers and improper fractions. Initially, the focus is solely on calculations that do not bridge whole numbers. Children are taught to use their knowledge of the composition of mixed numbers, and how these can be partitioned and combined, to support them in solving calculations. Once improper fractions have been introduced, further calculations that involve bridging whole numbers can then be taught.

Ensure children are comfortable with the fact that an improper fraction has a numerator which is greater than the denominator, whereas the proper fractions children have encountered so far have a numerator which is less than the denominator. While it will sometimes be important to refer to particular numbers using more precise language, such as fraction, mixed number, improper fraction or proper fraction, at other times do make sure that you refer to them simply as numbers. This will reinforce that – just as for whole numbers – fractions are numbers that can be positioned on a number line and used in calculation.

As in previous segments, it is important to continue to develop children's fraction sense. Number lines are used extensively across this segment to develop children's confidence in positioning fractions within the number system. This also includes estimating the position of mixed numbers on number lines that are only partially marked or labelled. The various methods for comparing mixed numbers that are covered in this segment will also aid children's development of fraction sense.

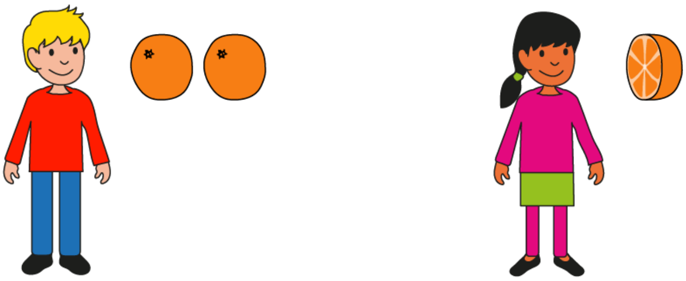
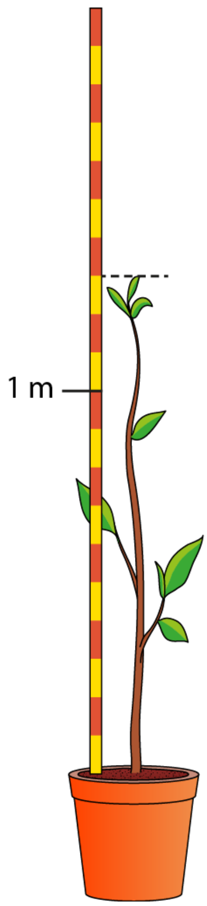
Three key models are used to help children visualise the concepts introduced within this segment: the area model, the number line and the part–part–whole model. The area models used in this segment are mostly in a linear form (similar to the bar model) or in a circular form (showing segments, similar to a pie chart). It is important that children are exposed to both of these forms of area model. The linear model is easier for children to draw, but does not clearly shown when fractional parts exceed one whole. Conversely, the circular model clearly shows when one whole has been made, but is difficult to draw accurately. The use of concrete manipulatives is crucial in ensuring area models can be understood and accessed by all; fraction tiles or a similar resource are useful for this. Number lines should be shown alongside area models as often as possible. A marked or unmarked number line is suggested at various stages, based on the focus of the question and whether the calculations bridge a whole number or not. The part–part–whole model is useful for encouraging children to connect back to previous addition and subtraction units, rather than perceiving fractions as stand-alone learning.

An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Teaching point 1:

Quantities made up of both wholes and parts can be expressed as mixed numbers.

Steps in learning

	Guidance	Representations
1:1	The purpose of this teaching point is to teach children to write numbers with both whole number and fractional parts as mixed numbers. This should be introduced to children through a familiar context, for instance: • *'Jonny has two oranges.'* • *'Ellen has half an orange.'* • *'How many oranges do Jonny and Ellen have altogether?'* Children may suggest various different ways that this could be described verbally. Allow them to share their ideas, and then explain that the correct way to describe it is *'two and a half'*, i.e. *'Jonny and Ellen have two and a half oranges altogether'*. Explain that this represents two wholes and one-half. Offer them a variety of opportunities to verbally describe other examples of mixed numbers in familiar but differing contexts, such as in the exemplars opposite. These could include length (for example, one and three-tenths of a metre), time (for example, two and a quarter hours), quantities (for example, four and two-thirds of a pizza), and so forth. When using this language, we will sometimes say, for example, *'two and a quarter hours'* or *'two and a half oranges'*, but in other instances we might say *'one and three-tenths **of a** metre'*, and *'four and two-thirds **of a** pizza'*. At this stage the most important thing is for children to become comfortable with the mixed number language of including both a whole	Real-life contexts: *'How many oranges do Jonny and Ellen have altogether?'*  • *'How tall is the plant in metres?'* 

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<td>number and a fraction; don't worry too much if they make a mistake with the precise use of 'of a'. However, do continue with the unitising language used in previous segments, for example alternating 'four and two-thirds' with 'four and two one-thirds'.

Children's ages provide a familiar context for discussing mixed numbers. Many children will be able to tell you that they are eight and one-half or eight and three-quarters. As you discuss ages more, they will enjoy working out their ages more precisely, for example eight and ten-twelfths or nine and a twelfth.

Provide examples where the fractional part is shown on the left-hand side of the whole part(s) or in the middle. Asking, 'What is the same? What is different?' will allow children to establish that the quantity is the same in each example. It should then be established that the most significant value (i.e. the number of wholes) should be said first, followed by the fractional part.</td>
<td>
'How long was spent reading?'
<table>
<tr><th>Day</th><th>Time spent reading</th></tr>
<tr><td>Monday</td><td>1 hour</td></tr>
<tr><td>Tuesday</td><td>1 hour</td></tr>
<tr><td>Wednesday</td><td>Quarter of an hour</td></tr>
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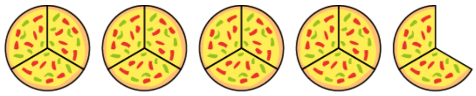
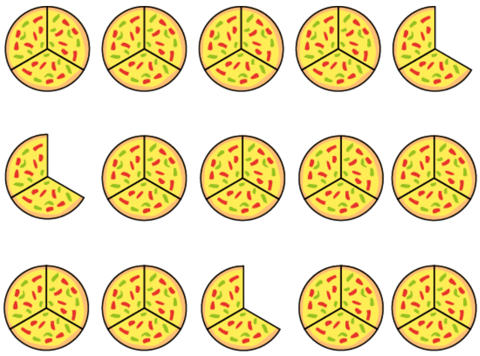
'How many pizzas are there?'

'What is the same? What is different?'

</td>
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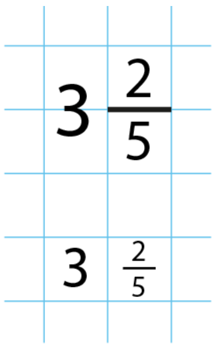
<tr>
<td>1:2</td>
<td>Once all children are confidently and accurately describing mixed numbers verbally, show them how mixed numbers should be written. First, as a school, establish how mixed numbers should be written in square-paper books in order to present a consistent approach across the school. The whole number could be written either spanning two squares (to show that it is separate from the numerator of the fraction) or the whole number could be written in one square, with the fraction in the adjacent square. See the examples opposite.

Use a part–part–whole model to show how the whole number and the fractional part are simply combined to</td>
<td>Options for recording on square paper:


</td>
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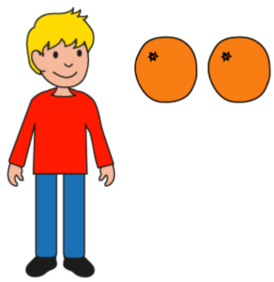
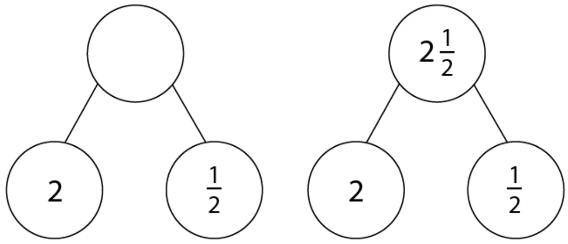
form a mixed number. Return to the images from the previous step and use them to help children practise writing mixed numbers.

As a result of verbalising and writing many different examples of mixed numbers, children should reach the point where they feel confident in the generalisation: ***'Quantities made up of both whole numbers and a fractional part can be expressed as mixed numbers.'***



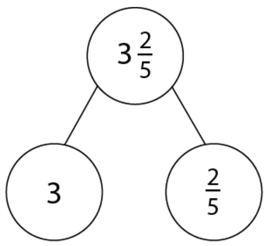


1:3 Once all children are secure in verbalising and writing mixed numbers, give them the opportunity to practise moving between the two forms. A simple activity to facilitate this practice is to write a mixed number in words or numerals on the board and ask children to read the mixed number aloud. Alternatively, say the mixed number aloud and ask the class to record it on paper or on whiteboards. As with previous work, ensure children are accurately expressing the fractional part, for example *'five and one-eigh**th**'*, not *'five and one-eigh**t**'*.

Teacher	**Pupil**
Write	**Write or say**
$5\frac{1}{8}$	*'five and one-eighth'*
Write or say	**Write**
'three and two-fifths' or *'three and two one-fifths'*	$3\frac{2}{5}$

1:4 Now link children's developing understanding of the composition of mixed numbers with addition and subtraction equations that express this composition.

In *Spine 1: Number, Addition and Subtraction*, children learnt that there are eight equations which can express the additive composition of a number into two parts. As a class, look at a mixed number represented in a part–part–whole model, and then write the equations which show how it can be split into a whole number part and a



$3\frac{2}{5} = 3 + \frac{2}{5}$ $\quad$ $3\frac{2}{5} = \frac{2}{5} + 3$

$3 + \frac{2}{5} = 3\frac{2}{5}$ $\quad$ $\frac{2}{5} + 3 = 3\frac{2}{5}$

$3\frac{2}{5} - 3 = \frac{2}{5}$ $\quad$ $\frac{2}{5} = 3\frac{2}{5} - 3$

$3 = 3\frac{2}{5} - \frac{2}{5}$ $\quad$ $3\frac{2}{5} - \frac{2}{5} = 3$

	fractional part. See the examples on the previous page. Use other mixed numbers and challenge children to practise constructing these equations for themselves.	
1:5	Once children have had ample practice writing additive equations to express the composition of a mixed number, proceed to addition and subtraction calculations. Present a range of addition equations, some with the whole number as the first addend and some with the fraction as the first addend. You might like to use the examples opposite as a starting point. Measurements provide a good context for fraction calculations. Note that measurements often use halves, quarters, fifths and tenths as the most common units for the fractional parts. These units also link to common intervals in graphing. To further deepen understanding of this concept, present dòng nǎo jīn problems like the one shown opposite.	Addition and subtraction calculations: $4 + \frac{2}{7} = \square$ $\frac{6}{9} + 5 \; \square$ $20 + \frac{8}{10} = \square$ $\frac{1}{15} + 7 \; \square$ $12\frac{2}{7} - 12 = \square$ $5\frac{1}{8} - \frac{1}{8} = \square$ $18\frac{3}{20} - \square = 18$ $\square = 3 - \frac{1}{4}$ Measurement context: • *'Mia walks $\frac{3}{4}$ km to her friend's house and then another 1 km to school. How far does Mia walk in total?'* • *'Imran needs to drink $2\frac{1}{2}$ litres of water during a training session. He has $\frac{1}{2}$ litre in his water bottle. How much more water does Imran need?'* • *'Archie is making cakes for the school fair. His recipe needs 4 kg of white sugar and $\frac{2}{5}$ kg of brown sugar. How much sugar does Archie need in total?'* • *'I have $3\frac{7}{10}$ m of ribbon. I use 3 m of it. How much ribbon do I have left over?'* Dòng nǎo jīn: *'Is the following statement always, sometimes or never true? Explain your answer.'* • A mixed number is greater than one.

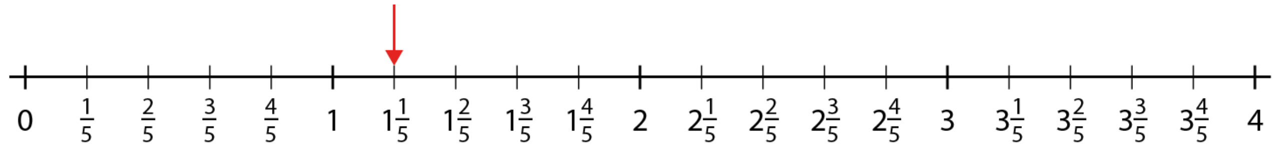
Teaching point 2:

Mixed numbers can be placed on a number line.

Steps in learning

	Guidance	Representations
2:1	In segment *3.3 Non-unit fractions: identifying, representing and comparing*, children learnt that fractions are numbers (e.g. the number $\frac{1}{4}$) that can be located on a number line, as well as operators (e.g. $\frac{1}{4}$ *of* something). Sometimes children can get quite far in their maths education only thinking of a fraction as a part of a whole, rather than as a number in its own right. Use of number lines throughout this spine is one way to reinforce that fractions are numbers. Up to this point, children have encountered examples of number lines displaying fractions between zero and one. We will now extend these number lines to display mixed numbers. Present a number line, such as the fifths number line below. Through their work with non-unit fractions, and addition and subtraction of fractions, children have learnt to identify the unit they are working in. Ask the children to discuss with a partner what unit this number line is marked in. Later in this segment, children will learn how to identify numbers on marked but unlabelled number lines. One of the common mistakes children make is to count the number of *marks* (also called tick marks or notches) between zero and one. In the fifths number line, there are four marks splitting the interval from zero to one into five parts. Some children will mistakenly conclude that four marks means the whole is divided into four equal parts instead of five. It is therefore important to teach the children to focus on the *parts* between each whole number – not the number of marks. Take the time to start drawing children's attention to this. You may find this stem sentence helpful: ***'There are ___ parts between zero and one. This means we are counting in units of ___.'*** Display the fifths number line and practise counting along it. On the first attempt, it is likely that some children will instinctively continue to count in fifths, for instance *'three-fifths, four-fifths, five-fifths'*. At this point stop the count and draw their attention to this. Remind children of their learning from segment *3.3 Non-unit fractions* – that when the numerator and the denominator in a fraction are the same, the fraction is equivalent to one whole. This is also likely to occur again when children count towards two: some children will say *'one and five-fifths'* and others will say *'two'*. Discuss with the children how both are structurally correct, but the convention is to say *'two'* rather than *'one and five-fifths'*. Continue to count forwards and backwards along the number line until all children are confident in their delivery. Once children's counting is secure, ask them what they notice about the numbers marked underneath the line. Responses may include: • *'Numbers smaller than one only have a fractional part.'* • *'Some numbers, such as 0, 1 and 2 do not have a fractional part.'* (This may be the appropriate time to also discuss why four-fifths is not followed by five-fifths as some children may have expected – although do note that five-fifths would also be accurate.) • *'The fraction one-fifth repeats after each whole number.'* (As do the other fractional notations.) As a class, discuss these points and why they occur. For example, you could write a '0' before the	

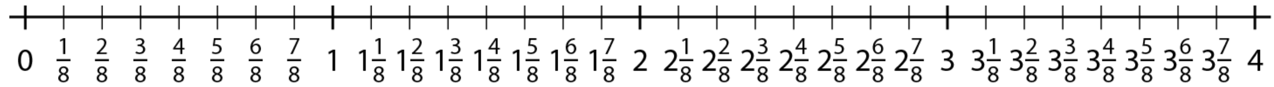
$\frac{1}{5}$ to record it as a mixed number, but this is not the convention. Make links to previous learning during your discussions.



2:2 Repeat the counting and dialogue but this time focus on a different mixed-number number line, such as eighths. You may wish to replicate the same process:

- Identify the unit the number line is marked in.
- Draw children's attention to the *seven marks* but the *eight parts* between each whole number.
- Practise counting forwards and backwards in eighths.
- Discuss equivalence, for example between $2\frac{8}{8}$ and 3, and remind children that the convention is to say *'three'*.

By now, children should be really confident saying mixed numbers aloud. You can check their understanding by pointing to random numbers on the number line and asking children to call them out.

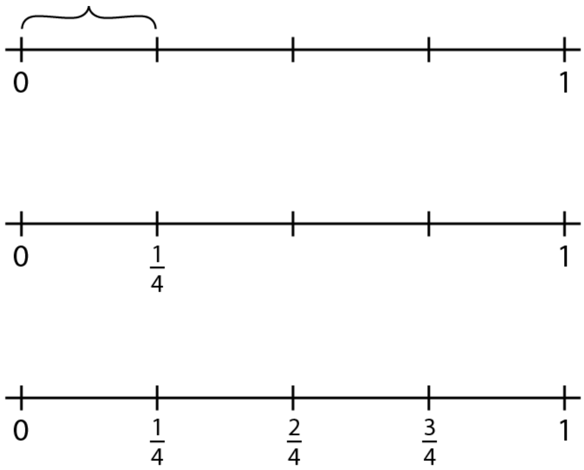


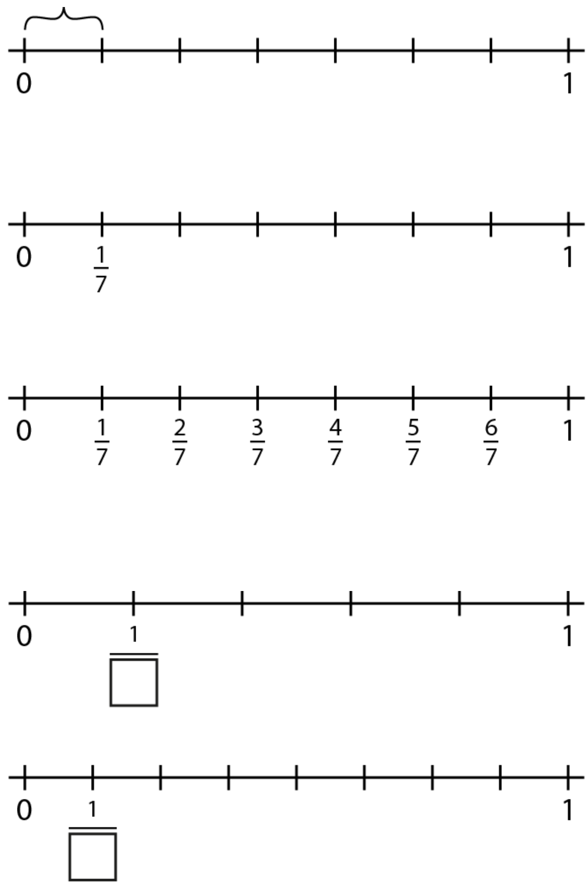
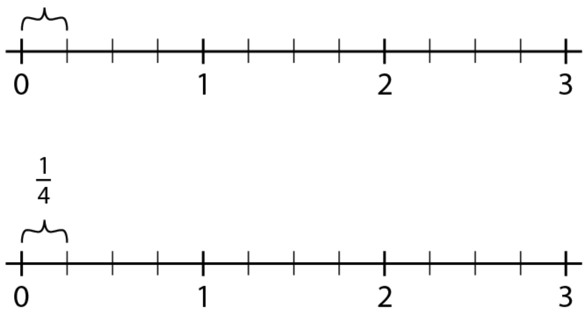
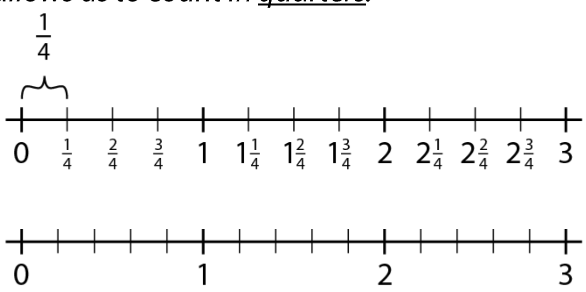
2:3 Children are familiar with number lines marked with mixed numbers. Next, look at number lines that are partitioned into equal parts but are *not* labelled. Provide verbal scaffolding in the form of a stem sentence: ***'The line is divided into ___ equal parts. This allows us to count in ___.'*** This will support children to correctly label the number lines.

Look at the first number line opposite. Ask the children to identify how many equal parts there are between zero and one. *'The line is divided into four equal parts. This allows us to count in quarters.'*

As discussed previously, it is important to emphasise the equal parts between zero and one, and not the division marks. A common misconception would be that there are three marks, so the unit is thirds.

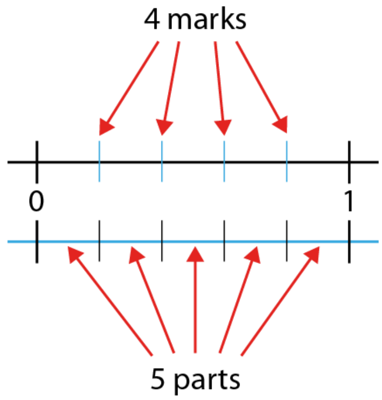
Once the unit has been defined as one-



	quarter, show how to label the number line with $\frac{1}{4}$, $\frac{2}{4}$ and $\frac{3}{4}$. $\frac{4}{4}$ could also be labelled on the number line but it is already labelled '1', which is equivalent to $\frac{4}{4}$. Present the children with further examples of number lines from zero to one that have marks but no numbers labelled. Ask them to identify the unit, label the first mark after '0', and then the subsequent marks. Remind them that the denominator is the number of equal parts that the zero-to-one interval has been divided into. Give children practice until you are confident that they can label a zero-to-one number line with consistent accuracy.	
2:4	Next, show a line segment from zero to three, split into quarters. Although there are 12 parts on the whole number line, children need to understand that it is the number of equal parts between zero and one (or indeed between any two adjacent whole numbers) that determine the unit of the number line. You may find it helpful to adapt the previous stem sentence to: ***'Each interval on the line is divided into ___ equal parts. This allows us to count in -___.'*** Label the number line as before, but this time extend beyond one. As children haven't been formally introduced to equivalent fractions yet, continue to mark the number line with $\frac{2}{4}$ rather than introducing the equivalent fraction of $\frac{1}{2}$. As the focus here is on identifying the unit the	 *'Each interval on the line is divided into <u>four</u> equal parts. This allows us to count in <u>quarters</u>.'* 

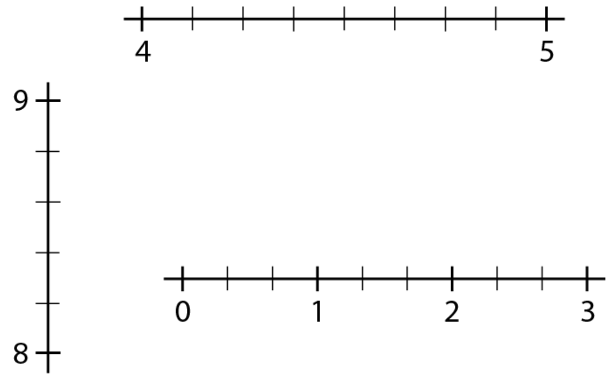
number line is marked in (i.e. quarters), staying with a denominator of $\frac{1}{4}$ will help to reinforce this point.

Continue to examine different number lines, for example a line marked in fifths. Again, discuss the number of parts that each whole has been split into and identify the unit. Check that children are really comfortable with the idea that there are four marks between each whole number but five parts. Each whole has been split into five parts and as a result the number line is marked in units of one-fifth.



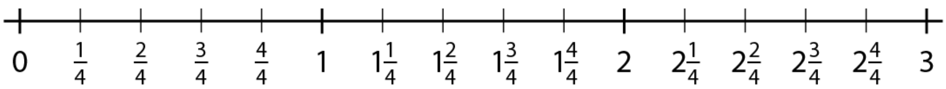
2:5 When you feel the children are secure in their understanding up to this point, give them a range of number lines (including in different orientations) and challenge them to practise labelling the number lines with mixed numbers.

To deepen understanding, introduce dòng nǎo jīn problems like the one below. Use this example to explore the common mistakes that children might make when identifying the unit a fractional number line is marked in. Can children see that every whole number is marked twice (e.g. '$1\frac{4}{4}$' and '2')? What incorrect reasoning has Sonny used? What unit is the number line actually marked in?



Dòng nǎo jīn:

'Sonny has labelled this number line incorrectly. Explain his mistake.'



2:6 Once the children have a solid understanding of how to identify the parts on a number line, show them two labelled number lines, such as those on the next page. Displaying both number lines, one above the other, will help the children see that the whole number sections of both number lines are the same length and in the same place, regardless of the number of fractional parts. However, the exact location of the mixed number in between two whole numbers (or integers), is a result of the value of the denominator and numerator of the fractional part.

Explain that all these mixed numbers could all be presented on one number line, but there would be both quarter and fifth marks, which would make it quite hard to read. In fact, the number of numbers between each integer on a number line is infinite. Here, the focus is on mixed numbers with denominators of four and five, but any number of equal parts could be used, and therefore any denominator.

The aim behind this sort of detailed discussion is to help children understand that, within a mixed number, the whole number is the most significant part. The whole number can be used to identify which integers a mixed number will be placed between on a number line. The fractional part can then be used to establish exactly where, in relation to the next whole number, it should be placed. Connect this to previous learning within this segment, where it was ascertained that the whole number is said and written before the fractional part because of its greater significance.

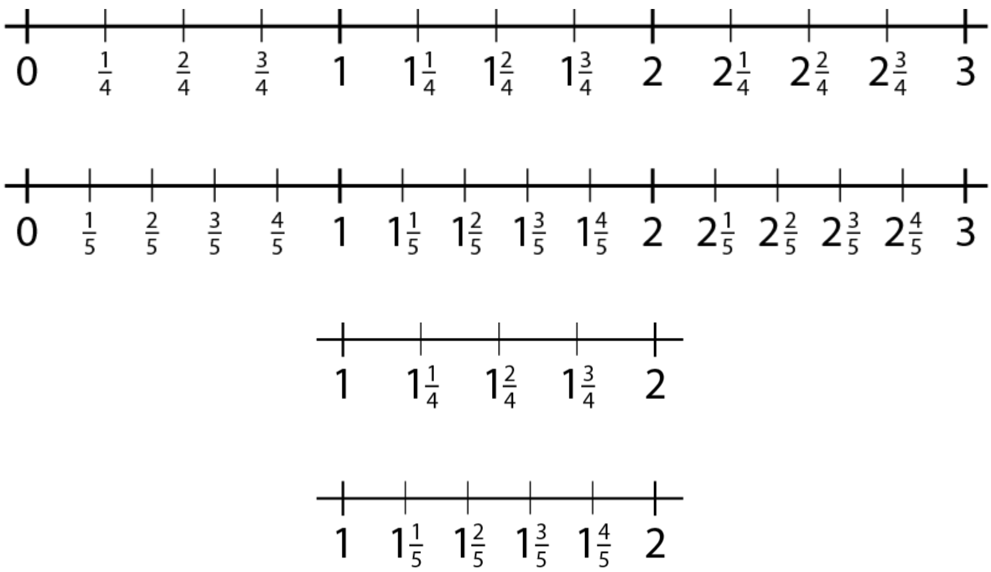
Ask open-ended questions that will allow you to evaluate the depth of children's understanding. For example:

- *'Tell me any number between two and three.'*
- *'Tell me a number very close to three.'*
- *'Tell me a number even closer to three than the previous number.'*
- *'Tell me a number greater than two that is very close to two.'*
- *'Tell me a number smaller than two that is very close to two.'*

Leave the two number lines on the board for support, although children may well visualise other number lines and offer various answers, e.g. $\mathbf{2\frac{9}{10}}$.

Also note that when posing questions, asking what *'number'* rather than what *'mixed number'*, will help to reinforce that fractions – including when presented as a part of a mixed number – are numbers.

You can explore this concept further using a dòng nǎo jīn problem. Allow children to discuss their ideas.



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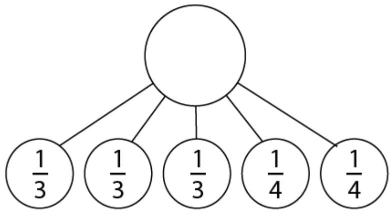
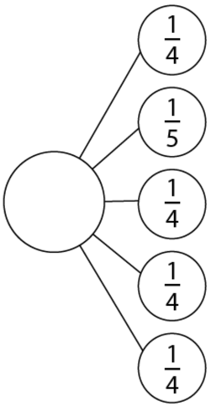
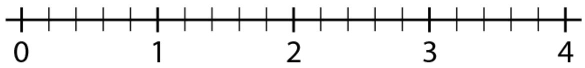
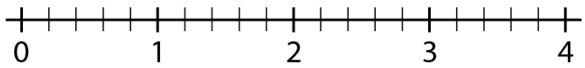
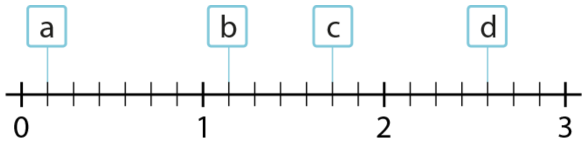
<tr><td></td><td colspan="2">Dòng nǎo jīn:
'What mixed number can be written from the following fractional parts?'



</td></tr>
<tr><td>2:7</td><td>By this stage, children should have grasped how mixed numbers can be located between two integers based on their whole-number part. They should recognise how the fractional part of the mixed number represents its exact location between two integers. If children's understanding of these concepts is secure, they can be asked to position given mixed numbers on a number line.
You may opt to provide the numbers in size order to begin with, before progressing to presenting a series of numbers that are not in size order.</td><td>'Position these numbers on the number line:'
$\frac{3}{5}$, $1\frac{2}{5}$, $2\frac{1}{5}$, $2\frac{4}{5}$


• 'Now try these:'
$\frac{4}{5}$, $2\frac{3}{5}$, $3\frac{1}{5}$, $1\frac{4}{5}$

</td></tr>
</table>

2:8	Proceed to identifying numbers denoted by letters marked on a number line. This is more challenging than the previous task: children will need to first ascertain what unit the number line is marked in. As before, examine the number of *parts* between consecutive integers and not the number of *marks*. You could potentially use the following stem sentence (as exemplified in step *2:4*) to reinforce this: ***'Each interval on the line is divided into ___ equal parts. This allows us to count in ___.'*** Children may have employed a variety of strategies in order to identify the numbers. As a class, discuss their chosen strategies. For instance, some children may have counted up from the smaller integer and others may have counted back from the greater integer. To identify '**c**' as $1\frac{5}{7}$, some children may have counted forward five parts from one, while others may have counted back two parts from two. Some children may have marked every number on the number line and read off the answer as $1\frac{5}{7}$. Establish why these decisions were made and discuss the efficiency of each strategy. Give children a variety of independent practice, similar to the example opposite, until you are confident they can consistently identify mixed numbers on marked but unlabelled number lines.	

2:9

One of the aims across this fractions spine, has been to develop 'fraction sense'. Just as children should be able to estimate the approximate position of, for example, 78 on a 0–100 number line labelled in tens, they should also be able to do this with fractions. Building a sense that, for example, $3\frac{4}{5}$ is just before 4, without having to rely on a marked number line to 'count up', requires children to think proportionally. It enables estimation and gives them a sense of the value of the number.

Give children a number line with integers labelled but no other parts marked. Provide a set of mixed numbers with different denominators in increasing size order, such as in the examples opposite.

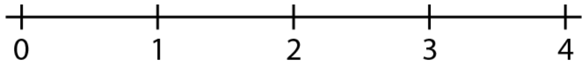
As a class, discuss each of the given numbers in turn. $\frac{1}{2}$ should be relatively easy to position. It lies halfway between zero and one. Now look at $1\frac{1}{3}$ and deal with it in two steps.

- **Step 1:**
  Ask the children first to identify which two integers $1\frac{1}{3}$ sits between.
  ('It sits between one and two.')
- **Step 2:**
  Now look at the fractional part of the number. Imagine the interval between one and two is split into three equal parts (or thirds), and we want one of those parts (or $\frac{1}{3}$). *('"$1\frac{1}{3}$" is "$\frac{1}{3}$" of the way between "1" and "2".')*

It might be tempting to explain the positioning of $1\frac{1}{3}$ by actually splitting the interval between one and two into three equal parts. However, the aim is to move children beyond the need to do this, for example, they should

'Estimate the position of the following numbers on this number line.'

$\frac{1}{2}$, $1\frac{1}{3}$, $\frac{3}{4}$, $3\frac{4}{5}$



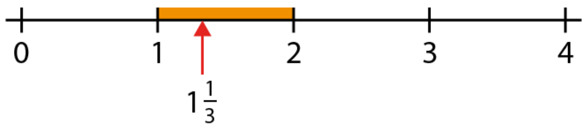
Estimating the position of $1\frac{1}{3}$:

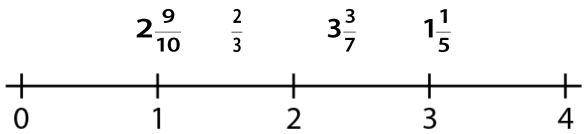
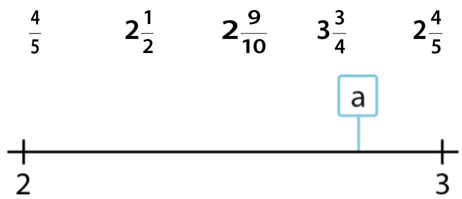
- Step 1:

'$1\frac{1}{3}$ sits between 1 and 2.'

- Step 2:

'Imagine the interval 1–2 is split into three equal parts and we want one of those parts, or $\frac{1}{3}$.'



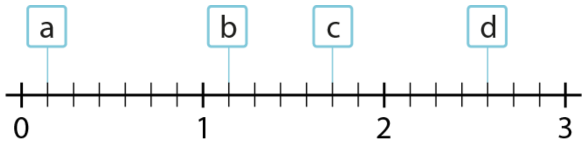
	develop a sense that $1\frac{18}{20}$ is quite close to 2, without having to rely on dividing the interval into 20 equal parts and then counting 18 of them. This is just the same as how children should be able to roughly position 78 on a number line, without needing to divide 70–80 into ten equal parts. Urge children to only *visualise* the partitioning and positioning of the number. They can draw on skills developed in prior work with estimation and pouring activities (from previous segments) to support them in this.	
2:10	Once you have examined several examples as a class and the children are working with increasing confidence, they can start to practise independently. Present them with examples along the lines of those shown opposite. The skill of estimating the position of mixed numbers on a number line requires children to identify the previous whole number and the next whole number. Provide children with practice in this skill, without the support of a number line. As a guide, see the example opposite.	*'Estimate the position of these numbers on this number line:'* 
2:11	Children's understanding can be deepened through dòng nǎo jīn problems. Ask children to justify which of a number of different options could be represented by '**a**' on an unmarked number line. They should provide reasons for their answer and explain why it could not be the other options. For example: • *'It could not be "$\frac{4}{5}$" or "$3\frac{3}{4}$" because **"a"** is found between 2 and 3.'* • *'It could not be "$2\frac{1}{2}$" because **"a"** is not halfway between 2 and 3.'*	Dòng nǎo jīn: *'Which of these numbers is represented by **"a"**?'* 

	• *'It could not be "* $\mathbf{2\frac{9}{10}}$ *" because that would mean the part between* ***"a"*** *and 3 represents* $\frac{1}{10}$ *and nine more parts of the same size would not fit in between 2 and* ***"a"****.'* • *'The answer must be that* ***"a"*** *is "*$\mathbf{2\frac{4}{5}}$*".'*	

Teaching point 3:

Understanding how to compare and order proper fractions supports the comparison and ordering of mixed numbers.

Steps in learning

	Guidance	Representations
3:1	Now that children have scrutinised the composition of mixed numbers and can place and identify them on a number line, comparing and ordering becomes a manageable next step. Children should be confident using the signs <, > and = to compare numbers. They can now apply these symbols to mixed numbers. It may initially help to use a number line alongside questions in order to reinforce the position of each number in the number system. As a starting point, you may wish to look at the number line shown opposite. Identify the mixed numbers represented by each of the letters. Then use <, > or = to complete the statements, discussing each one as a class as you complete them. Summarise some of the discussion points that are likely to arise, such as: • *'If the whole-number part of the numbers being compared is different (e.g.* $2\frac{4}{7}$ *and* $1\frac{5}{7}$*), use this to compare the numbers.'* • *'If the whole-number part of the numbers being compared is the same (e.g.* $1\frac{1}{7}$ *and* $1\frac{5}{7}$*), use the fractional part to compare the numbers.'* Children should apply caution with examples where the mixed number is not given in the 'correct' notation, such as in comparing '$1\frac{7}{7}$' with '2.' They should avoid applying rules or generalisations without thought; they	*'Identify the numbers represented by the letters. Use them to help you complete the missing symbols (<, > or =) in the comparison statements.'*  $\frac{1}{7}$ ○ $1\frac{1}{7}$ $1\frac{1}{7}$ ○ $1\frac{5}{7}$ $1\frac{7}{7}$ ○ 2 $2\frac{4}{7}$ ○ $1\frac{5}{7}$ $1\frac{1}{7}$ ○ $1\frac{5}{7}$ ○ $2\frac{4}{7}$

<table>
<tr><td></td><td>need to make sense of what they are doing.</td><td></td></tr>
<tr><td>3:2</td><td>Challenge children to start working without the number line. Encourage them to visualise the number line and picture which whole numbers each mixed number is sitting between. By beginning with comparing mixed numbers to whole numbers, more children should be able to access increasingly demanding questions. Offer examples like those opposite.

Progress to comparing and sequencing numbers with different whole-number parts.</td><td>Missing-symbol problems:
'Fill in the missing symbols (<, > or =).'

$1\frac{1}{2}$ ◯ 2
3 ◯ $3\frac{2}{3}$
4 ◯ $3\frac{2}{3}$

$2\frac{3}{8}$ ◯ $1\frac{5}{7}$
$5\frac{99}{100}$ ◯ $8\frac{1}{100}$
$10\frac{2}{3}$ ◯ $9\frac{5}{6}$
$2\frac{2}{3}$ ◯ $3\frac{1}{3}$ ◯ $4\frac{1}{3}$

Ordering:
'Put the numbers in order from smallest to largest.'
8 $4\frac{5}{7}$ 7 $5\frac{7}{8}$ 5 $8\frac{4}{7}$</td></tr>
<tr><td>3:3</td><td colspan="2">Move on to comparing mixed numbers where the whole numbers are the same. In segment 3.3 Non-unit fractions, children learnt how to compare fractions that have the same denominator, and how to compare fractions that have the same numerator. Review this previous learning.

In segment 3.3, children met the generalisation: 'When we compare fractions with the same denominator, the greater the numerator, the greater the fraction.'

Children used area models, number lines and the following chain of reasoning to support this conclusion, as shown below. Review these models with children.

Tell the children that you have a theory: 'Because we know that $\frac{3}{8} < \frac{5}{8}$ we must also know that $1\frac{3}{8} < 1\frac{5}{8}$.'

Do they agree or disagree with you? Ask the children to discuss your theory in pairs, then explain their thinking. You might like to provide some empty supporting models on the board, such as those below, or you may wish to challenge children to create their own.</td></tr>
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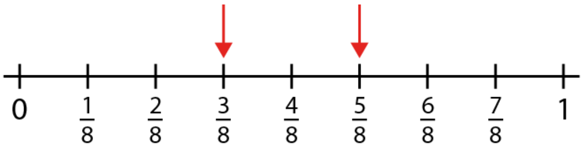
Comparing mixed numbers – same whole number:

Method 1 – on a diagram

Method 2 – on a number line

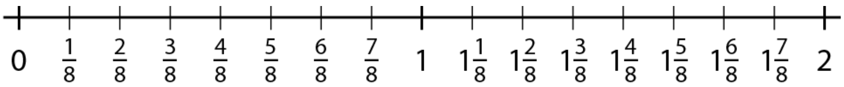


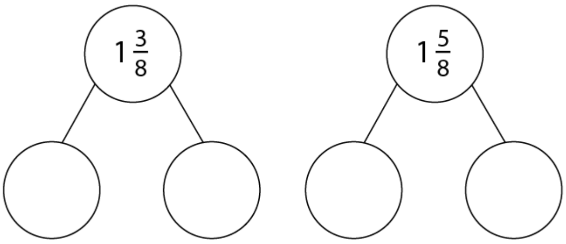
$\frac{3}{8} < \frac{5}{8}$

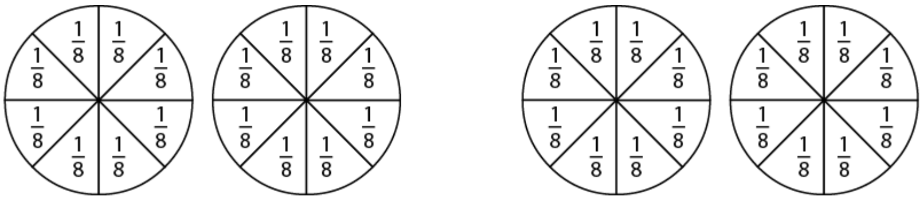
Method 3 – verbal reasoning

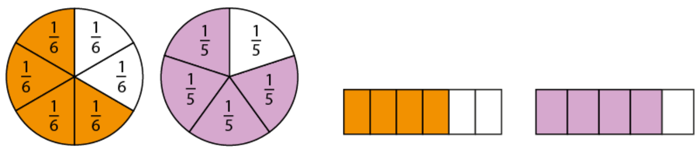
- *'$\frac{3}{8}$ is three lots of $\frac{1}{8}$.'*
- *'$\frac{5}{8}$ is five lots of $\frac{1}{8}$.'*
- *'I know that three is less than five…'*
- *'…so $\frac{3}{8}$ is less than $\frac{5}{8}$.'*

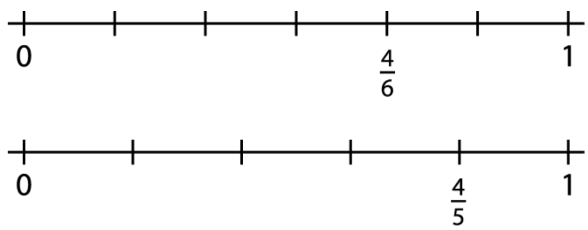
Supporting models:



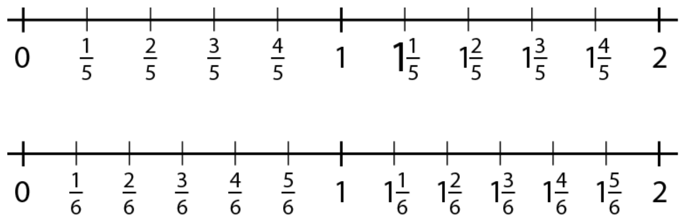
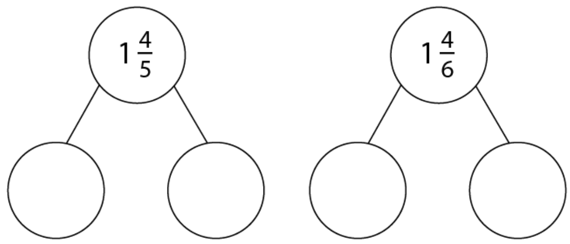
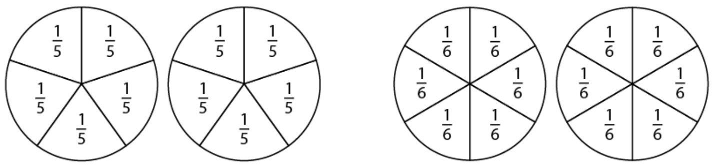




<table>
<tr>
<td>3:4</td>
<td>Establish that your theory is valid and then present a chain of statements, like those opposite. Discuss that the 'same denominator' method can be used to compare these pairs of mixed numbers because they each have the same whole-number part. Use this knowledge to complete the statements. Draw children's attention to the following points:
• The number line above shows that whenever we count between two whole numbers in eighths, $\frac{3}{8}$ always comes before $\frac{5}{8}$.
• The composition of the numbers opposite shows that if we have the same whole-number parts, then it is the size of the fractional parts that determines which number is bigger. $1\frac{3}{8}$ is made of one and three-eighths and $1\frac{5}{8}$ is made of one and five-eighths, so the latter is bigger.</td>
<td>'Fill in the missing symbols (<, > or =).'
$\frac{3}{8} \bigcirc \frac{5}{8}$
$1\frac{3}{8} \bigcirc 1\frac{5}{8}$
$2\frac{3}{8} \bigcirc 2\frac{5}{8}$
$3\frac{3}{8} \bigcirc 3\frac{5}{8}$
$100\frac{3}{8} \bigcirc 100\frac{5}{8}$</td>
</tr>
<tr>
<td>3:5</td>
<td>Now compare pairs of numbers using children's 'same numerator' knowledge. Briefly review their learning from segment 3.3 Non-unit fractions, Teaching point 8. The models and justifications that the children used in this teaching point are shown opposite.
Probe to see whether the children think it might be possible to extend this knowledge as well. For example, you could ask:
'Can I conclude that, because I know that $\frac{4}{6} < \frac{4}{5}$, I must also know that $1\frac{4}{6} < 1\frac{4}{5}$?'
Again, you could provide empty models, such as those opposite, to help children scaffold their responses. Or you could leave them to work in pairs and develop their own models.</td>
<td>Comparing numbers – same numerator:
Method 1 – on a diagram


$\frac{4}{6} < \frac{4}{5}$ $\frac{4}{6} < \frac{4}{5}$
Method 2 – on a number line


$\frac{4}{6} < \frac{4}{5}$</td>
</tr>
</table>

		Method 3 – verbal reasoning • *'$\frac{4}{6}$ is four lots of $\frac{1}{6}$'* • *'$\frac{4}{5}$ is four lots of $\frac{1}{5}$'* • *'I know that $\frac{1}{6}$ is less than $\frac{1}{5}$…'* • *'…so, I know that four lots of $\frac{1}{6}$ is less than four lots of $\frac{1}{5}$.'* Empty models:   
3:6	Again, after establishing the validity of this conclusion, extend this to different whole-number parts, as in the examples opposite. Children who are successfully developing a depth of understanding will be able to think of these comparisons both in terms of their relative positions on a number line, and in terms of the composition of each mixed number.	*'Fill in the missing symbols (<, > or =).'* $\frac{4}{5}$ ◯ $\frac{4}{6}$ $1\frac{4}{5}$ ◯ $1\frac{4}{6}$ $2\frac{4}{5}$ ◯ $2\frac{4}{6}$ $3\frac{4}{5}$ ◯ $3\frac{4}{6}$ $100\frac{4}{5}$ ◯ $100\frac{4}{6}$

3:7	To conclude this teaching point, children should consolidate and further deepen their understanding through practice with a range of questions. Include dòng nǎo jīn problems, such as the one shown opposite.	Missing-symbol problems: *'Fill in the missing symbols (<, > or =).'* $4\frac{3}{8}$ ◯ $3\frac{5}{8}$ $1\frac{11}{12}$ ◯ $1\frac{11}{15}$ $6\frac{4}{9}$ ◯ $6\frac{7}{9}$ $10\frac{99}{100}$ ◯ $11\frac{1}{100}$ Ordering: *'Put these numbers in order from smallest to largest.'* 3 $3\frac{3}{5}$ $3\frac{4}{5}$ $3\frac{3}{8}$ 5 Dòng nǎo jīn: *'Explain what numbers could be added to the expression below to make it true.'* $2 < \square < 3$

Teaching point 4:

Mixed numbers can be partitioned and combined in the same way as whole numbers.

Steps in learning

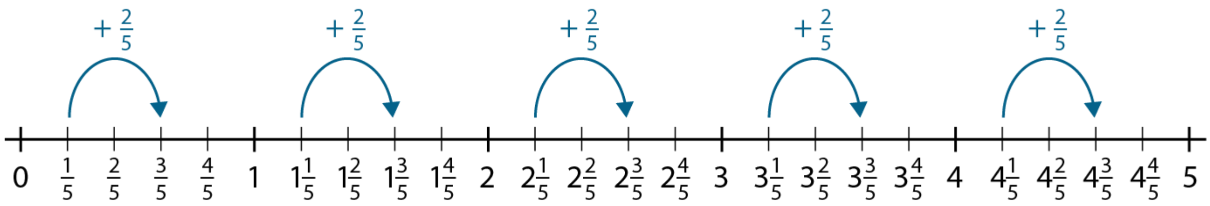
	Guidance	Representations
4:1	Once children understand the composition of mixed numbers and their position in the linear number system (*Teaching points 1–3*), they can then begin to combine (add) and partition (subtract) mixed numbers, whole numbers and fractional parts in a variety of different ways. Start by looking at examples which do *not* bridge a whole number. Three key representations are used in this teaching point: • number lines – building on directly from the previous teaching point • area models – these could be represented as bar models using rectangular parts, or as pie charts using segments (Note: a linear 'bar' area model is used extensively later in the spine, but as it is a continuous model it isn't always obvious when one whole has been completed. Using a pie model avoids this potential confusion as each whole forms a complete circle. Pie models are therefore used within this teaching point.) • part–part–whole models – these models allow children to relate this work to their whole number addition and subtraction understanding. Children have seen these models used to represent the additive composition of a number so extensively that they should be able to apply the model to this new context. (Note: think carefully about the language used, as a 'part' could contain a whole number, and a 'whole' that is a mixed number will have a whole number and a fractional 'part'. You may wish to use the language of 'total' alongside 'whole' when discussing the numbers in the model. 'Whole' might refer to the number '1', one complete thing, or one whole unit of measure. For example, in the context of $1\frac{1}{2}$ litres of squash, the emphasis can be put on the concept of a whole represented as one 'whole' litre and one-half of a litre more. Alternatively, the emphasis can be put on the 'whole' represented as all of the squash: $1\frac{1}{2}$ litres in 'total'.) This teaching point will build on work covered in segment *3.4 Adding and subtracting within one whole*, where children learnt to add and subtract fractions within one. To begin with, focus solely on addition. Provide children with a sequence of completed equations showing addition of a fraction to a mixed number, as with the examples below. This will allow children to focus on the representations that can be used to model each equation and to identify patterns between equations resulting from the commonalities in structure, rather than focusing on finding the solution. Start by asking: • *'What is the same?'* • *'What is different?'* In your discussions, make sure you go beyond *pattern* and consider the underlying *structure* that gives rise to this pattern. For example, a valid observation might be that the totals all have a fractional part of three-fifths, but the whole number part goes up by one each time. This is an accurate description of the *pattern* in the sequence. The *structure* that causes this pattern is the	

fact that two-fifths are added to a fractional part of one-fifth each time, i.e. as $\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$; the whole number is irrelevant to the size of the fractional part – the fractional part must always be $\frac{3}{5}$.

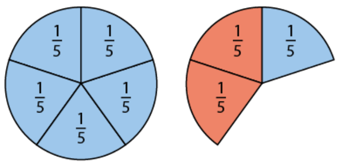
Useful questions to prompt discussion and deepen understanding include:

- *'What would the next equation in the sequence be?'*
- *'How would each of these equations be shown on a number line?'*
- *'How might each of these equations be shown using an area model?'*
- *'How could each of these equations be shown using a part–part–whole model?'*

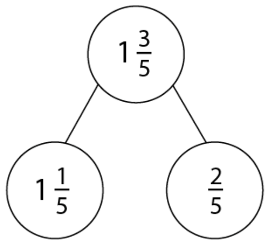
Number line:



Area model:



Part–part–whole model:



Sequence of completed equations:

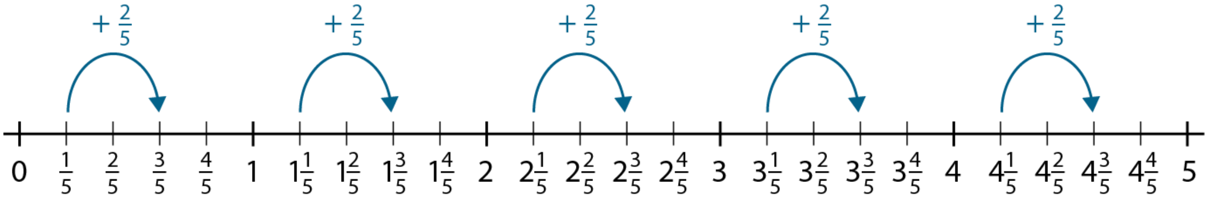
$\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$

$1\frac{1}{5}+\frac{2}{5}=1\frac{3}{5}$

$2\frac{1}{5}+\frac{2}{5}=2\frac{3}{5}$

$3\frac{1}{5}+\frac{2}{5}=3\frac{3}{5}$

Completed models showing patterns:



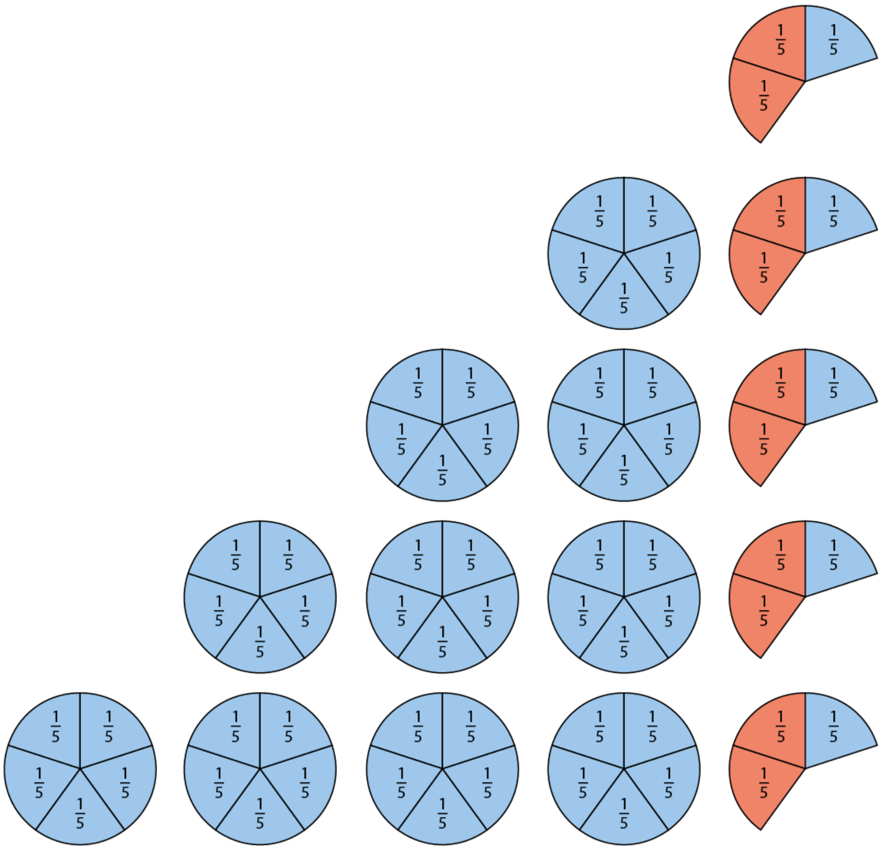
$\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$

$1\frac{1}{5} + \frac{2}{5} = 1\frac{3}{5}$

$2\frac{1}{5} + \frac{2}{5} = 2\frac{3}{5}$

$3\frac{1}{5} + \frac{2}{5} = 3\frac{3}{5}$

$4\frac{1}{5} + \frac{2}{5} = 4\frac{3}{5}$



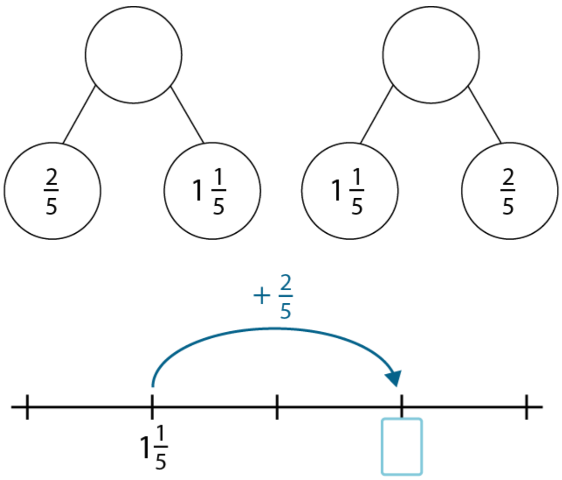
4:2 When children have a firm grasp of this concept, look at other generalisations and approaches they have encountered in whole-number arithmetic, and apply these to fractions. Look at the following calculation again:

$1\frac{1}{5} + \frac{2}{5} =$

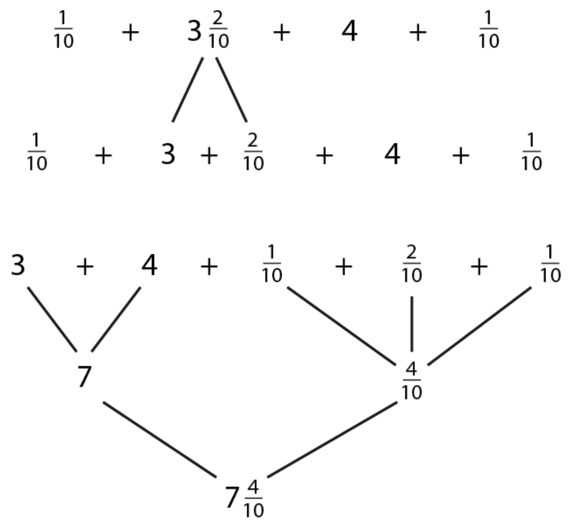
Similar to whole numbers, fractional parts can be added in any order (commutativity):

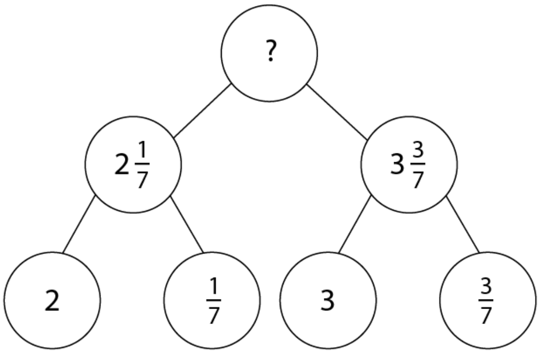
$\frac{2}{5} + 1\frac{1}{5} =$

Showing this first on a part–part–whole model will allow children to see that both $\frac{2}{5}$ and $1\frac{1}{5}$ are parts. You can



*'The **parts** are $\frac{2}{5}$ and $1\frac{1}{5}$. The **total,** or **whole,** is $1\frac{3}{5}$.'*

	verbalise this with a stem sentence: ***'The <u>parts</u> are ___ and ___. The <u>total</u>, or <u>whole</u>, is ___.'*** For example: *'The **parts** are $\frac{2}{5}$ and $1\frac{1}{5}$. The **total** or **whole** is $1\frac{3}{5}$.'* This will help children to see how the same representation previously used on a number line can still be used. Thinking of $\frac{2}{5}+1\frac{1}{5}=$ as '*$\frac{2}{5}$ more than $1\frac{3}{5}$*' is probably simpler than thinking of it the other way around, as '*$1\frac{3}{5}$ more than $\frac{2}{5}$*.'	
4:3	Present children with new examples and challenge them to justify the order in which they negotiate them. Examples of possible orders might include: • Start with the mixed number as the first addend and the fractional part as the second addend. • Start with the fractional part as the first addend and the mixed number part as the second addend. • Where there is a mixed number and more than one fractional part, it would be more natural to begin with the mixed number and then add the fractional parts. Extend this further by providing questions that require children to add proper fractions, mixed numbers and whole numbers, such as: $\frac{1}{10}+3\frac{2}{10}+4+\frac{1}{10}$ Again, ensure that none of the questions bridge a whole when the fractional parts are added. Children may find it easier to use their understanding of the composition of mixed numbers and break down the question further, separating out all	*'Justify the order you use when finding the answer.'* $1\frac{2}{7}+\frac{4}{7}$ $\frac{2}{9}+4\frac{3}{9}$ $3\frac{1}{8}+\frac{3}{8}$ 

	the whole numbers and fractional parts. This separation can also be displayed through an area model, as provided in *3.5 Representations*, slide *33*.	
4:4	The final step is to combine two mixed numbers that have the same denominator in their fractional parts. As children learnt to partition mixed numbers into the whole number and fractional parts in *Teaching point 4.3*, this is an appropriate method to repeat. $2\frac{1}{7}+3\frac{3}{7}$ As children have not yet met improper fractions, avoid additions where the fractional parts add to more than one. Using area and part–part–whole models will assist children in visualising how the smaller parts can be added in any order. We are simply combining the whole number parts and then combining the fractional parts to make the total.	Part–part–whole model: 
4:5	Once children have a secure understanding of how to add fractions to mixed numbers, and vice versa, subtraction can be introduced. Again, it is important for children to see this concept across the three key representations: number line, area model and part–part–whole model. Select questions that do not bridge a whole number. A similar sequence to *Teaching point 4:1* can be used so that children are familiar with what is required. Look at the examples below and start by asking: • *'What is the same?'* • *'What is different?'* Again, focus not just on the *pattern* within the chain of equations, but also the underlying *structure* (of always subtracting $\frac{2}{8}$ from $\frac{7}{8}$, regardless of the size of the whole-number part) that gives rise to the patterns seen. Present and debate similar questions to step *4:1*: • *'What would the next equation in the sequence be?'* • *'How would each of these equations be shown on a number line?'* • *'How would each of these equations be shown using an area model?'* • *'How could each of these equations be shown using a part–part–whole model?'*	

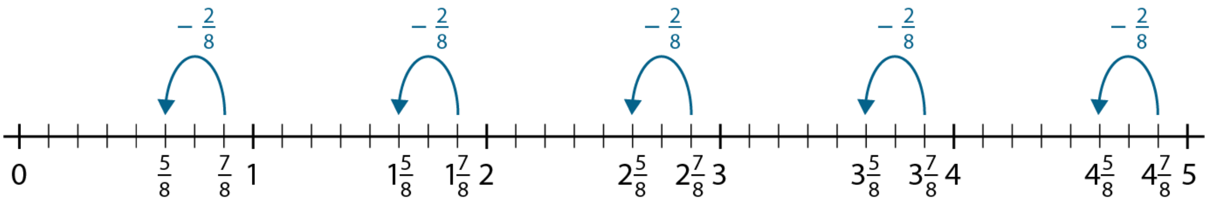
Sequence of completed equations:

$\frac{7}{8} - \frac{2}{8} = \frac{5}{8}$

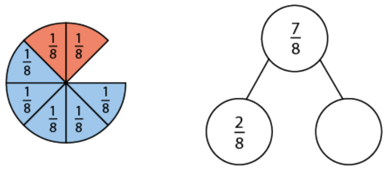
$1\frac{7}{8} - \frac{2}{8} = 1\frac{5}{8}$

$2\frac{7}{8} - \frac{2}{8} = 2\frac{5}{8}$

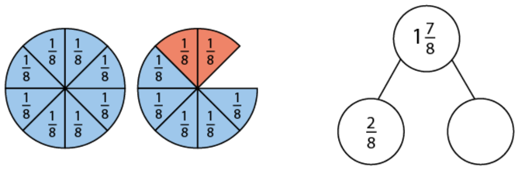
$3\frac{7}{8} - \frac{2}{8} = 3\frac{5}{8}$



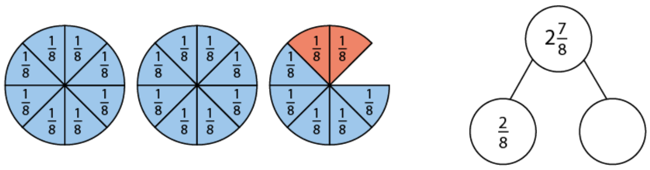
$\frac{7}{8} - \frac{2}{8} = \frac{5}{8}$



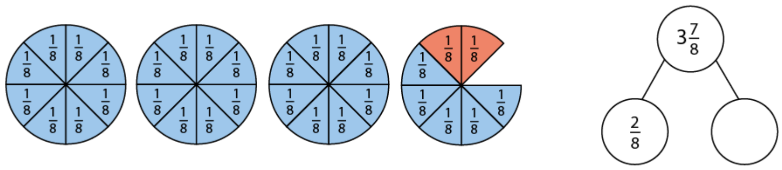
$1\frac{7}{8} - \frac{2}{8} = 1\frac{5}{8}$



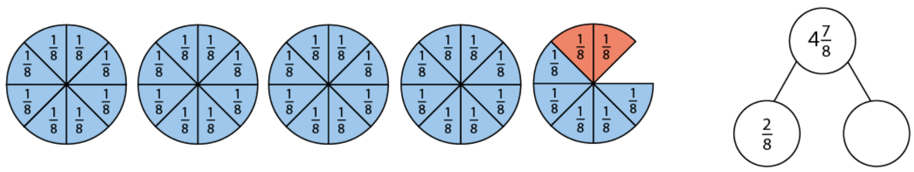
$2\frac{7}{8} - \frac{2}{8} = 2\frac{5}{8}$

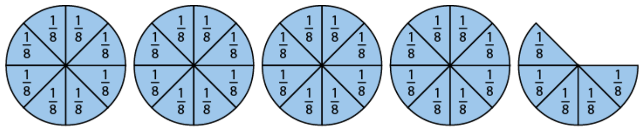
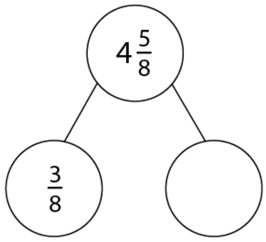
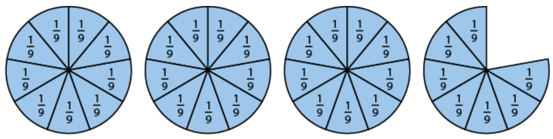
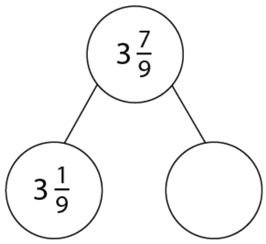
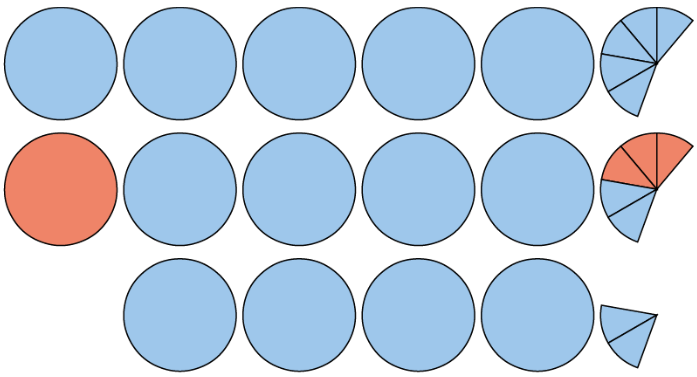


$3\frac{7}{8} - \frac{2}{8} = 3\frac{5}{8}$



$4\frac{7}{8} - \frac{2}{8} = \square$



4:6	Present children with new, unfamiliar examples and ask them to justify the different ways they worked to find their solution. Start by only subtracting whole number or fractional parts from a mixed number. For example: • $4\frac{5}{8} - \frac{3}{8} = ?$ • $3\frac{7}{9} - ? = 3\frac{1}{9}$ • $5\frac{1}{8} - 4 = ?$ • $2\frac{3}{7} - ? = 1\frac{3}{7}$ Children should use the part–part–whole and area models to show or check their calculations. The use of the number line may be less useful; when subtracting a whole number from a mixed number, the subtraction will go across a whole number, thus adding a layer of complexity.	   
4:7	Once you think children have had sufficient practice, progress to subtracting mixed numbers from other mixed numbers (without needing to bridge to solve the fractional subtraction). For example: • $5\frac{5}{9} - 1\frac{3}{9} = ?$ • $2\frac{3}{4} - ? = \frac{1}{4}$ The area model is the clearest representation for this calculation as it visually demonstrates what must be subtracted from the whole and the fractional part of the mixed number.	$5\frac{5}{9} - 1\frac{3}{9} = ?$ 
4:8	Provide plenty of varied practice in adding and subtracting mixed numbers, involving the partitioning or aggregation of different parts. All of this should help children to see that mixed number quantities can be put together and taken apart in exactly the same way that whole number quantities can.	

Teaching point 5:

Mixed numbers can be written as improper fractions.

Steps in learning

	Guidance	Representations
5:1	Until now, children have only met fractions where the numerator is smaller than the denominator (e.g. $\frac{3}{4}$) and where the numerator is equal to the denominator (e.g. $\frac{4}{4}$). However, the numerator can also be greater than the denominator. This is called an '*improper fraction*'. To introduce children to improper fractions, return to the oranges context that was used to introduce mixed numbers in step *1:1*. For example: • *'Jonny has two oranges.'* • *'Ellen has half an orange.'* • *'How many oranges do Jonny and Ellen have altogether?'* After children have given the answer $2\frac{1}{2}$, cut two whole oranges in half. Again ask, *'How many oranges are there?'* to ensure children understand that this has not changed. Ask the class if there is another way they could describe how many oranges there are. If they need prompting, ask *'How many halves are there?'*. Count the number of halves together: *'One-half, two-halves, three-halves, four-halves, five-halves. There are five-halves altogether.'* Discuss how this might be written as a fraction. Children may be inclined to write $\frac{2}{5}$, as all the examples they have encountered until now have had a numerator smaller than the denominator. Recap how a fraction is written – see segment *3.2 Unit Fractions.*	

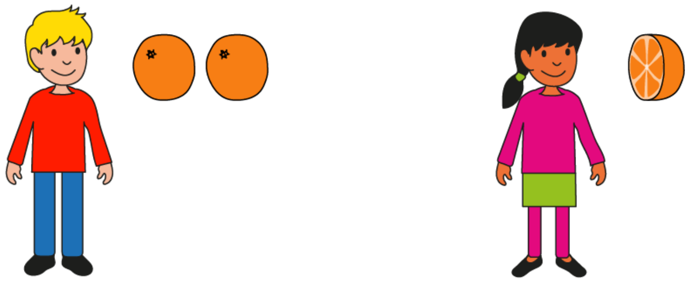
Say	Write
'Each whole orange is divided…'	Draw the fraction bar.
'…into two equal parts.'	Write the denominator: 2
'And we have five of those parts.'	Write the numerator: 5

It is important that children are secure with the fact that the denominator represents how many parts *one whole orange* has been split into, not how many parts the whole group of oranges has been split into.

To summarise, write the following:

$$2\frac{1}{2} = \frac{5}{2}$$

'How many oranges are there altogether?'



— fraction bar

$\overline{2}$  each whole orange splits into 2 parts

$\frac{5}{2}$

 5 halves

 each whole orange splits into 2 parts

5:2 Advance to showing an unfamiliar example in a familiar context. Show one orange, this time cut into quarters. To help children visualise, it would help to do this practically using real oranges. Ask, *'How many quarters are there?'*. Children will respond that there are $\frac{4}{4}$. Show this on a 0–5 number line by writing $\frac{4}{4}$ at the point marked and labelled '1'. Repeat with a second orange. *'How many quarters are there now?'* There are two groups of $\frac{4}{4}$, which is $\frac{8}{4}$.

Write $\frac{8}{4}$ on the number line using the same language as before:

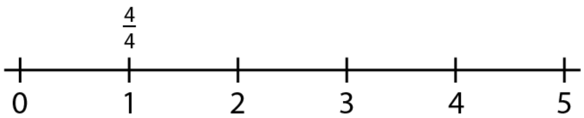
Say	Write
'Each whole orange is divided…'	Draw the fraction bar.
'…into four equal parts.'	Write the denominator: 4
'And we have eight of those parts.'	Write the numerator: 8

Now cut a third orange into quarters. *'How many quarters are there now?'* There are three groups of $\frac{4}{4}$, which is $\frac{12}{4}$. Write $\frac{12}{4}$ on the number line using the same language as before:

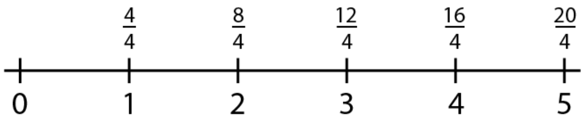
Say	Write
'Each whole orange is divided…'	Draw the fraction bar.
'…into four equal parts.'	Write the denominator: 4
'And we have twelve of those parts.'	Write the numerator: 12

Repeat for four and five oranges, constructing the number line as shown below.

Showing $\frac{4}{4}$ on a number line:



Showing up to $\frac{20}{4}$ on a number line:

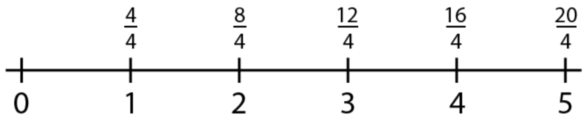


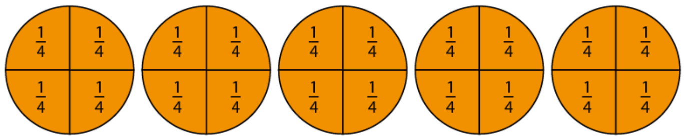
5:3

Take some time to have a close look at the number line you have constructed. What do the children notice about it? Features they draw-out may include:

- A whole number can be written in more than one way.
- We can have fractions where the numerator is larger than the denominator.
- In these examples, all of the numerators are multiples of four.

Discuss these points. When you reach the final point, make a link to the diagram of the quartered oranges, opposite. You can use this image to scaffold skip counting in groups of $\frac{4}{4}$.





For example, count: *'One orange, two oranges, three oranges…'* (Point as you count.) Then count in different ways:

- *'One group of four-quarters, two groups of four-quarters, three groups of four-quarters…'*
- *'Four-quarters, eight-quarters, twelve-quarters…'*

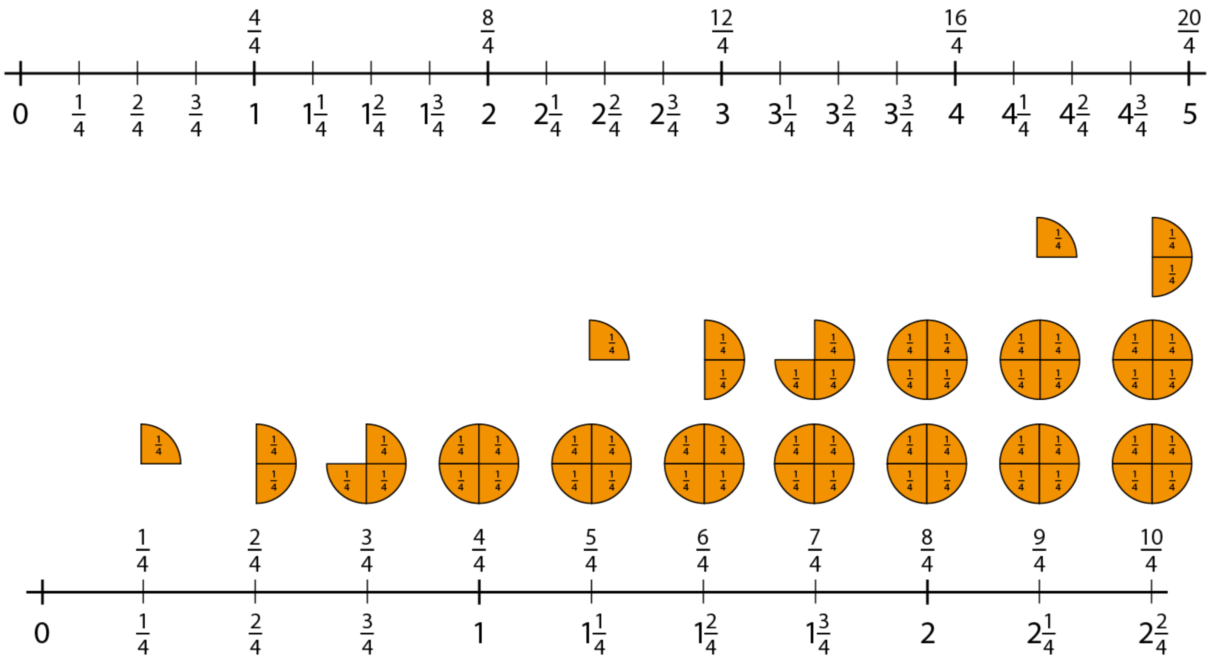
The numerator will always be a multiple of four in this instance because each orange is cut into four quarters.

5:4 Once children are comfortable with the equivalence of each whole number to its improper fractional representation, you can then introduce them to the improper fractions that fall between each whole number. To do this, add the orange quarters one-by-one to establish progressively how many oranges there are. One possible method might be to have several children at the front of the class, and hand four orange quarters to the first child, then four to the second child, and so on, until each child ends up with a whole orange.

To help them understand the equivalence of each improper fraction to its mixed number, show both representations at the same time, with the number line alongside the pictorial representation. See the example below.

You may choose to use a stem sentence, for example: ***'Each whole is divided into four equal parts. We have ___ of these equal parts. This represents ___ quarter(s).'***

(Note: Leave the mixed numbers un-simplified, that is, don't change $1\frac{2}{4}$ to $1\frac{1}{2}$. This will ensure that the focus remains with the equivalence between mixed numbers and improper fractions.)



5:5

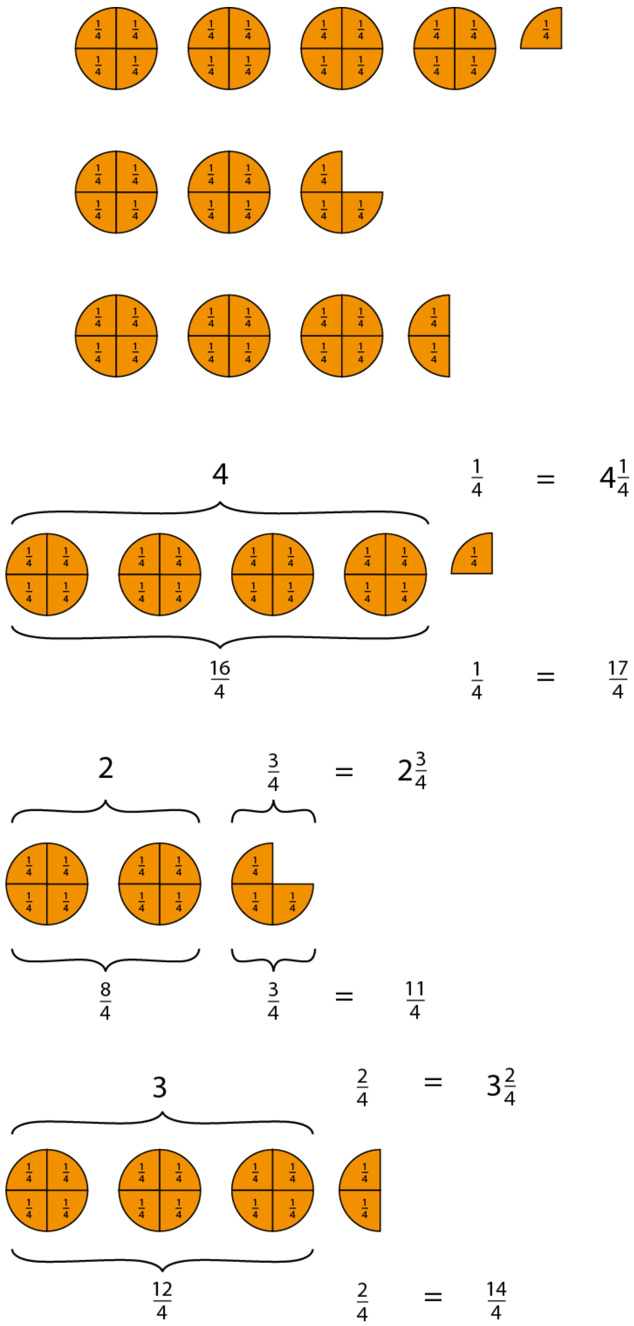
Still using the context of quartered oranges, give children the opportunity to practise writing a given amount using both mixed-number and improper-fraction notation. At this stage, it is probably more practical to move to images of quartered oranges on the board, rather than manipulating real oranges. Some possible examples are shown opposite.

Look at each quantity in turn. Ask the children to think of different ways to describe each quantity. Ask them to point at the image on the board as they describe what they see. For example:

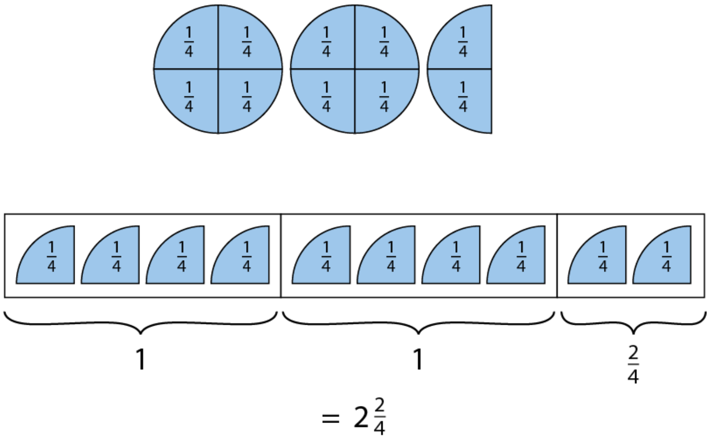
- Seeing a mixed number:
  'I can see 4 whole oranges, and then $\frac{1}{4}$ of an orange, so that is $4\frac{1}{4}$*.'*
- Using counting up in groups of four-quarters to find an improper fraction:
  'I can see four-quarters, eight-quarters, twelve-quarters, sixteen-quarters and then an extra one makes seventeen-quarters.'
- Using multiplication to find an improper fraction:
  'I can see four groups of four-quarters, so that is sixteen-quarters, and one more quarter, so that is seventeen-quarters.'

During discussion, the children might draw jottings on the images to support their working out when writing each quantity as a mixed number and improper fraction.

Provide practice until children can confidently express a quantity represented with an image, as both a mixed number and as an improper fraction.



5:6	Once children can use a diagram to write a mixed number and an improper fraction, progress to converting between the two *without* the aid of a diagram. Stay with quarters to introduce this, and write a mixed number question on the board, for example: $2\frac{1}{4} = \frac{\square}{4}$ Encourage the children to visualise the oranges cut into four parts. Use the following stem sentence as a scaffold: ***'There are ___ groups of four-quarters which is ___-quarters, and ___ more quarters, so that is ___-quarters.'*** For $\mathbf{2\frac{1}{4}}$ the completed stem sentence would be: *'There are two groups of four-quarters which is eight-quarters, and one more quarter, so that is nine-quarters.'*	
5:7	Until now, improper fractions have been formed from mixed numbers. It is important that children can work in both directions. This time, present children with a sequence of improper fractions and ask them to find the mixed number equivalent. Children can refer back to previous images for support. Some examples are provided opposite. Ask what patterns and generalisations they can see. These might include: • When the numerator is a multiple of the denominator, it is equivalent to a whole number. (As before, discuss the structure that gives rise to this pattern, i.e. because there are $\frac{4}{4}$ in one whole, every time there is a multiple of $\frac{4}{4}$; there is an exact multiple of one whole orange.)	$\frac{4}{4} = \square$ $\frac{5}{4} = \square$ $\frac{6}{4} = \square$ $\frac{7}{4} = \square$ $\frac{8}{4} = \square$ $\frac{9}{4} = \square$ $\frac{10}{4} = \square$

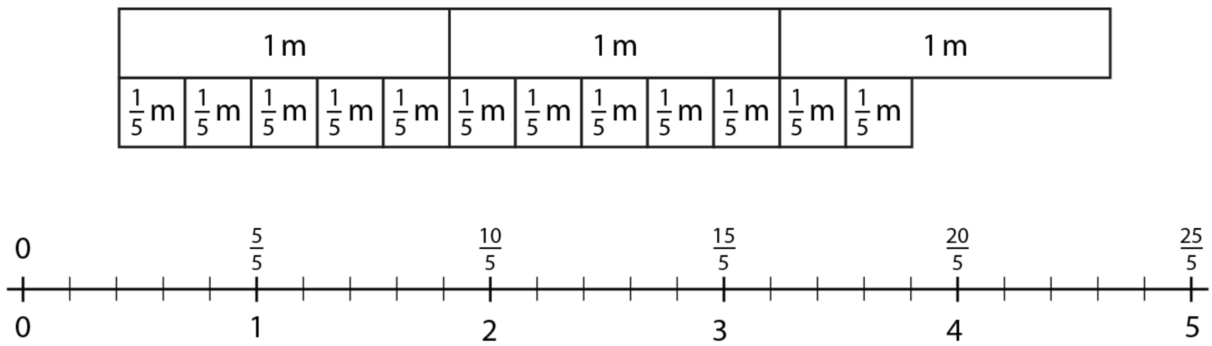
	• The denominator of the fractional part of the mixed number and the denominator of the improper fraction are the same. (This is an important point, so if children don't notice it themselves, be sure to raise it for discussion.)	$\frac{11}{4} = \square$ $\frac{12}{4} = \square$
5:8	Once children understand these patterns, it is important that they learn to apply this understanding to calculating other conversions. During this process, children should make the link to division by grouping. Take a fraction, for example $\frac{10}{4}$, and break down how this can be converted to a mixed number. • Identify what the denominator is and what this actually means: *'The denominator is four. This means that each whole has been split into four equal parts.'* • This also means that: *'Four parts make each whole.'* • Identify what the numerator is: *'The numerator is ten. This means that there are ten equal parts.'* Use these facts to see how many full groups of $\frac{4}{4}$ can be made out of the $\frac{10}{4}$ *'It is possible to make two full groups of* $\frac{4}{4}$ *and there are two more quarters.'* The number of parts that are left over is the numerator of the fractional part of the mixed number. The denominator is still the number of equal parts the whole has been split into (the pattern identified and discussed in step *5:7*) Once this has been broken down into steps, give children the opportunity to use stem sentence scaffolds to convert other improper fractions to mixed numbers. Continue to use quarters.	

- ***'The denominator is ___. This means that each whole has been split into ___ equal parts. ___ parts make each whole.'***
- ***'The numerator is ___. The means there are ___ equal parts.'***
- ***'It is possible to make ___ full groups of ___-quarters and there are ___ more quarters.'***

(Note: This links directly to quotative division and the understanding of what remains, i.e. the remainder is expressed as a fraction.)

5:9 So far, we have only used quarters. Now, repeat this sequence with a different unit.

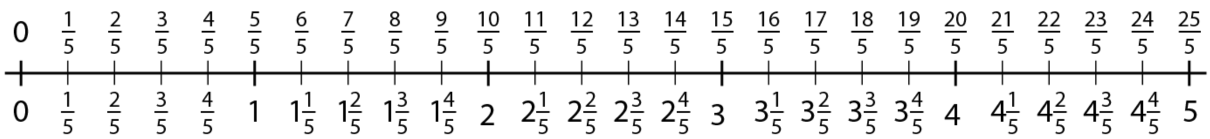
In the example below, fifths are shown using a linear model in the context of metre-long ribbons cut into $\frac{1}{5}$ m lengths. Follow the same progression as for the oranges, by showing a real 1 m length of ribbon. Cut the ribbon into five equal parts and arrange the parts length-ways. This could be done on the classroom floor, with the children standing around so they are all able to see. Repeat this with an additional 1 m length of ribbon. Place metre rules above the pieces of ribbon so they can clearly see when each metre has been completed. In the example below, $2\frac{2}{5}$ metres of ribbon have been laid out, but you may choose to continue up to 5 m. As each piece of ribbon is placed, complete the new information on a number line. Initially, just mark the improper fractions which are equivalent to the whole number.



5:10 You may find it helpful to follow the same progression previously used for quarters:

- Mark intermediate improper fractions and mixed numbers on the number line.
- Write a quantity of fifths, and present this quantity visually, as an improper fraction and as a mixed number.
- Convert between mixed numbers and improper fractions, without the support of a visual model.

For detailed guidance, refer back to steps *5:4–5:7*.

	 	
5:11	Before moving on to other units, give children further practice in converting between improper fractions and mixed numbers in quarters and fifths. You may wish to use some of the examples opposite. Remind children that they will need to think carefully about whether they are working in groups of four (quarters) or groups of five (fifths). Have number lines available as support, but it is important that you use these as a scaffold to build conceptual understanding, rather than something children use to 'get an answer' by reading it off the number line. Restrict the examples you offer to ones that the children can easily solve with times table facts. This is so they focus on the link between mixed numbers and improper fractions, rather than on performing unnecessarily complex calculations. Children should consolidate and further deepen their understanding through practice with a range of questions, such as a dòng nǎo jīn problem.	• *'Convert these mixed numbers to improper fractions.'* $2\frac{3}{5}$ $9\frac{1}{5}$ $6\frac{4}{5}$ $3\frac{1}{4}$ $7\frac{2}{4}$ $10\frac{3}{4}$ • *'Convert these improper fractions to mixed numbers.'* $\frac{8}{5}$ $\frac{24}{5}$ $\frac{31}{5}$ $\frac{6}{4}$ $\frac{15}{4}$ $\frac{29}{4}$ Dòng nǎo jīn: *'What is the same? What is different? Explain your thoughts as clearly as you can.'* $\frac{17}{3} = \square\frac{\square}{\square}$ $17 \div 3 = \square$ remainder $\square$
5:12	By this stage, children have worked with units of quarters and fifths in some depth. Now develop their confidence in working with any fractional unit to understand equivalences of mixed numbers and improper fractions. The key to this is accessing links to their times tables knowledge. For example, when working in sixths, children need to be thinking in groups of six:	

	• We *multiply* by six to convert mixed numbers to improper fractions: $4\frac{1}{6}$ is four groups of $\frac{6}{6}$, plus $\frac{1}{6}$. • We *divide* by six to convert improper fractions to mixed numbers, for example $\frac{20}{6}$: 20 divided into groups of six is three groups, with two remaining, which gives $3\frac{2}{6}$. Explain that they are now going to work with units other than quarters and fifths. You might wish to introduce this as follows: • *'When our unit was quarters, we thought about groups of four. There are $\frac{4}{4}$ in one whole.'* • *'When our unit was fifths, we thought about groups of five. There are $\frac{5}{5}$ in one whole.'* • *'I wonder what groups we will think about if our unit is sixths? We will be thinking about groups of ___.'* • *'And what about if our unit is sevenths? We will be thinking about groups of ___.'* Explain that each group is going to be working with a different unit, but each group will have 21 of these units (e.g. $\frac{21}{10}$ or $\frac{21}{7}$). Show the table opposite with just the first column completed, to help clarify this. They will be converting each of these improper fractions to a mixed number.	(table below)
5:13	Hand each group 21 blank counters and ask them to label each counter with the unit they are working in (for example, if they are working in eighths, they label each counter $\frac{1}{8}$). Wipeable whiteboard pens work well here. Alternatively, you may opt to label counters in advance with a permanent	

Improper fraction	**Prompt question**	**Mixed number**
$\frac{21}{10}$		
$\frac{21}{9}$		
$\frac{21}{8}$		
$\frac{21}{7}$		
$\frac{21}{6}$		
$\frac{21}{5}$		
$\frac{21}{4}$		
$\frac{21}{3}$		
$\frac{21}{2}$		

marker. Each group should end up with a set of 'unitising counters'. The counters for the eighths group are shown opposite.

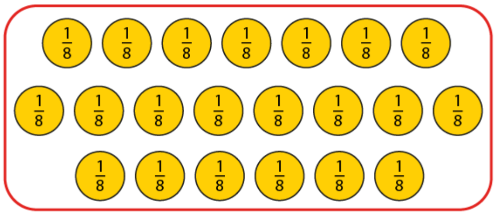
Reintroduce the following completed stem sentence, which you modelled in the previous step:

- *'Our unit is ___ so we will be thinking about groups of ___.'*
- *'There are ___ in one whole.'*

Ask each group to rearrange their counters so that they are organised into groups with a value of one. Repeatedly use the language from the stem sentence. For example:

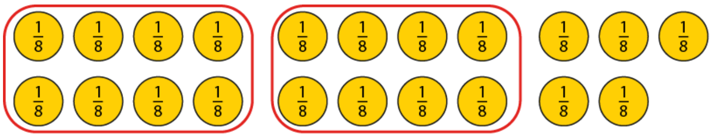
- *'Our unit is sixths so you are thinking about groups of ___?'*
- *'Six.'*
- *'Yes, there are six-sixths in one whole.'*

If the children work with the counters on a mini whiteboard, they can draw around each group, as shown opposite, to demonstrate their total.



$\frac{21}{8}$

- *'Our unit is eighths so we will be thinking about groups of eight.'*
- *'There are $\frac{8}{8}$ in one whole.'*



1 1 $\frac{5}{8}$

5:14 Go to each group in turn. At this point, you might find it helpful to display *3.5 Representations*, slide *51* of the counters. To check children's understanding, you could ask questions such as:

- *'How many groups of $\frac{10}{10}$ in $\frac{21}{10}$?'*

  'There are two groups.'
- *'And how many more/extra tenths are there?'* (See the prompt questions in the table.)
  'One-tenth.'

Show the children how to link their response to writing the mixed number $2\frac{1}{10}$.

Repeat for each of the other units with the support of the representation slide – discuss with each group in turn.

Continue to use the language which

Improper fraction	Prompt question	Mixed number
$\frac{21}{10}$	How many groups of $\frac{10}{10}$ in $\frac{21}{10}$? (2 groups and 1 more tenth.)	$2\frac{1}{10}$
$\frac{21}{9}$	How many groups of $\frac{9}{9}$ in $\frac{21}{9}$? (2 groups and 3 more ninths.)	$2\frac{3}{9}$
$\frac{21}{8}$	How many groups of $\frac{8}{8}$ in $\frac{21}{8}$? (2 groups and 5 more eighths.)	$2\frac{5}{8}$

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<td>links the unit (e.g. ninths) to the group size (groups of nine). As you work through the progression, the link between converting improper fractions to mixed numbers and division with remainders, should become clear.

For $\frac{21}{7}$ and $\frac{21}{3}$, the conversion results in a whole number rather than a mixed number. For these examples, look at the relationship between the denominator and numerators. In both of these cases the numerator is a multiple of the denominator. As shown in step 5:3, when the numerator is a multiple of the denominator, the fraction is equivalent to a whole number.</td>
<td>
<table>
<tr><td>$\frac{21}{7}$</td><td>How many groups of $\frac{7}{7}$ in $\frac{21}{7}$?
(3 groups and no sevenths left over.)</td><td>3</td></tr>
<tr><td>$\frac{21}{6}$</td><td>How many groups of $\frac{6}{6}$ in $\frac{21}{6}$?
(3 groups and 3 more sixths.)</td><td>$3\frac{3}{6}$</td></tr>
<tr><td>$\frac{21}{5}$</td><td>How many groups of $\frac{5}{5}$ in $\frac{21}{5}$?
(4 groups and 1 more fifth.)</td><td>$4\frac{1}{5}$</td></tr>
<tr><td>$\frac{21}{4}$</td><td>How many groups of $\frac{4}{4}$ in $\frac{21}{4}$?
(5 groups and 1 more quarter.)</td><td>$5\frac{1}{4}$</td></tr>
<tr><td>$\frac{21}{3}$</td><td>How many groups of $\frac{3}{3}$ in $\frac{21}{3}$?
(7 groups and no thirds left over.)</td><td>7</td></tr>
<tr><td>$\frac{21}{2}$</td><td>How many groups of $\frac{2}{2}$ in $\frac{21}{2}$?
(10 groups and 1 more half.)</td><td>$10\frac{1}{2}$</td></tr>
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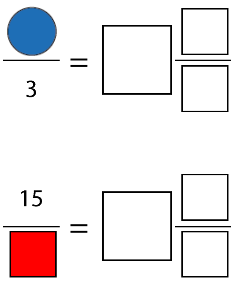
<tr>
<td>5:15</td>
<td>Children should now be able to start generalising how to convert an improper fraction to a mixed number. You may wish to use questions such as those opposite to support discussions around this.

To convert an improper fraction to a mixed number, we need to consider:
How many groups of the denominator can be made out of the numerator? This gives us the whole number part.</td>
<td>
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- What is remaining? This gives us the proper fraction part.

The ability to generalise in this way indicates a solid understanding, but do not encourage children to rely on a rule for the conversion. It is important that they understand *why* this generalisation works, so that they have a firm foundation for future learning.

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<td>**5:16**</td>
<td colspan="2">

Now look at sequences of mixed numbers and make sense of how they can be converted into improper fractions. Each of the conversions in the examples below will expose a different aspect for the children to focus on.

Begin by noting the denominator in the mixed number. This defines the unit we are working in, and so will also be the denominator for the improper fraction. When looking at the whole number, think about it as being made up of groups of the unit we are working in.

Look at *Example 1* below, $3\frac{1}{6}$, and the associated area model and number line. Confirm that our unit is *sixths*. Return to, and adapt, the stem sentence from step *5:6*: ***'There are ___ groups of ___- sixths which is ___ -sixths, and ___ more sixths, so that is ___-sixths.'***

Now look at the next fraction in the series, $3\frac{2}{6}$. Pose questions to probe their understanding, such as:

- *'How will the models change? What will be different about the area model compared to* $3\frac{1}{6}$*?'*
- *'What will be different about the number line?'*

You might find it helpful to display *3.5 Representations*, slide *53*.

Repeat for $3\frac{3}{6}$ and $3\frac{4}{6}$, talking about what will be the same and what will be different each time.

Children may quickly notice the pattern that the numerator is one more each time, but it is important to go beyond simply 'getting the answer', by continuing the pattern. Focus on:

- What is the structure that causes that pattern?
- What is the relationship between each mixed number and its equivalent improper fraction?
- What is the relationship between successive numbers?

Return to the stem sentence for each example to help the children gain confidence in moving from the mixed number to the improper fraction.

You might also ask what happens as the sequence continues, noting that when you get to $3\frac{6}{6}$ this is equivalent to 4 and also to $\frac{24}{6}$.

Now present a second set of examples (*Example 2* below), where the denominator rather than the numerator changes. As before, look at the first number in the sequence on the area model and number line. Use a stem sentence to help determine the equivalent improper fraction. Moving to the second fraction, there will be more changes to the models. Each whole is now split into eight equal parts rather than seven, and the equivalent fraction for three 'wholes' is $\frac{24}{8}$

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</table>

instead of $\frac{21}{7}$. Ask children what they notice about these two numbers:

- *'Why are they both equivalent to three?'*
- *'What is the relationship between the denominator and numerator in each of these?'*

Example 3 returns to working with a single denominator (sevenths). Work through the same process, calculating the first equivalence (supported by the models) and then comparing the subsequent numbers to the first. As the whole number increases by one each time, the numerator of the improper fraction will increase by seven each time. Note the change to $\frac{6}{7}$ in the final example.

After discussing each of the three examples, you may wish to display the exemplar as an opportunity to generalise. We can calculate the number of parts in an improper fraction from a mixed number by:

- multiplying the whole number by the denominator
- adding the numerator.

As mentioned in step *5:15*, developing the ability to generalise like this shows a good understanding, and enables children to apply the structure when converting any mixed number to an improper fraction. But do not be tempted to get children to just rely on a rule for the conversion. It is important for their future learning that they understand why this works.

Example 1:

Equation

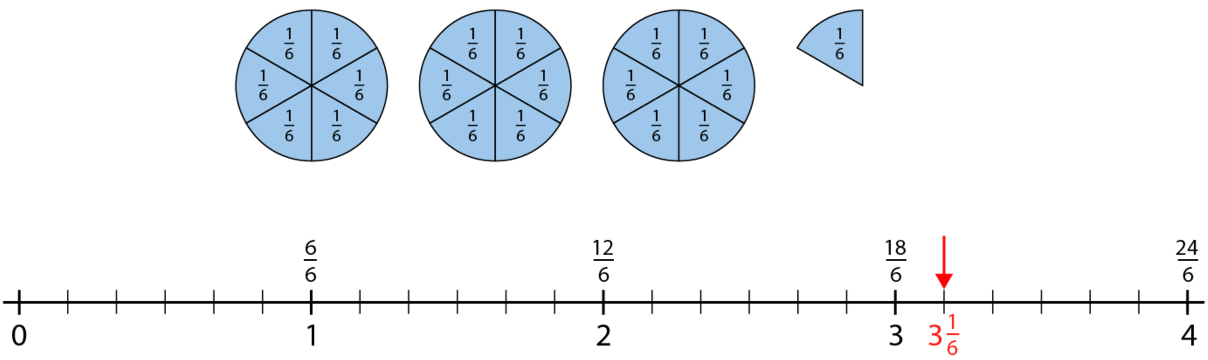
$3\frac{1}{6} = \frac{\square}{6}$ *'There are three groups of $\frac{6}{6}$ which is $\frac{18}{6}$, and one more sixth; that's $\frac{19}{6}$.'*

$3\frac{2}{6} = \frac{\square}{6}$ *'What will be different about the area model compared to $3\frac{1}{6}$?'*

$3\frac{3}{6} = \frac{\square}{6}$ *'What is the same? What is different?'*

$3\frac{4}{6} = \frac{\square}{6}$ *'What is the same? What is different?'*

Representations



Example 2:

Equations

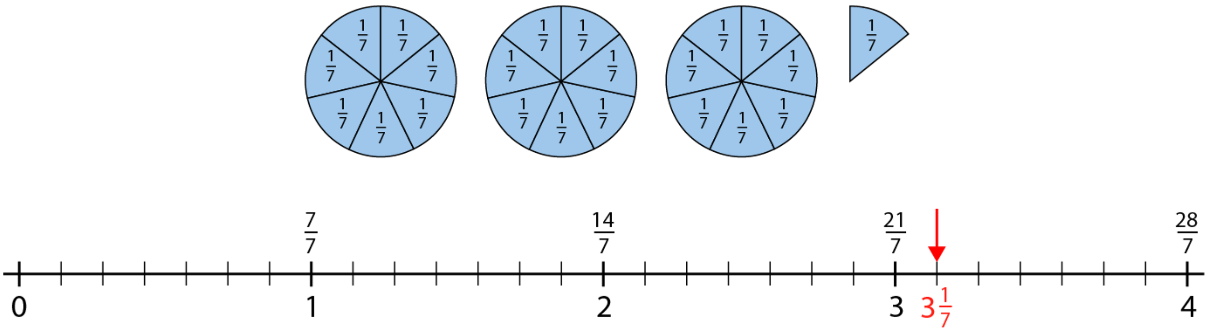
$3\frac{1}{7} = \frac{\square}{\square}$ *'There are three groups of $\frac{7}{7}$ which is $\frac{21}{7}$, and one more seventh, so that is $\frac{22}{7}$.'*

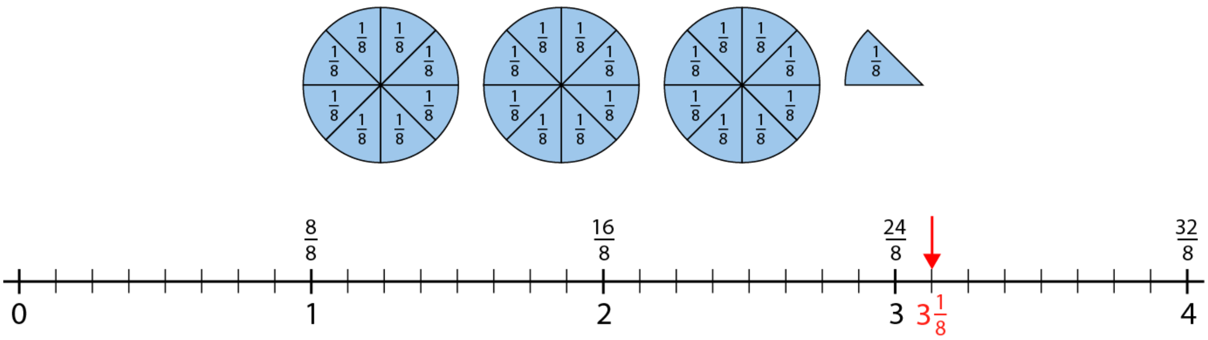
$3\frac{2}{8} = \frac{\square}{\square}$

$3\frac{3}{9} = \frac{\square}{\square}$

$3\frac{4}{10} = \frac{\square}{\square}$

Representations





Example 3:

Equations

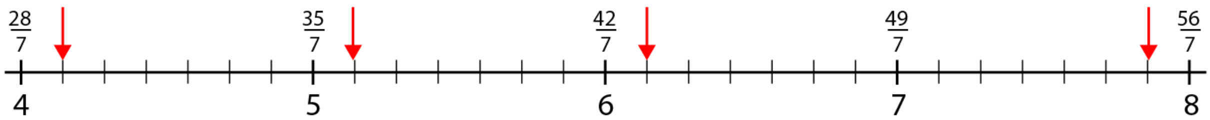
$4\frac{1}{7} = \frac{\square}{\square}$

$5\frac{1}{7} = \frac{\square}{\square}$

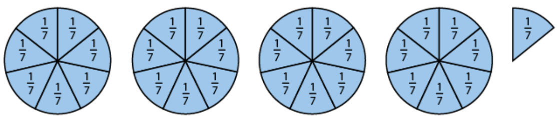
$6\frac{1}{7} = \frac{\square}{\square}$

$7\frac{6}{7} = \frac{\square}{\square}$

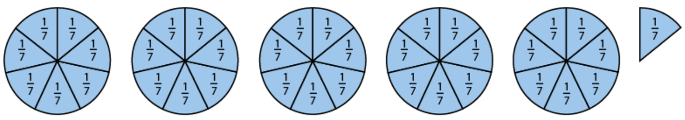
Representations



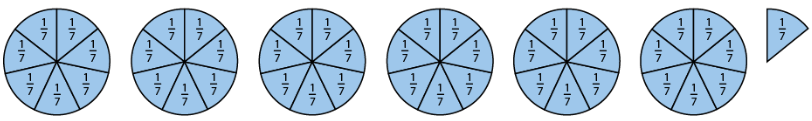
$4\frac{1}{7} = \frac{\square}{\square}$



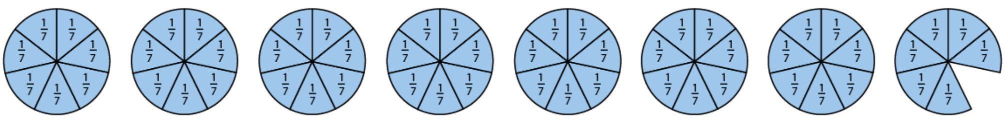
$5\frac{1}{7} = \frac{\square}{\square}$



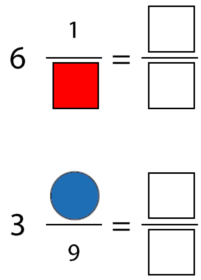
$6\frac{1}{7} = \frac{\square}{\square}$



$7\frac{6}{7} = \frac{\square}{\square}$



Exmplar:



5:17

By this stage, children should be able to convert between any improper fraction and mixed number. Provide varied practice to consolidate these skills, such as shown opposite.

To further deepen understanding of this concept, present dòng nǎo jīn problems like the ones presented here.

- *'Express the following improper fractions as mixed numbers.'*

  $\frac{17}{2}$ $\frac{13}{6}$ $\frac{28}{10}$ $\frac{41}{7}$

- *'Express the following mixed numbers as improper fractions.'*

  $4\frac{1}{8}$ $6\frac{4}{9}$ $3\frac{11}{12}$ $8\frac{2}{3}$

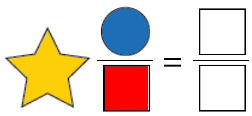
Dòng nǎo jīn:

- *'Look at the completed fractions. What do you know about fractions where the numerator is a multiple of the denominator?'*
- *'Fill in the missing numbers.'*

$\frac{14}{5} = \frac{\square}{4}$ $\frac{18}{9} = \frac{\square}{7}$

$\frac{\square}{8} = \frac{12}{3}$ $\frac{10}{\square} = \frac{24}{12}$

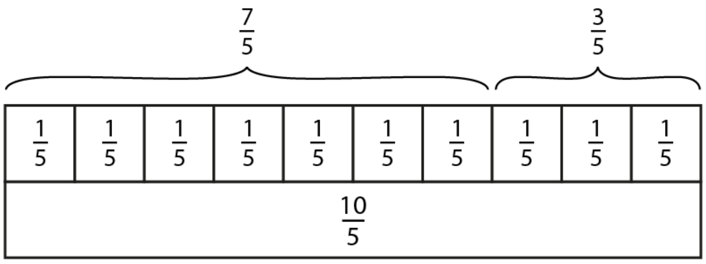
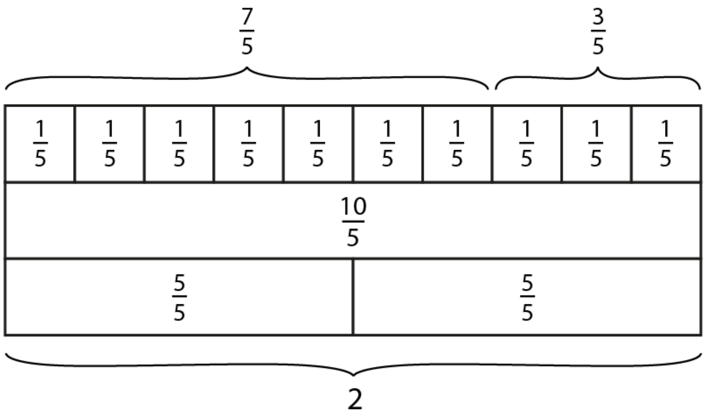
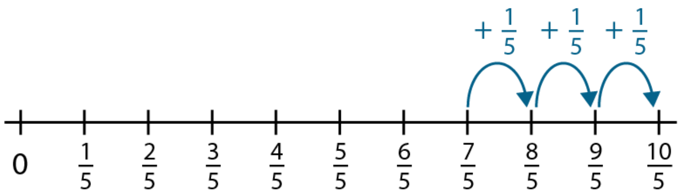
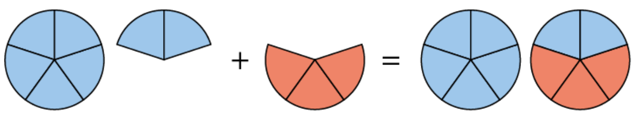
- *'What would the denominator be in this improper fraction? Write an equation to show how you would calculate the numerator.'*

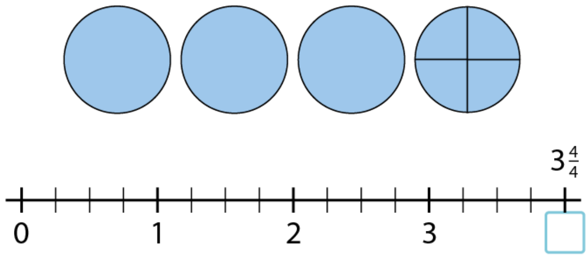
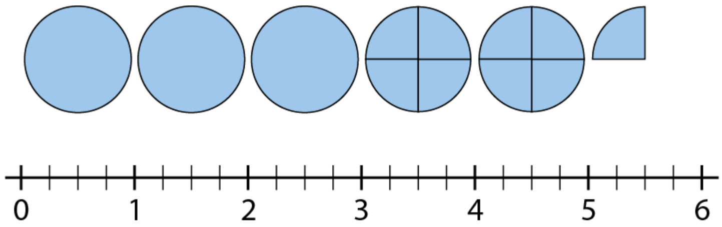
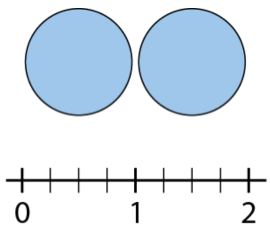


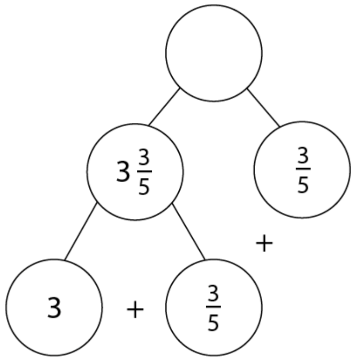
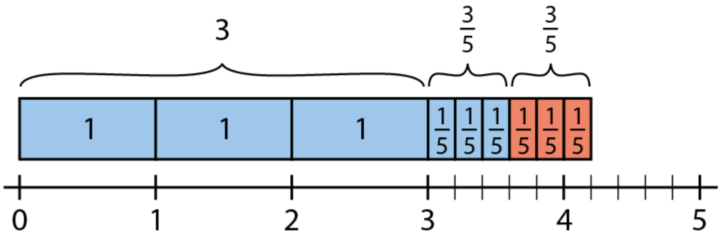
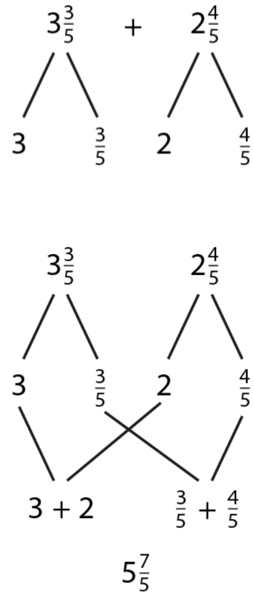
Teaching point 6:

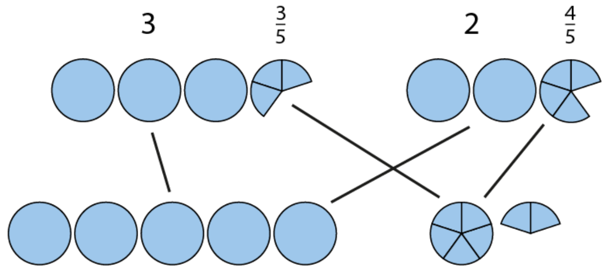
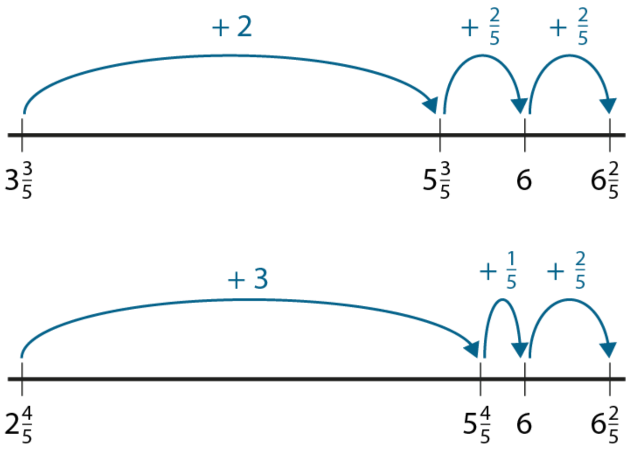
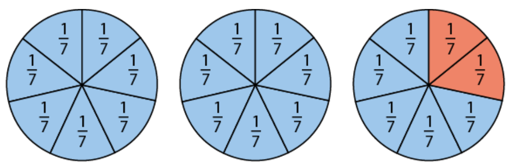
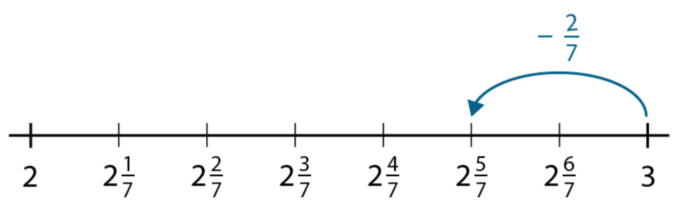
Improper fractions can be added and subtracted in the same was as proper fractions.

Steps in learning

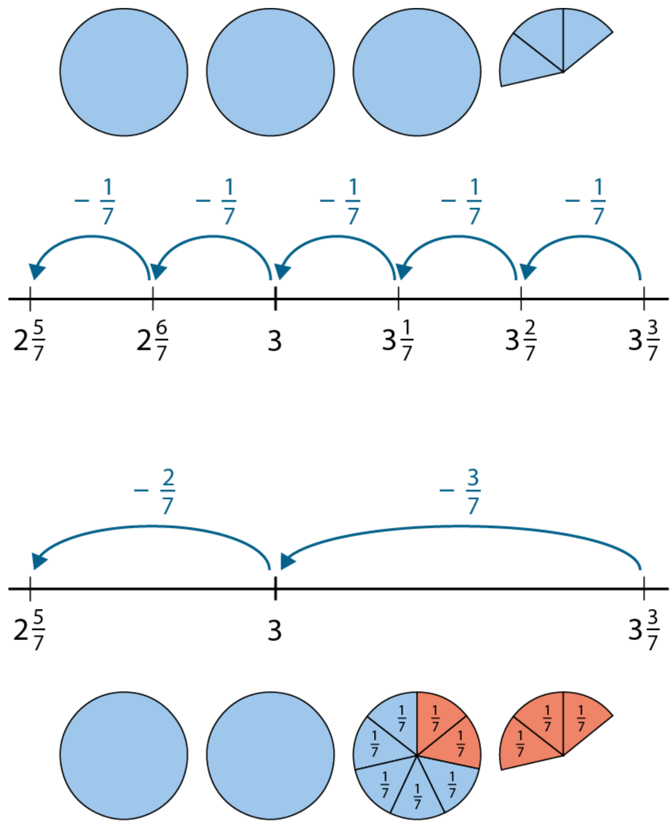
	Guidance	Representations
6:1	As a part of *Teaching point 3*, children practised adding and subtracting fractions that did not bridge. Children should now be confident in converting fractions between their improper and mixed number forms, so more complex calculations that involve bridging can be introduced. Begin with addition calculations that do not require conversions to solve but where solutions can be represented in different ways. This is one way to make children aware of how conversions can be used. For example, ask children if they agree or disagree with a calculation such as: $\frac{7}{5}+\frac{3}{5}=\mathbf{2}$ Instinctively, children may think that this calculation looks incorrect because fractions are added together but a whole number is given as the total. Asking children what type of fractions are added together, i.e. improper and proper fractions, may help more children to understand that this could be correct. The visual way to prove this calculation is correct is to use a pictorial representation and include an intermediate step: $\frac{7}{5}+\frac{3}{5}=\frac{10}{5}=\mathbf{2}$ Children may prefer different ways to view this calculation pictorially. A linear model is easier to draw, but a pie chart model makes it easier to see whether a calculation is correct. Both models could be displayed alongside each	$\frac{7}{5}+\frac{3}{5}=\mathbf{2}$  $\frac{7}{5}+\frac{3}{5}=\frac{10}{5}=\mathbf{2}$    

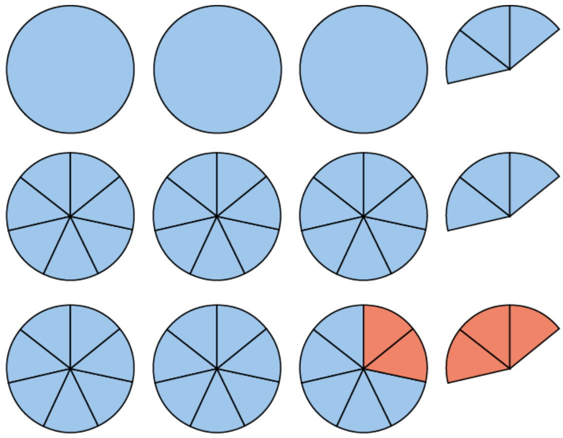
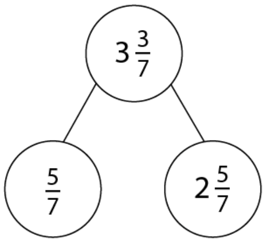
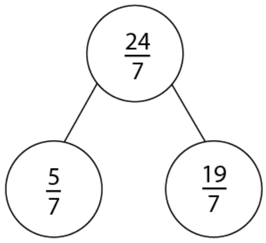
	other to show different ways to represent the same calculation. Explain that while it is mathematically accurate to leave the answer as $\frac{10}{5}$, it is usually written in its whole-number form, so in this case the answer is '2'.	
6:2	Now proceed to questions where a mixed number is added to a proper fraction so that it bridges a whole number. This will provide a situation where the total of the numerators will be greater than the denominator, i.e. the fractional part will be an improper fraction (e.g. $3\frac{3}{5}+2\frac{4}{5}=5\frac{7}{5}$) Explain to children that while the conventions of maths allow us to write $\frac{3}{5}+\frac{4}{5}=\frac{7}{5}$, it is not an accepted convention to write $3\frac{3}{5}+2\frac{4}{5}=5\frac{7}{5}$, with an improper fraction within a mixed number. *In a mixed number, the numerator must always be smaller than the denominator.* Before solving any calculations, allow the children to practise writing incorrectly-recorded mixed numbers in the correct form, e.g. • $5\frac{7}{5}=6\frac{2}{5}$ • $3\frac{4}{4}=$ • $3\frac{9}{4}=$ • $2\frac{0}{4}=$ Although this last calculation would not occur during addition, it could occur when subtracting proper or improper fractions from mixed numbers. It is therefore useful for children to understand what a numerator of zero means in the context of mixed numbers. Note: this is built on further in Spine 2: Multiplication and Division, *segment 2.12,* where the concept that a	$3\frac{4}{4}=$  $3\frac{9}{4}=$  $2\frac{0}{4}=$ 

	remainder cannot be greater than the divisor is explored.	
6:3	Once children are confident converting mixed numbers with improper fractional parts into an accepted format, provide opportunities for them to do this when adding mixed numbers and proper fractions together (e.g. $3\frac{3}{5}+\frac{3}{5}=?$). As before, the part–part–whole model is useful in showing how both the mixed number and proper fraction are parts. An area model and number line can be used to emphasise how the improper fractional part of the mixed number can be written in an alternative way.	
6:4	The final addition step is adding two mixed numbers together. This will again result in the fractional part of the mixed number being improper. There are different ways that children may choose to solve calculations of this type. Allow children to explore these different methods. Providing fraction tiles or an alternative physical manipulative will allow children to see different ways the calculation can be completed for a 'count all' aggregative model. Whereas a number line is more likely to show how it can be completed for a 'count on' augmentative model. Present an example calculation, such as $3\frac{3}{5}+2\frac{4}{5}=?$	$3\frac{3}{5}+2\frac{4}{5}=$ Aggregation model: 

	Aggregation model Children can use their understanding of mixed numbers to partition each mixed number into its whole number and fractional parts. As a result, it is possible to add the parts together in different orders. Adding the whole numbers together and then the fractional parts is a logical next step. These can then be combined to form a mixed number solution. The parts could be added together in any order, but this method is the most logical based on the children's understanding that the whole number is the most significant part. **Augmentation model** Children can also use a number line to support addition of mixed numbers. As they already know, addition is commutative, so they can start at either number and then add on the other mixed number. The fractional part could, of course, be added before the whole number part and this will give the same answer, but here, adding the whole number is shown first. Notice how the fractional part is partitioned to support bridging through the whole number. This method negates the need to complete a final conversion as it avoids being left with an improper fractional part of the mixed number.	• *Final conversion needed:* $5\frac{7}{5}=6\frac{2}{5}$  • *Final conversion needed:* $5\frac{7}{5}=6\frac{2}{5}$ Augmentation model: $3\frac{3}{5}+2\frac{4}{5}=$ 
6:5	When children are sufficiently confident with addition, move on to subtraction problems. Within these questions, children will need to bridge to and then back from a whole number. To prepare them for this, begin with questions where a fraction is subtracted from a whole number (e.g. $3-\frac{2}{7}=?$). Children will need to be secure with this before moving on, and it is a step which is surprisingly easy to neglect.	$3-\frac{2}{7}=?$  

	Starting with an area model showing this calculation will allow children to see why the solution is a mixed number, and why the denominator will be the same as the denominator of the proper fraction subtrahend. The same calculation can then also be shown on a number line. It may also help to encourage children to think of three as $2\frac{7}{7}$. For instance, $3-\frac{2}{7}=$ can be rewritten as $2\frac{7}{7}-\frac{2}{7}=$.	
6:6	Work through some similar examples, subtracting a proper fraction from a whole number using area models and number lines. Some potential questions have been provided opposite. The variation in these examples will focus attention on different aspects of subtracting a fraction from a whole number. Variation is a pedagogical tool which helps expose underlying structures and build understanding. Children can identify patterns within a sequence, but their attention should be drawn to the underlying structure. This can lead to generalisation and the ability to make sense of any similar calculation. As a class, work through each column of examples in turn, discussing the patterns within them, and the structures which give rise to these patterns.	*'Solve these equations.'* $3-\frac{1}{7}=$ $\quad$ $5-\frac{1}{6}=$ $6-\frac{1}{4}=$ $3-\frac{1}{6}=$ $\quad$ $5-\frac{2}{6}=$ $\quad$ $7-\frac{1}{4}=$ $3-\frac{1}{5}=$ $\quad$ $5-\frac{3}{6}=$ $8-\frac{1}{4}=$ $3-\frac{1}{4}=$ $\quad$ $5-\frac{4}{6}=$ $\quad$ $10-\frac{2}{4}=$

6:7	Once children have a solid understanding of the concept, provide practice (similar to that shown opposite) with examples shown out of sequence. For variation, include examples set in a real-life context.	*'Solve these equations.'* $5-\frac{2}{3}=$ $12-\frac{5}{9}=$ $3-\frac{7}{10}=$ Real-life contexts: • *'It is a 4 km cycle ride to my friend's house. I have cycled $\frac{3}{4}$ km. How much further do I have to go?'* • *'I have 5 m of rope. I cut off $\frac{4}{10}$ m. How much rope is left?'*
6:8	Once children understand how to subtract a fraction from a whole number, progress to questions where this skill will be required during the calculation (e.g. $3\frac{3}{7}-\frac{5}{7}$). Within this calculation, children will be exposed to the same calculation they encountered in the previous step of learning. Initially, present it with an area model to allow children to see how the calculation will bridge through a whole number. The steps of the calculation could be shown on the area model with the number line alongside. Complete the calculation, subtracting unit fractions one-by-one until the correct amount has been subtracted. Each time a unit fraction is subtracted, recap how much has been subtracted so far, e.g. *'One-seventh, two-sevenths, three-sevenths, four-sevenths, five-sevenths'*. The final step will then be to see what the solution is on both the area model and the number line. Ask children if they think the calculation could be completed in fewer steps. Children should see that the subtrahend can be split into two parts to jump back to the previous whole number in one jump and then the remaining part.	$3\frac{3}{7}-\frac{5}{7}$ 

6:9	An alternative way to complete subtraction problems involving mixed numbers is to convert the mixed number to an improper fraction. Introduce children to this method by asking showing them a calculation (e.g. $3\frac{3}{7}-\frac{5}{7}$) and ask *'How could I rewrite the mixed number* $3\frac{3}{7}$*?'* (It can be rewritten as $3\frac{3}{7}=\frac{24}{7}$) Then show how the calculation can be written using improper fractions: $3\frac{3}{7}-\frac{5}{7}=\frac{24}{7}-\frac{5}{7}=\frac{19}{7}$ The difference of $\frac{19}{7}$ can then be converted back to a mixed number: $2\frac{5}{7}$. Refer back to the area model to help the children make sense of the steps they have followed. Summarise the calculation in both forms (mixed-number and improper fraction) on part–part–whole models, as shown opposite. Discuss the two methods with the class, and the merits and points of difficulty of each. The class could discuss where mistakes might occur, based on these points of difficulty. In the mixed-number bridging approach, the hardest part is probably the subtraction from the whole number back to the final answer, where the children have to, for example, think of three as $2\frac{7}{7}$. In the improper-fraction approach, the actual subtraction is probably more straightforward. An error is most likely to occur when converting between a mixed number and an improper fraction.	  

6:10

As with the subtraction of whole numbers, subtractions on a number line and mental subtractions can sometimes be most efficiently performed by counting on to 'find the difference'. For example, a common way to think about 72 – 68 is, *'two more to get to 70 and another two to get to 72. That is a difference of four.'* The same idea applies to subtracting mixed numbers.

Start by explaining how either of the methods learnt so far will still work. Use the example $2\frac{1}{3}-1\frac{2}{3}$.

Reduction model

Begin with $2\frac{1}{3}$. Subtract the whole number (1) to make $1\frac{1}{3}$. Then subtract the fractional part ($\frac{2}{3}$); this can be made easier by splitting it into two jumps of $\frac{1}{3}$. As with addition, showing this on an empty number line will allow the jump over the whole number to be shown more clearly.

Converting to improper fractions

$2\frac{1}{3}-1\frac{2}{3}$

$2\frac{1}{3}=\frac{7}{3}$

$1\frac{2}{3}=\frac{5}{3}$

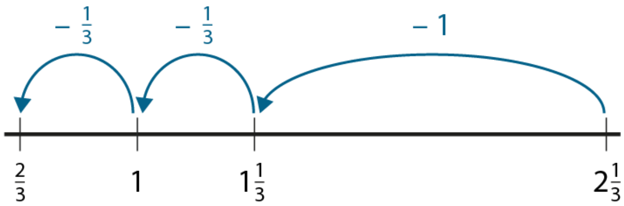
$2\frac{1}{3}-1\frac{2}{3}=\frac{7}{3}-\frac{5}{3}=\frac{2}{3}$

Subtraction by finding the difference

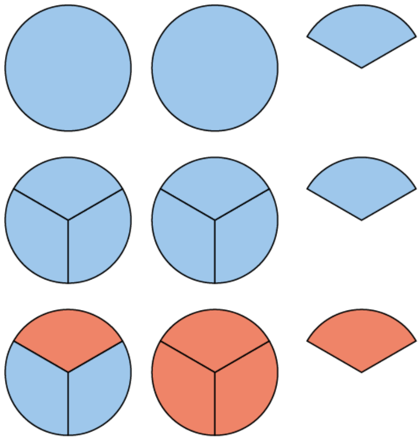
Although both the reduction model and converting-to-improper-fractions methods work, there is another, more efficient method. Look at the minuend and subtrahend in the equation $2\frac{1}{3}-1\frac{2}{3}$, and show it on a part–part–whole model. When children recognise that these numbers are quite close

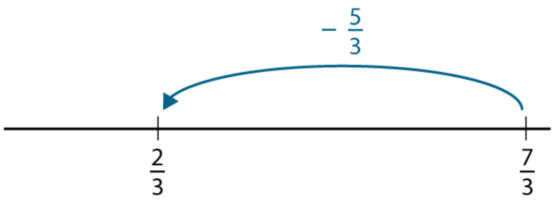
$2\frac{1}{3}-1\frac{2}{3}$

Reduction model:



Converting to improper fractions:

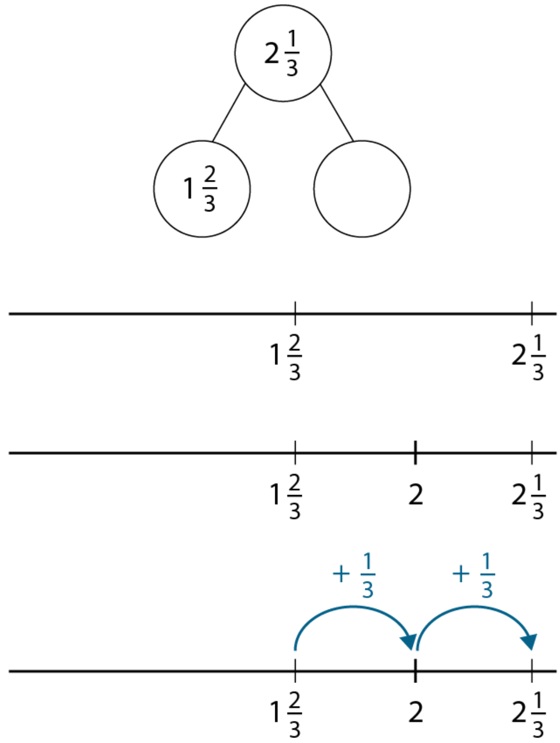




together, they might realise that the difference can be found by counting forward on the number line. This is a subtraction approach that children previously encountered in *Spine 1: Number, Addition and Subtraction*, segment *1.19*.

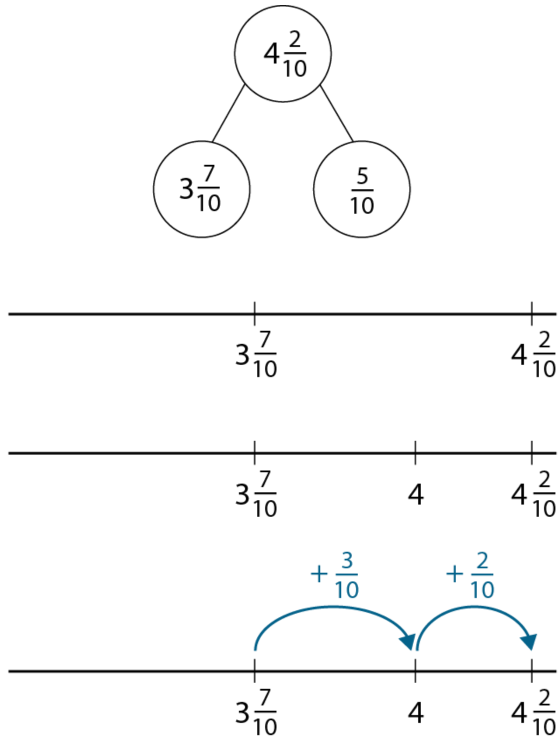
Work through this on a number line for $2\frac{1}{3} - 1\frac{2}{3}$, and then work through other examples where this strategy would be most appropriate, for example: $4\frac{2}{10} - 3\frac{7}{10}$.

Subtraction by finding the difference:



$2\frac{1}{3} - 1\frac{2}{3} = \frac{2}{3}$

$4\frac{2}{10} - 3\frac{7}{10} =$



6:11 When two mixed numbers are added, one possible method to complete the calculation is to treat the whole number and fractional parts separately, and then recombine them to find the solution. However, this approach is more problematic in calculations such as $2\frac{1}{3} - 1\frac{2}{3}$:

$2 - 1 = 1$

$\frac{1}{3} - \frac{2}{3} =$ a negative number.

If carried through, this method *does* work, as shown here, but it is beyond the programme of study at primary.

$2 - 1 = 1$

$\frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$

$1 + (-\frac{1}{3}) = \frac{2}{3}$

Just as when children learn to subtract two-digit numbers, they often incorrectly solve a subtraction in this way:

$53 - 39$

$50 - 30 = 20$

$9 - 3 = 6$

so,

$53 - 39 = 26$ ✗

The same error quite often occurs when subtracting mixed numbers if children partition the numbers and then subtract incorrectly.

$2 - 1 = 1$

$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

so,

$2\frac{1}{3} - 1\frac{2}{3} = 1\frac{1}{3}$ ✗

Now take a different example such as $4\frac{1}{6} - 3\frac{5}{6}$, shown below, and discuss it explicitly with the children. If children simply partition the numbers and start subtracting, this is what often happens:

$4 - 3 = 1$

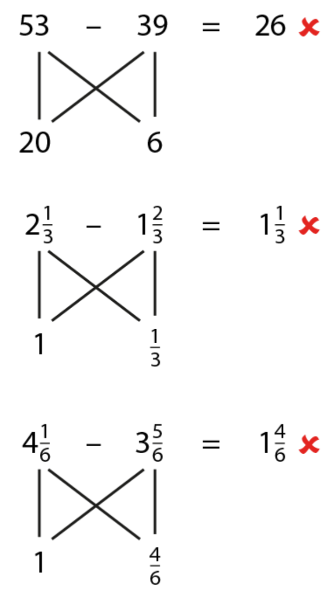
$\frac{5}{6} - \frac{1}{6} = \frac{4}{6}$

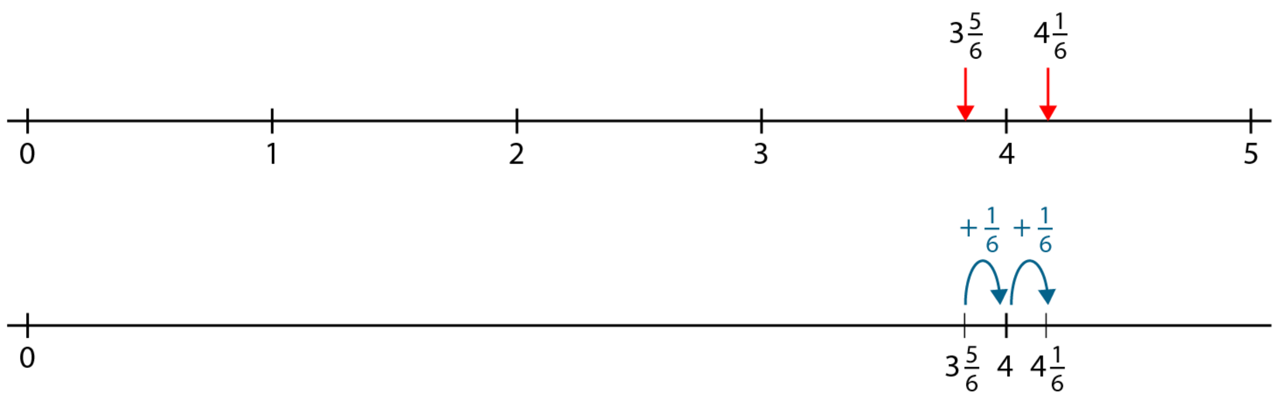
so

$4\frac{1}{6} - 3\frac{5}{6} = 1\frac{4}{6}$ ✗

Look at the position of $4\frac{1}{6}$ and $3\frac{5}{6}$ on a number line (see the example below). It will become evident that the difference can't possibly be $1\frac{4}{6}$.

Unpick what has gone wrong here: the strategy would have worked if both the whole-number part and fractional part were larger in the minuend. However, this isn't the case here; the error was subtracting the fractional part of the *minuend* ($\frac{1}{6}$) from the subtrahend ($\frac{5}{6}$). Using either of the other strategies that have been learnt would have avoided this error. In this particular case, using a 'counting up' approach and recognising that the difference between the minuend and subtrahend is just $\frac{2}{6}$ would have been the simplest.





6:12	Once children are confident with completing different addition and subtraction calculations, provide varied practice, including different contexts and word problems, as shown opposite.	Solving equations: *'Solve these calculations.'* $10\frac{2}{4}+4\frac{2}{5}=$ $4\frac{1}{7}+2\frac{6}{7}=$ $3\frac{9}{10}+2\frac{4}{10}=$ $12\frac{7}{8}-4\frac{1}{8}=$ $9\frac{3}{7}-8\frac{4}{7}=$ $6\frac{2}{5}-1\frac{4}{5}=$

Missing-number sequences:

'Fill in the missing numbers.'

$2\frac{1}{7}$	$2\frac{4}{7}$			$3\frac{6}{7}$	

8		$6\frac{2}{4}$		5	$4\frac{1}{4}$

Word problems:

- *'I walked* $10\frac{1}{4}$ *km one day and* $9\frac{3}{4}$ *km the next day. How far did I walk altogether?'*
- *'The table below shows how many hours Josie read each day for a week. How long in total did she spend reading during the week?'*

Mon	Tues	Wed	Thurs	Fri
$1\frac{3}{4}$ hours	1 hour	$1\frac{1}{4}$ hours	$1\frac{1}{4}$ hours	$2\frac{3}{4}$ hours

- *'An athlete records how much water she drinks over the course of a day. Before her final training session, she drank* $1\frac{2}{5}$ *litres of water. She drinks another* $\frac{4}{5}$ *litre during the final session. How much water does she drink altogether?'*
- *'A school day is* $6\frac{1}{4}$ *hours long.* $1\frac{3}{4}$ *hours are spent at break. How many hours of learning are there in a school day?'*
- *'A tailor has* $3\frac{7}{10}$ *m of ribbon. She uses* $1\frac{9}{10}$ *m to complete a dress. How much ribbon is left?'*
- *'At the beginning of snack break there are 15 oranges. After break there are* $3\frac{2}{6}$ *oranges left. How many oranges were eaten?'*