Mastery Professional Development

2 Operating on number

2.1 Arithmetic procedures

Guidance document | Key Stage 3

Making connections

The NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of 'core concepts'.

The second of these themes is *Operating on number*, which covers the following interconnected core concepts:

2.1 **Arithmetic procedures**

2.2 Solving linear equations

This guidance document breaks down core concept *2.1 Arithmetic procedures* into five statements of knowledge, skills and understanding:

2.1.1 Understand and use the structures that underpin addition and subtraction strategies

2.1.2 Understand and use the structures that underpin multiplication and division strategies

2.1.3 Know, understand and use fluently a range of calculation strategies for addition and subtraction of fractions

2.1.4 Know, understand and use fluently a range of calculation strategies for multiplication and division of fractions

2.1.5 Use the laws and conventions of arithmetic to calculate efficiently

Then, for each of these statements of knowledge, skills and understanding we offer a set of key ideas to help guide teacher planning.

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Overview

An understanding of and ability to use standard arithmetic procedures for all four operations with integers and decimals, as well as procedures for some calculations with fractions, should be well established at Key Stage 2. Work in Key Stage 3 should develop this both conceptually and procedurally by:

- ensuring students have a strong understanding of the mathematical structures that underpin these standard procedures
- ensuring students generalise these standard procedures with integers, extending to use with decimals, and appreciate that the structures are the same
- extending work on fractions to include multiplication of any two fractions and the division of any fraction by another
- building on students' Key Stage 2 experiences of positive and negative numbers to develop a full understanding and fluency with procedures for all four operations with directed numbers.

Students should also develop fluency with a range of calculation approaches and techniques involving combinations of numbers (positive and negative integers, decimals and fractions) and operations. They should develop an ability to exploit number relationships and structures in order to calculate efficiently. For example, students should notice that the calculation $0.43 \times 26.2 + 2.62 \times 5.7$ can be transformed into $0.43 \times 26.2 + 26.2 \times 0.57$ and so simplified to $(0.43 + 0.57) \times 26.2$, which is equal to 26.2.

Key to students' development in Key Stage 3 is not only a secure proficiency with arithmetic procedures, but also a connected understanding of the underlying concepts and an ability to think and calculate creatively with complex and multi-faceted calculations.

Prior learning

Before beginning to teach *Arithmetic procedures* at Key Stage 3, students should already have a secure understanding of the following from previous study:

Key stage	Learning outcome
Upper Key Stage 2	- Add and subtract whole numbers with more than four digits, including using formal written methods (columnar addition and subtraction) - Multiply and divide whole numbers and those involving decimals by 10, 100 and 1 000 - Multiply multi-digit numbers up to four digits by a two-digit whole number using the formal written method of long multiplication - Divide numbers up to four digits by a two-digit whole number using the formal written method of long division, and interpret remainders as whole number remainders, fractions, or by rounding, as appropriate for the context - Divide numbers up to four digits by a two-digit number using the formal written method of short division where appropriate, interpreting remainders according to the context

- Use their knowledge of the order of operations to carry out calculations involving the four operations
- Solve addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why
- Solve problems involving addition, subtraction, multiplication and division
- Use estimation to check answers to calculations and determine, in the context of a problem, an appropriate degree of accuracy
- Add and subtract fractions with different denominators and mixed numbers, using the concept of equivalent fractions
- Multiply simple pairs of proper fractions, writing the answer in its simplest form (e.g. $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$)
- Divide proper fractions by whole numbers (e.g. $\frac{1}{3} \div 2 = \frac{1}{6}$)
- Multiply one-digit numbers with up to two decimal places by whole numbers
- Use written division methods in cases where the answer has up to two decimal places
- Use negative numbers in context, and calculate intervals across zero

You may find it useful to speak to your partner schools to see how the above has been covered and the language used.
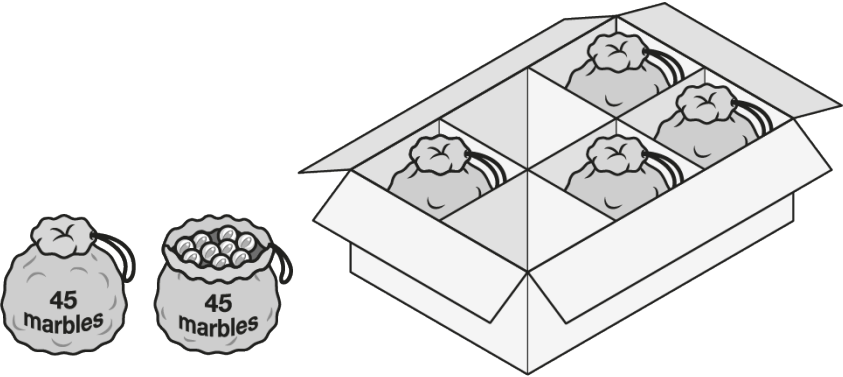
The prior learning is covered in a number of the segments from Spines 1, 2 and 3 of the NCETM primary mastery professional development materials[1].
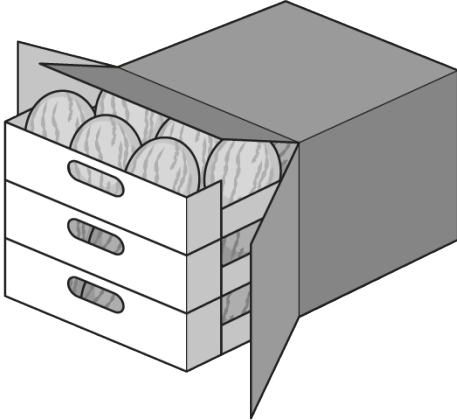
Checking prior learning

The following activities from the NCETM primary assessment materials[2] and the Standards & Testing Agency's past mathematics papers[3] offer useful ideas for assessment, which you can use in your classes to check whether prior learning is secure:

Reference	Activity
Year 6 page 15	*Find numbers to complete these number sentences.* $736 \div 23 = \square$ $\square \times 100 = 2\,400$ $\square \times 100 = 10 \times \square$ $7\,360 \div 230 = \square$ $25 \times \square = 200$ $25 \times \square = 4 \times \square$ $230 \times 24 = \square$ $23 \times \square = 161$ $23 \times \square = 161 \times \square$ $240 \times 23 = \square$ $24 \times \square = 168$ $24 \times \square = 168 \times \square$ $1\,668 \div 8 = \square$ $161 \div \square = 23$ $161 \div \square = 23 \times \square$ $208.5 \times 8 = \square$ $\square \div 25 = 9$ $\square \div 25 = 9 \times \square$

Year 6 page 16	*Work out:* • *8.4 × 3 + 8.4 × 7* • *6.7 × 5 − 0.67 × 50* • *93 × 0.2 + 0.8 × 93* • *7.2 × 4 + 3.6 × 8*
Year 6 page 22	*In each number sentence, replace the boxes with different whole numbers less than 20 so that the number sentence is true.* $$\frac{1}{\square} \times \frac{3}{\square} = \frac{\square}{\square}$$ $$\frac{\square}{\square} \times \frac{\square}{\square} = \frac{8}{15}$$ $$\frac{2}{\square} \times \frac{5}{\square} < \frac{10}{\square}$$ $$\frac{\square}{\square} \div 3 = \frac{1}{\square}$$ $$\frac{\square}{\square} \div 3 > \frac{1}{4}$$
2016 Key Stage 2 Mathematics Paper 1: arithmetic Question 33	$$\frac{3}{5} \div 3 = \boxed{}$$ Source: *Standards & Testing Agency* Public sector information licensed under the Open Government Licence v3.0

2016 Key Stage 2 Mathematics Paper 3: reasoning Question 11	*A toy shop orders 11 boxes of marbles.* *Each box contains 6 bags of marbles.* *Each bag contains 45 marbles.* *How many **marbles** does the shop order in total?* *Show your method.*
2016 Key Stage 2 Mathematics Paper 3: reasoning Question 21	*5 542 ÷ 17 = 326* *Explain how you can use this fact to find the answer to **18 × 326***
2017 Key Stage 2 Mathematics Paper 1: arithmetic Question 15	$\dfrac{4}{6} \times \dfrac{3}{5} = \boxed{}$
2018 Key Stage 2 Mathematics Paper 1: arithmetic Question 22	$43\overline{)645}$
2018 Key Stage 2 Mathematics Paper 1: arithmetic Question 29	$\begin{array}{r} 5\ \ 4\ \ 1\ \ 3 \\ \times \qquad 8\ \ 6 \\ \hline \\ \hline \end{array}$

2018 Key Stage 2 Mathematics Paper 1: arithmetic Question 31	$\frac{1}{4} \div 2 = \boxed{}$
2018 Key Stage 2 Mathematics Paper 1: arithmetic Question 32	$9^2 - 36 \div 9 = \boxed{}$
2018 Key Stage 2 Mathematics Paper 1: arithmetic Question 35	$4\,\frac{2}{3} - 1\,\frac{6}{7} = \boxed{}$
2018 Key Stage 2 Mathematics Paper 2: reasoning Question 15	*A box contains trays of melons.* *There are 15 melons in a tray.* *There are 3 trays in a box.* *A supermarket sells **40** boxes of melons.* *How many melons does the supermarket sell?* *Show your method.*

2018 Key Stage 2 Mathematics Paper 2: reasoning Question 23	*The length of a day on Earth is 24 hours.* *The length of a day on Mercury is $58\frac{2}{3}$ times the length of a day on Earth.* *What is the length of a day on Mercury, in **hours**?* *Show your method.* Source: *Standards & Testing Agency* Public sector information licensed under the Open Government Licence v3.0
2018 Key Stage 2 Mathematics Paper 3: reasoning Question 4	*Write the three missing digits to make this **addition** correct.* 5 3 2 ☐ 9 + ☐ 7 4 2 ☐ ——————— ☐ 0 6 7 6 Source: *Standards & Testing Agency* Public sector information licensed under the Open Government Licence v3.0
2018 Key Stage 2 Mathematics Paper 3: reasoning Question 18	*This is a diagram of a vegetable garden.* *It shows the fractions of the garden planted with potatoes and cabbages.* **potatoes** $\frac{2}{3}$ **cabbages** $\frac{1}{4}$ **carrots** *Not to scale* *The remaining area is planted with carrots.* *What **fraction** of the garden is planted with carrots?* *Show your method.* Source: *Standards & Testing Agency* Public sector information licensed under the Open Government Licence v3.0

Key vocabulary

Term	Definition
additive identity	An identity is a number such that when another number is combined with it (using a given operation) it does not change that number. The additive identity (i.e. the identity for addition and subtraction) is 0.
associative	A binary operation $*$ on a set S is associative if $a * (b * c) = (a * b) * c$ for all a, b and c in the set S.
	Addition of real numbers is associative, which means $a + (b + c) = (a + b) + c$ for all real numbers a, b and c. It follows that, for example, $1 + (2 + 3) = (1 + 2) + 3$.
	Similarly, multiplication is associative.
	Subtraction and division are not associative because $1 - (2 - 3) = 1 - (-1) = 2$, whereas $(1 - 2) - 3 = (-1) - 3 = -4$ and $1 \div (2 \div 3) = 1 \div \frac{2}{3} = \frac{3}{2}$, whereas $(1 \div 2) \div 3 = \left(\frac{1}{2}\right) \div 3 = \frac{1}{6}$.
commutative	A binary operation $*$ on a set S is commutative if $a * b = b * a$ for all a and $b \in S$.
	Addition and multiplication of real numbers are commutative where $a + b = b + a$ and $a \times b = b \times a$ for all real numbers a and b. It follows that, for example, $2 + 3 = 3 + 2$ and $2 \times 3 = 3 \times 2$.
	Subtraction and division are not commutative since, as counter examples, $2 - 3 \neq 3 - 2$ and $2 \div 3 \neq 3 \div 2$.
distributive	One binary operation $*$ on a set S is distributive over another binary operation $\bullet$ on that set if $a * (b \bullet c) = (a * b) \bullet (a * c)$ for all a, b and $c \in S$.
	For the set of real numbers, multiplication is distributive over addition and subtraction since $a(b + c) = ab + ac$ for all a, b and c real numbers. It follows that $4(50 + 6) = (4 \times 50) + (4 \times 6)$ and $4 \times (50 - 2) = (4 \times 50) - (4 \times 2)$.
	For division, $\frac{(a+b)}{c} = \frac{a}{c} + \frac{b}{c}$ (division is distributive over addition), but $\frac{c}{(a+b)} \neq \frac{c}{a} + \frac{c}{b}$ (addition is not distributive over division).
	Addition, subtraction and division are not distributive over other number operations.
multiplicative identity	An identity is a number such that when another number is combined with it (using a given operation) it does not change that number. The multiplicative identity (i.e. the identity for multiplication and division) is 1.

rational number	A number that is an integer or that can be expressed as a fraction whose numerator and denominator are integers, and whose denominator is not zero. Examples: -1, $\frac{1}{3}$, $\frac{3}{5}$, 9, 235. Rational numbers, when expressed as decimals, are recurring decimals or finite (terminating) decimals. Numbers that are not rational are irrational. Irrational numbers include $\sqrt{5}$ and π, which produce infinite, non-recurring decimals.
reciprocal	The multiplicative inverse of any non-zero number. Any non-zero number multiplied by its reciprocal is equal to 1. In symbols, $x \times \frac{1}{x} = 1$, for all $x \neq 0$. Multiplying by $\frac{1}{x}$ is the same as dividing by x, and – since division by zero is not defined – zero has to be excluded from all other numbers that have a reciprocal.

Collaborative planning

Below we break down each of the five statements within *Arithmetic procedures* into a set of key ideas to support more detailed discussion and planning within your department. You may choose to break them down differently depending on the needs of your students and timetabling; however, we hope that our suggestions help you and your colleagues to focus your teaching on the key points and avoid conflating too many ideas.

Please note: We make no suggestion that each key idea represents a lesson. Rather, the 'fine-grained' distinctions we offer are intended to help you think about the learning journey irrespective of the number of lessons taught. Not all key ideas are equal in length and the amount of classroom time required for them to be mastered will vary, but each is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

The following letters draw attention to particular features:

D   Suggested opportunities for **deepening** students' understanding through encouraging mathematical thinking.

L   Examples of shared use of **language** that can help students to understand the structure of the mathematics. For example, sentences that all students might say together and be encouraged to use individually in their talk and their thinking to support their understanding (for example, *'The smaller the denominator, the bigger the fraction.'*).

R   Suggestions for use of **representations** that support students in developing conceptual understanding as well as procedural fluency.

V   Examples of the use of **variation** to draw students' attention to the important points and help them to see the mathematical structures and relationships.

PD   Suggestions of questions and prompts that you can use to support a **professional development** session.

For selected key ideas, marked with an asterisk (*), we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches, together with suggestions and prompts to support professional development and collaborative planning. You can find these at the end of the set of key ideas.

Key ideas

2.1.1 Understand and use the structures that underpin addition and subtraction strategies

Adding and subtracting integers and, to some extent, decimals using the standard columnar format will be familiar to students from Key Stage 2.

The focus in Key Stage 3 is on deeply understanding the structures underpinning the standard columnar format and generalising fully to decimals (i.e. not regarding calculation with decimals as a separate method).

A key idea is that of 'unitising' – adding quantities of the same 'unit'. For example, the standard columnar method with decimal numbers exploits the idea that hundreds can be added to hundreds, tens to tens, ones to ones, tenths to tenths, hundredths to hundredths, etc., and this gives meaning to why decimals need to be aligned as they do in the standard method.

The use of negative numbers extends the domain in which students can explore and deepen their understanding of the additive structure. This can bring to the surface misconceptions based on previous experiences of addition and subtraction with positive numbers (i.e. adding a number always increases and subtracting a number always decreases).

Broadening the range of possible examples students explore and work with will deepen their understanding of the underlying additive structures.

2.1.1.1* Understand the mathematical structures that underpin addition and subtraction of positive and negative integers

2.1.1.2* Generalise and fluently use written addition and subtraction strategies, including columnar formats, with decimals

2.1.2 Understand and use the structures that underpin multiplication and division strategies

A key feature of the standard algorithm for the multiplication of integers is that it involves sequences of multiplications of single-digit numbers; place-value considerations and the lining up of columns ensure that the product is of the correct order of magnitude. When using the method with decimals, it is important that the underlying mathematical structure is thoroughly understood, e.g. $300 \times 7\,000$ can be considered as $3 \times 100 \times 7 \times 1\,000 = 3 \times 7 \times 100 \times 1\,000$. This awareness supports informal calculation methods and underpins the columnar methods. When multiplying decimals, it is important for students to understand, for example, that $0.3 \times 0.007 = 3 \times 7 \times 0.1 \times 0.001$ and, therefore, how 3×7 and 0.3×0.007 are connected.

When dividing one decimal by another it will be important for students to understand how multiplying and dividing the dividend and the divisor by 10, 100, etc. changes the quotient, i.e. that, for example, $74 \div 3 = 7.4 \div 0.3 = 0.74 \div 0.03$, etc., and that, for example, $7.4 \div 3$ is ten times smaller than $74 \div 3$; $74 \div 0.3$ is ten times bigger than $74 \div 3$ and $74 \div 0.003$ is one thousand times bigger than $74 \div 3$.

These various awarenesses come together to give meaning to the idea that a calculation such as 3.14×5.6 can be calculated as $(314 \times 56) \div 1\,000$ and that $25.7 \div 0.32$ can be calculated as $2\,570 \div 32$.

Multiplication and division involving negative integers is also introduced in this set of key ideas. It is important to explore why the rules for combining positive and negative numbers work and to avoid rote learning of the rules without meaning. For example, the structure $-a \times 0 = -a \times (+b + -b)$ together with the application of the distributive law can be used to give meaning to the fact that the product of two negative numbers is a positive number.

2.1.2.1* Understand the mathematical structures that underpin multiplication and division of positive and negative integers

2.1.2.2   Factorise multiples of 10^n in order to simplify multiplication and division of both integers and decimals, e.g. $300 \times 7\,000$, 0.3×0.007, $0.9 \div 0.03$, etc.

2.1.2.3* Generalise and fluently use written multiplication strategies to calculate accurately with decimals

2.1.2.4   Generalise and fluently use written division strategies to calculate accurately with decimals

2.1.3 Know, understand and use fluently a range of calculation strategies for addition and subtraction of fractions

Students will have been taught strategies for the addition and subtraction of fractions with same and different denominators and mixed numbers during Key Stage 2. The focus in this set of key ideas is to use addition and subtraction of fractions to further expand the range of possible examples that students are able to explore as their understanding of additive structures grows and matures.

Here, unitising is again a key idea and one that is particularly evident when working with fractions. For example, adding halves and thirds is not using the same 'unit'; however, converting both to sixths means that both have the same unit and addition is relatively straightforward.

Students should develop an understanding of the additive structures underpinning the operations, as well as fluency with strategies for adding and subtracting a wide range of types of fractions (including improper fractions).

2.1.3.1   Understand the mathematical structures that underpin the addition and subtraction of fractions

2.1.3.2   Generalise and fluently use addition and subtraction strategies to calculate with fractions and mixed numbers

2.1.4 Know, understand and use fluently a range of calculation strategies for multiplication and division of fractions

There is a danger that students see the mathematics curriculum as a set of separate topics, each with its own set of rules and techniques. This unconnected view of the curriculum can result in an entirely instrumental and procedural approach to mathematics, with no sense of conceptual coherence. It is, therefore, important that students see fractions or rational numbers as a part of a unified number system and that the operations on such numbers are related and connected to previously taught and learnt concepts for integers. For instance, the area model used for multiplication with integers can also be used for fractions.

For example, $\frac{1}{7} \times \frac{1}{5}$ can be represented as:

$$\frac{1}{7} \times \frac{1}{5}$$
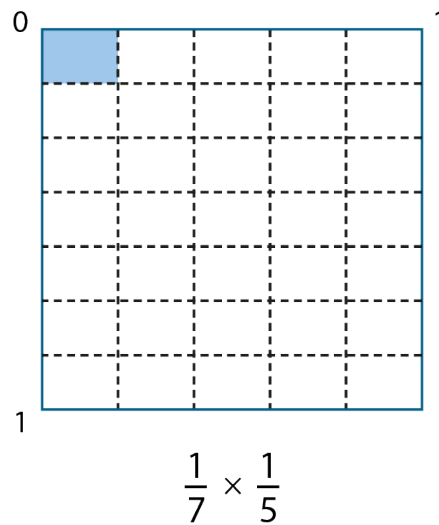
In the key ideas below, multiplication with mixed numbers has been given a separate focus. The rationale for this is that the different possible representations for multiplying mixed numbers – such as converting to improper fractions or partitioning as a pair of binomials, e.g. $\left(2+\frac{3}{4}\right)\left(1+\frac{2}{3}\right)$ – may offer different and deeper insights into multiplication.

2.1.4.1* Understand the mathematical structures that underpin the multiplication of fractions

2.1.4.2* Understand how to multiply unit, non-unit and improper fractions

2.1.4.3   Generalise and fluently use strategies to multiply with mixed numbers (e.g. $2\frac{3}{4} \times 1\frac{2}{3}$)

2.1.4.4   Understand the mathematical structures that underpin the division of fractions

2.1.4.5   Divide a fraction by a whole number

2.1.4.6   Divide a whole number by a fraction

2.1.4.7   Divide a fraction by a fraction

2.1.5   Use the laws and conventions of arithmetic to calculate efficiently

Previous statements of knowledge, skills and understanding in this core concept have developed students' awareness and understanding of ideas such as unitising when working additively and scaling when multiplying. The focus has been on broadening the domain of examples that students can draw on when calculating and, through this, deepening their understanding of these operations. This set of key ideas is focused on ways in which these operations fit together and the structures that they have in common.

Students should both know and notice examples of the commutative [$ab = ba$, $a + b = b + a$], associative [$abc = (ab)c = a(bc)$; $a + b + c = (a + b) + c = a + (b + c)$] and distributive laws [$a(b + c) = ab + ac$] and need to be able to calculate fluently with the full range of different types of numbers in a wide range of contexts and problem-solving situations, exploiting these laws to increase the efficiency of calculation.

2.1.5.1   Know the commutative law and use it to calculate efficiently

2.1.5.2   Know the associative law and use it to calculate efficiently

2.1.5.3   Know the distributive law and use it to calculate efficiently

2.1.5.4   Calculate using priority of operations, including brackets, powers, exponents and reciprocals

2.1.5.5* Use the associative, distributive and commutative laws to flexibly and efficiently solve problems

2.1.5.6   Know how to fluently use certain calculator functions and use a calculator appropriately

Exemplified key ideas

2.1.1.1 Understand the mathematical structures that underpin addition and subtraction of positive and negative integers

Common difficulties and misconceptions
Although students have been introduced to negative numbers at Key Stage 2, their experience is likely to have been set in a context, and it is unlikely that they will have carried out operations with negative numbers.

Students are now working with a new 'type' of number and, in doing so, challenging and extending their understanding of additive operations. Until the introduction of negative numbers, students' experience will have been that addition makes larger and subtraction makes smaller. The inclusion of situations in which this is not the case can be problematic.

When assessing students' understanding of operations with negative numbers, Hart (1981)[†] records that, when subtracting a positive integer from either a positive or negative number, many students simply subtract the numerals and then attempt to determine the sign for the answer while, when subtracting a negative integer, many students used the rule that 'two negatives make a positive'. Examining the structure of such calculations (using classroom examples, such as the ones offered below) rather than teaching such rules will help students overcome these difficulties.

What students need to understand	Guidance, discussion points and prompts
Use 'zero pairs' to partition a number. *Example 1:* *Fill in the blanks to make the calculations correct.* a) $3 + 4 - 4 - \boxed{} = 0$ b) $3 - 3 + 1 - \boxed{} = 0$ c) $15 + 7 - 15 = \boxed{}$ d) $10 - 7 - \boxed{} = 0$ e) $182 - 82 - \boxed{} = -1$	**V** *Example 1* is designed to draw students' attention to the way that pairs of numbers can be used so that an answer can be found without the need for calculating. In parts a), b) and c), the pairing of numbers is clear. In part d), students need to identify that the calculation can be thought of as $(3 + 7) - 7 - \rule{1cm}{0.4pt} = 0$. It is awareness of this partitioning that is key.
Example 2: *These counters show −2.* 	**R** By using counters, and particularly zero pairs of counters (where a positive and negative counter combine to give a value of zero), students can extend their understanding of partitioning numbers to include negative numbers.

[†] K. M. Hart (ed.), 1981, *Children's Understanding of Mathematics: 11–16*, London, John Murray

These counters also show −2.
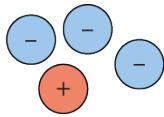
Describe how you found these arrangements.

The intention in *Example 2* is for students to have a flexible understanding of numbers and be able to exploit mathematical structures to manipulate numbers in order to calculate efficiently.

V You might follow *Example 2* by asking students to form arrangements of counters to show +2. By working with positive and negative numbers of the same absolute value, students have access to the symmetry of the number line around zero, should you wish to explore this with them.

Expand understanding of addition and subtraction to include calculation with negative numbers.
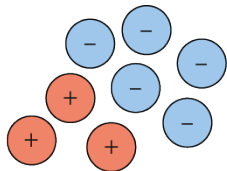
Example 3:

These counters show −2.

a) *Another counter is **added** and the value changes to −1.*
   (i) *Was the counter a positive or a negative? Explain how you know.*
   (ii) *Write a number sentence to describe this.*
b) *One of the counters is **taken away** and the value changes to −1.*
   (i) *Was the counter a positive or a negative? Explain how you know.*
   (ii) *Write a number sentence to describe this.*
c) *Find two different ways to change the total of the counters to −3.*
   (i) *Explain how you know that you are correct.*
   (ii) *Write a number sentence to describe each of your strategies.*

R Students' understanding that numbers can be partitioned into zero pairs can be built on to see the equivalence of adding +1 and subtracting −1. A key point to be drawn out of *Example 3* is that the two operations give an equivalent result.

It is important that the symbols are used alongside the representation, and that the symbols are seen as describing the manipulations that have been made. This connection must be explicit if the representation is to support students' understanding of addition and subtraction with negative numbers.

PD Mathematics websites and blogs, such as those from [NRICH][4], can offer some strategies for calculating with negative numbers. Reflecting on the blogs, your own experiences of teaching, or a combination of the two, consider:

- Do any of these strategies, or other strategies that you have used, offer access to the additive structure that underpins this topic?
- Do any of these strategies, or other strategies that you have used, support your students in calculating with negative numbers?
- What are the benefits and disadvantages of each?

Example 4:

*Each of these calculations is missing **at least one** negative sign in order to make the calculation correct.*

Place the negative sign(s) to make each calculation correct.

a) $12 + 3 = 9$

b) $12 + 3 = -9$

c) $772 + 158 = 930$

d) $772 - 158 = 614$

e) $2\,937.2 - 10.71 = 2\,947.91$

V By specifying that the calculations in *Example 4* have been carried out correctly (i.e. the answers would be correct if the signs were), students are able to focus on the relationships between the numbers and the implications for the position of the negative sign(s).

Parts c), d) and e) increase the value of the numbers in order to reduce the temptation for students to calculate rather than use their understanding of the relative values of the numbers.

D It is worth noting that some of these calculations can be made correct in more than one way (in fact, only parts b) and d) have a unique solution). Students might explore this and conclude why this is the case.

Example 5:

$$-6 - -4 = \boxed{}$$

Mia says, 'Two negatives make a positive, so this is the same as 6 + 4 and the answer is 10.'

Sasha says, 'Two negatives make a positive, so this is the same as 6 − 4 and the answer is 2.'

Mia and Sasha are both wrong.

How would you help them to understand that the correct answer is −2?

PD In *Example 5*, students discuss and unpick a common misconception. It might be seen that offering a misconception to a class runs the risk of students remembering an incorrect method. Alternatively, it might be thought that students in a class are likely to have this misconception and it is important to raise it in order to address it. What do you do when you know a common misconception is likely?

Solve problems where there is more than one answer and there are elements of experimentation, investigation, checking, reasoning, proof, etc.

Example 6:

Find some possible values for a and b (where a and b are integers).

a) $a + b = -4$ *and* $a < b$

b) $a - b = 3$ *and* $a < b$

D *Example 6* challenges students to think more deeply about the relationship between *a* and *b* under certain conditions. In part a), reasoning that $-2 + -2 = -4$ is a solution when $a = b$ will help them reason the values for which this is true when $a < b$. In part b), students have to think about how *a* and *b* are related if $a - b$ is a positive number and reason that *a* cannot be less than *b*.

PD Can you create some other 'find all possible values'-type questions that force your students to think about relationships?

Example 7: *Is it always, sometimes or never true that subtracting one negative number from another will give a negative answer?*	**D**	*Example 7* prompts students to generate some examples for themselves, test them against the given criteria and generalise, and thus to think mathematically. It requires a higher thinking skill than just the ability to subtract directed numbers; it helps students focus on the relationship between numbers.

2.1.1.2 Generalise and fluently use addition and subtraction strategies, including columnar formats, with decimals

Common difficulties and misconceptions

Students may be proficient with the standard column algorithms when adding and subtracting with integers, but using these methods with decimals can prove challenging to those students who do not understand the structures that underpin them. For example, when working with integers, students may be able to align the numbers being operated on 'from the right', but the inclusion of decimals means that this now needs to be refined to an understanding that the decimal points need to be aligned.

Aligning decimal points can also bring the additional challenge of using zero as a place holder. For example, when calculating 173.61 − 28.35082, students need to understand both the need for using zero as a place holder, and the equivalence of the given calculation and 173.61000 − 28.35082. Similarly, for examples such as 17 − 12.34 where there are 'no decimal places'.

The use of representations, such as place-value counters, can support students in understanding the structures that underpin the standard algorithms for both integers and decimal numbers.

What students need to understand	Guidance, discussion points and prompts
Partition a number in multiple ways.	**R** The Gattegno chart offers a useful representation to support students in partitioning integers and decimals in order to identify those elements that can be easily added or subtracted. A possible pre-cursor to students working on *Example 1* could be provided by the teacher tapping with a finger or a pointer on a sequence of cells – 300, 20, 9, 0.5 and 0.01 as the students chant 'three hundred and twenty-nine point five one'. The teacher could then ask the class to write this as 300 + 20 + 9 + 0.5 + 0.01, or to identify other partitions, such as 320 + 9.51.

Example 1:

1 000	2 000	3 000	4 000	5 000	6 000	7 000	8 000	9 000
100	200	300	400	500	600	700	800	900
10	20	30	40	50	60	70	80	90
1	2	3	4	5	6	7	8	9
0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09

Using the chart above, how would you partition the following numbers?

a) 3 033

b) 33.03

c) 3.14

d) 31.4

R By careful variation in the numbers chosen in *Example 1*, students practise using place value to partition decimal numbers, which is a key idea when adding or subtracting.

L This representation is useful to ensure that students are able to use number names correctly. Make sure that students say the decimal part of the number correctly ('three hundred and twenty-nine point five one' rather than saying 'three hundred and twenty-nine point fifty-one'), as the imprecise use of language can support misconceptions concerning the size of numbers (particularly that, for example, 0.29 > 0.8 because 'twenty-nine is greater than eight').

PD Compare and contrast the way that tenths, hundredths and thousandths are introduced in segments *1.23 Composition and calculation: tenths* and *1.24 Composition and calculation: hundredths and thousandths* of the NCETM primary mastery professional development materials[1] with your current practice.

Example 2:

What is the same and what is different about these sets of place-value counters?
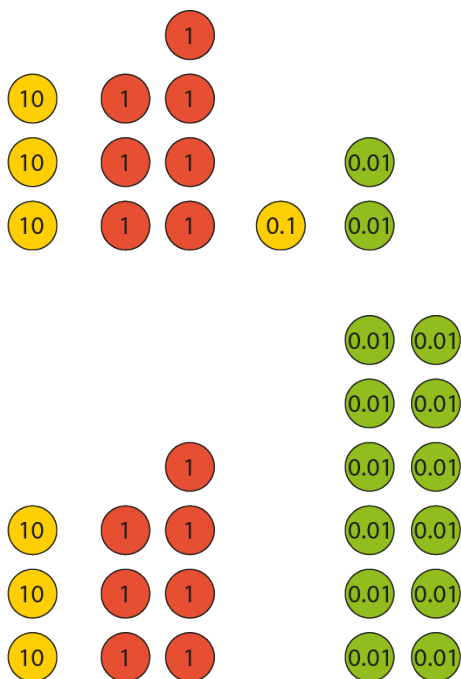
R Place-value counters offer a flexible representation of multi-digit numbers; they allow students to see partitioning of the same number in different ways.

In *Example 2*, the number 37.12 is represented. By manipulating the counters, students can see that this is $3 \times 10 + 7 \times 1 + 1 \times 0.1 + 2 \times 0.01$, but also that it is $3 \times 10 + 7 \times 1 + 12 \times 0.01$.

The understanding that the same number can be partitioned in these different ways is essential in efficient calculation, particularly when subtracting using the columnar method.

Example 3:

Complete the following calculations.

a) $12.653 = 12 + \boxed{}$

b) $12.653 = 12.6 + \boxed{}$

c) $12.653 = 12.503 + \boxed{}$

d) $12.653 = 12.64 + \boxed{}$

V In *Example 3*, the number 12.653 is partitioned in a range of ways. The partitions used in parts c) and d) are of particular importance, since these are the kinds of partitioning used when carrying out a subtraction using the column method of decomposition.

For example, in the calculation:

$$
\begin{array}{r}
1\ 2\ .\ 6\ 5\ 3 \\
-\quad\ 1\ .\ 0\ 7\ 2 \\
\hline

\end{array}
$$

one is subtracted from the '6' in the tenths column and transferred to the '5', giving 15 in

the hundredths column, in order that the '7' hundredths can be subtracted from this, and so 12.653 is rewritten as

$12.\overset{5}{\cancel{6}}\overset{1}{}5\,3$

This is essentially the same as writing 12.653 as 12.503 + 0.15.

Similarly, in the calculation:

```
    1 2 . 6 5 3
 −     1 . 1 3 6
    _____
```

rewriting 12.653 as 12.64 + 0.013 allows for the subtraction of 0.006.

The variation in the different partitions can be used to draw students' attention to the fact that each calculation has the same value but a different appearance.

PD Tasks similar to *Example 3* feature in segment *1.24 Composition and calculation: hundredths and thousandths* (*Teaching point 4*) of the [NCETM primary mastery professional development materials](1). Do you think that there is value in revisiting this type of partitioning before calculating with decimal numbers? Are there any possible drawbacks?
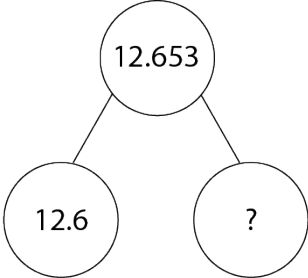

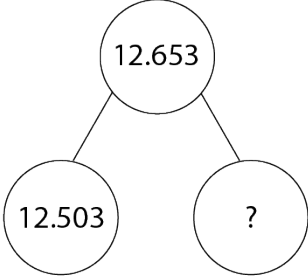*Example 4:*

What are the missing numbers?

a)
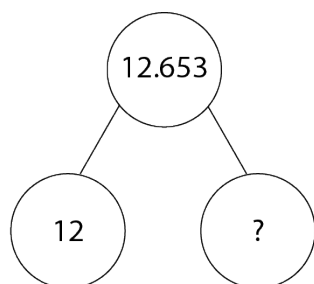
R Students should be familiar with the part–part–whole model from Key Stages 1 and 2. This representation allows students to generalise their understanding of partitioning integers to partitioning with decimals.

b)

```
        ( 12.653 )
         /      \
   ( 12.6 )    ( ? )
```

c)

```
        ( 12.653 )
         /       \
  ( 12.503 )    ( ? )
```

d)

```
        ( 12.653 )
         /       \
   ( 12.64 )    ( ? )
```
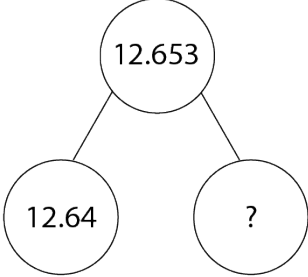
Understand partitioning as a part of the columnar method with decimals.

Example 5:

Fill in the missing digits in these calculations.

a)

	3	1	7	.	☐	☐	☐
−	1	1	9	.	1	5	1
	☐	☐	☐	.	8	1	1

D In challenging students to fill in the blanks in *Example 5*, the routine algorithm is disrupted, and students need to think more deeply about the structures that underpin it; in particular, the process of exchange (e.g. exchanging one ten for ten ones).

The questions here are sequenced so that the nature of the exchanges is varied, resulting in the complexity of the problem increasing.

b)

```
      ☐  ☐  ☐  .  2  5  3
   −  1  4  9  .  1  5  1
   ─────────────────────────
      2  2  9  .  ☐  ☐  ☐
```

c)

```
      4  0  3  .  1  ☐  ☐
   −  3  1  ☐  .  ☐  2  5
   ─────────────────────────
   ☐  ☐        8  .  9  9  6
```

Example 6:

a) *Use place-value counters to model the calculation 21 314 − 9 117.*

b) *Use place-value counters to model the calculation 21.314 − 9.117.*

c) *What is the same and what is different about these two calculations?*

R Here, a subtraction (21 314 − 9 117 = 12 197) is modelled using place-value counters, in preparation for extending the algorithm to include decimals.

Asking, *'What is the same and what is different?'* at each stage of the manipulation of the counters and the corresponding calculation will help students see the common structure behind the algorithm.

The numbers here have been chosen so that there are three instances of exchange, including one instance in which zero is needed as a place holder.

PD What classroom strategies might you use to support students through the initial complexity of the problems?

If, as is suggested, you show consecutive stages in the calculation, what would you draw students' attention to?

Manipulate place value correctly when calculating with integers and decimals.

Example 7:

$$15.2 - 2.17 = 13.03$$

a) Can you use the answer to this calculation to write down an answer to any of the following? *Explain your answer.*

    (i)   *1.52 – 0.217*

    (ii) *15.2 – 0.217*

    (iii) *15.2 – 2.017*

    (iv) *15.2 – 2.16*

b) Which calculations are not easily answered using the original example?
Explain your answer.

V   *Example 7* draws attention to the digits in the numbers which have the same place value. Questions such as the following will support students in reasoning about the value of the digits in these numbers:

- *'What is the same and what is different about the calculations in part a) and the original example?'*
- *'In which parts will the answer include the digits 1, 3, 0 and 3, in that order?'*
- *'How does the answer to part (iv) relate to the answer in the example?'*

2.1.2.1   Understand the arithmetic structures that underpin multiplication and division of positive and negative integers

Common difficulties and misconceptions

As discussed in *2.1.1.1 Understand the mathematical structures that underpin addition and subtraction of positive and negative integers*, students are likely to have met negative numbers at Key Stage 2, but this may only have been in context, and they are unlikely to have carried out operations involving negative values.

Many Key Stage 3 students will have the misconception that 'multiplication makes bigger' when working with positive numbers. There may be additional layers to this when considering the difference between magnitude and value when working with negative numbers. For example, until students met negative numbers their experience would be that numbers closer to zero are 'smaller' both in terms of value and magnitude: thus, they may think that −591 is larger than −3 because it's further from zero but agree that it's a lower value. Modelling consistent language around this (e.g. by using *'greater or less'* rather than the more ambiguous *'bigger or smaller'*) may help deepen their understanding.
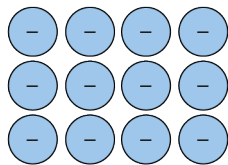
Some students will come to working with negative numbers knowing informal rules such as 'two negatives make a positive'. Exploring the limits of this sort of rule with a class may be of benefit in moving them to a deeper and more connected understanding.

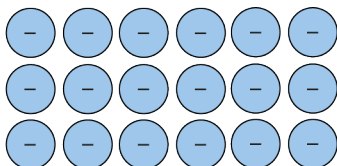| What students need to understand	Guidance, discussion points and prompts
Understand multiplication and division with one negative factor. *Example 1:* *This diagram shows some counters in an array.* *Robyn says, 'I can see three groups of negative four' and writes* $3 \times -4 = -12$. *Taylor says, 'I can see four groups of negative three.'* a) *Write a number sentence for Taylor's description of the array.* b) *Write two number sentences that describe this array:*	**R** Here, an array is used to connect an existing representation for multiplication and extend it to include products with one negative factor. Robyn and Taylor both use the array to identify groups of values and you might like to work with students to write these as repeated additions (for example, writing Robyn's statement as $-4 + -4 + -4$) to also extend this interpretation of multiplication to negative products. **PD** The array is a powerful model of multiplication but can be challenging when working with negative numbers. It is used here with only one negative factor, but the narrative might be continued (using zero pairs) to extend this representation to allowing for calculations with two negative factors.

For example, consider -3×-4. An array of matching pairs might be used.
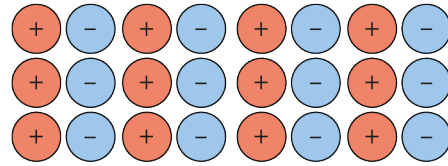
With the calculation being read as *'subtract three groups of negative four'*, the representation becomes this, which can be seen as positive twelve.
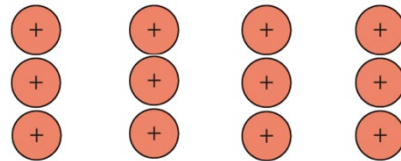
The coherent use of representation can support students in understanding mathematical structure, but it is important that the representation is eventually put to one side. It might be argued that this example is also inconsistent in its use of the interpretation of the minus sign.

Because of these inconsistencies, the array has not been used in these materials as a model to multiply two negative numbers. Instead, students are asked to reason with mathematical structure in order to justify the outcome for the product of two negative numbers (see *Example 4*).

There is an argument that if the array is not a productive model for all multiplications involving negative numbers, then it should not be introduced. Do you agree with this interpretation?

Example 2:

1. Latoya calculates that $3\,719 \times 418 = 1\,554\,542$. Use Latoya's work to fill in the gaps.

a) $3\,719 \times -418 =$ ☐

b) $-3\,719 \times 418 =$ ☐

c) $1\,554\,542 \div 418 =$ ☐

V   *Example 2* gives the numerical outcome of a calculation. The intention is that students are then able to focus on the structure of the calculation and reason about whether the result should be positive or negative.

In question 1 part c), students are introduced to division with negative numbers. From earlier work, they should understand multiplicative relationships and know that if $3 \times 4 = 12$, then $12 \div 3 = 4$ and $12 \div 4 = 3$ and

d) $1\,554\,542 \div -418 =$ ☐

e) $-1\,554\,542 \div 418 =$ ☐

f) $-1\,554\,542 \div 3719 =$ ☐

2.   *Marijke calculates that $23.1 \times 9.7 = 224.07$.*

Write down as many multiplication and division facts as possible that use these three numbers and their negatives

---

it is this structural awareness that is being used here to move beyond multiplication.

The use of large numbers in 1 and decimals in 2 again discourages students from attempting to calculate.

Example 3:

a)   *Is it always, sometimes or never true that a **negative number divided by a positive number** results in a positive number?*

b)   *Is it always, sometimes or never true that a **positive number divided by a negative number** results in a positive number?*

D   After the informal generalisation in *Example 2* that uses 'peculiar' numbers (Mason, Watson, Bills and Bills, 2004), *Example 3* uses an always, sometimes, never structure to move towards a more formal generalisation of the outcomes when multiplying and dividing with negative numbers.

Understand multiplication and division with two negative factors.

Example 4:

Pete writes a calculation.

$$-1 \times (1 + -1) = -1 \times 1 + -1 \times -1$$

*He says, 'This shows that negative one multiplied by negative one **must** be positive one.'*

a)   *What is the value of $1 + -1$?*

b)   *What is the value of $-1 \times (1 + -1)$?*

c)   *What is the value of -1×1?*

d)   *Do you agree with Pete that his calculation shows $-1 \times -1 = 1$?*

In *Example 4* students are asked to reason with two known properties (the zero property and the distributive law) to create a chain of reasoning that justifies why $-1 \times -1 = 1$.

R PD   As discussed in *Example 1*, a pictorial representation is not offered here; rather, students are asked to reason with a symbolic representation. You may like to consider the advantages and disadvantages of extending the pictorial representation when working with two negative numbers. Are there situations or particular classes you teach where this is more appropriate?

PD   A suggested sequence of questions is offered in *Example 4* to allow students to explore and access some of the key elements of the required reasoning. Consider how you might support students to link what they may see as separate questions into one coherent whole.

Example 5:

Look at the calculation.

$$3 \times -4 = 12$$

It is not correct.

V   The intention here is that by allowing students an opportunity to explore the dimensions of possible variation they will work with the structure that underpins multiplication with negative numbers.

---

a) Place just one minus sign into the calculation to make it correct. b) There are at least two different correct solutions. How many can you find?	**D** The prompt states that there are at least two different correct solutions. The intended correct solutions are $-3 \times -4 = 12$ and $3 \times -4 = -12$ but you might like to challenge students with, or they might raise, the situation of $3 \times --4 = 12$. Discussing the validity of this suggestion may offer useful insights into students' understanding and give a context to think deeply about multiplication and negative numbers.

Example 6:
 	
Each of these calculations is correct, but some of the negative numbers are missing their minus sign.
 	
*Place **at least one** minus sign in every calculation so that it is correct.*
 	
    a) $13 \times -27 = 351$
 	
    b) $-47 \times 58 = 2\,726$
 	
    c) $13 \times 19 = 247$
 	
    d) $7.4 \times 3.9 = -28.86$
	As in the previous examples, the numerical value of the calculation is already given so that students focus their attention on the relationship between the positive and negative numbers. Part c) may be particularly valuable for gaining an insight into students' understanding of multiplication with negative numbers, as there are several possible solutions.
	
	PD Each of the questions in *Example 6* has at least two correct solutions, and part c) has more than two. Is there a benefit to asking students to find **all** possible solutions rather than using the prompt as it is written? Although the numbers here have been chosen to discourage calculation, and include the use of decimal numbers, you might like to adapt or add a question involving calculating. It is useful for students to understand that the multiplicative structures work with all positive and negative numbers, whether they are whole numbers or presented as decimals or fractions. Would you include fractions in the question for the classes you teach?

Example 7:
 	
Each of these calculations is correct, but some of the negative numbers are missing their minus sign.
 	
*Place **at least one** minus sign in every calculation so that it is correct.*
 	
    a) $351 \div 13 = -27$
 	
    b) $2\,726 \div -47 = 58$
 	
    c) $247 \div 19 = 13$
 	
    d) $-28.86 \div 7.4 = 3.9$
	V *Example 7* follows on from *Example 6* and uses the same relationships in a different form. The intention here is to allow a context in which students become aware of the multiplicative relationship in all its forms, so they understand that there are not different 'sets of rules' for multiplication and division.

Use the associative law to calculate the products of positive and negative numbers.

Example 8:

Ursula says, 'All multiplication and division calculations have an even number of minus signs.'

Is Ursula correct? Explain how you know.

All the examples used so far have been of the form factor × factor = product.

Example 8 offers a context in which students can reason with known facts to generalise and explain a relationship.

PD It might be said that *Example 8* is not dissimilar to offering 'two negatives make a positive' as it gives students a rule to follow. To what extent do you agree with this?

2.1.2.3 Generalise and fluently use written multiplication strategies to calculate accurately with decimals

Common difficulties and misconceptions

Although students should be proficient with the standard written multiplication algorithm for calculating with whole numbers in Key Stage 2, they do not always make the connection with using the standard written multiplication algorithm for calculating with decimals in Key Stage 3. Some students will try to use the standard written algorithm with decimals. This can give the correct answer if the multiplier is a whole number (see *Example 1*) but soon fails with decimal multipliers (see *Example 2*). They see multiplying decimals as 'new' learning rather than thinking about the underlying mathematical structure of a calculation such as $24.3 \times 1.2 = (243 \times 12) \times 0.1 \times 0.1$ or $(243 \times 12) \div 10 \div 10$.

R

$$
\begin{array}{r}
2\ \ 4\ .\ 3 \\
\times\quad\ \ 1\ .\ 2 \\
\hline \\
\\
\hline
2\ \ 9\ .\ 1\ \ 6
\end{array}
\qquad
\begin{array}{c}
\times\,10 \\
\rightarrow \\
\times\,10 \\
\\
\\
\\
\leftarrow \\
\div10\ \div10
\end{array}
\qquad
\begin{array}{r}
2\ \ 4\ \ 3 \\
\times\quad\ \ 1\ \ 2 \\
\hline
4\ \ 8\ \ 6 \\
2\ \ 4\ \ 3\ \ 0 \\
\hline
2\ \ 9\ \ 1\ \ 6
\end{array}
$$

V Using 'If I know … then I also know …' mathematical thinking helps students understand that if they know $243 \times 12 = 2916$ then they also can find the product of other calculations such as 24.3×1.2, 2.43×0.12, 0.243×1.2, etc. *Examples 2, 3* and *4* explore how to avoid mechanical practice of exclusively standard questions by varying the value of the multiplicand and multiplier.

It is also useful to include some non-examples (see *Examples 1* and *6*) for students to critique and reason about. Students should be encouraged to estimate the product first to check they have transformed the calculation correctly using powers of 10.

What students need to understand	Guidance, discussion points and prompts
Appreciate the limitations of the (standard) written method for multiplying decimals. *Example 1:* Richard thinks $4.56 \times 3 = 13.68$, because: $$\begin{array}{r} 4\ .\ 5\ \ 6 \\ \times \qquad\quad 3 \\ \hline 1\ \ 3\ .\ 6\ \ 8 \\ {\scriptstyle 1}\qquad{\scriptstyle 1} \end{array}$$ -- *Nicola thinks $2.5 \times 1.1 = 27.5$, because:*	**V** The 'what it's not' calculations in *Example 1* will help students understand that the standard written algorithm for multiplying whole numbers (short/long multiplication) can't be applied to calculations with decimals. This is particularly the case when the multiplier is a decimal number, in the case of Nicola's calculation. Encourage the students to give meaning to the idea that calculations such as 4.56×3 and 2.5×1.1 can be calculated as $(456 \times 3) \div 100$ and $(25 \times 11) \div 100$.

```
        2  .  5
  ×     1  .  1
  ─────────────
        2  .  5
    2   5  .  0
  ─────────────
    2   7  .  5
```

Use estimation to explore if Richard and Nicola are both correct.

PD Why does 'lining up' the decimal point work in the standard columnar written methods for addition and subtraction of decimal numbers (see *2.1.1.2*) but not for multiplication?

Why what is your department or school's calculation policy for multiplying (and dividing) whole numbers?

Use place value to manipulate a calculation involving decimals to an equivalent calculation involving whole numbers.

Example 2:

Given $456 \times 12 = 5\,472$, work out:

a) 45.6×12

b) 4.56×12

c) 45.6×1.2

d) 4.56×1.2

e) 4.56×0.12

V *Example 2* draws attention to the digits in a calculation rather than the individual numbers. Asking *'what is the same and what is different?'* about the five calculations will encourage students to notice the connections between the calculations and an *'If I know … then I also know …'* way of thinking.

D Deepen students' thinking by asking more probing questions. For example, use $456 \times 12 = 5\,472$ to find the products of other pairs of numbers.

PD Students need to be confident and fluent with multiplying and dividing by powers of 10. How could you assess this before starting on this example?

Appreciate that the (standard) written method for multiplying two whole numbers can be used as part of the strategy for multiplying decimals.

Example 3:

Fill in the gaps.

$$
\begin{array}{ccc}
& 2 & 4 & 5 \\
\times & & 1 \;.\; 3 \\
\hline
\end{array}
\quad \times \;\square \quad \rightarrow \quad
\begin{array}{ccc}
& 2 & 4 & 5 \\
\times & & 1 & 3 \\
\hline
\end{array}
$$

$245 \times 13 = \boxed{}\boxed{}\boxed{} . \boxed{}$

Example 4:

Fill in the gaps:

$$
\begin{array}{cccc}
& 2 & 4 \;.\; & 5 \\
\times & & 1 \;.\; & 3 \\
\hline
\end{array}
\quad \times \;\square \quad \rightarrow \quad
\begin{array}{ccc}
& 2 & 4 & 5 \\
\times & & 1 & 3 \\
\hline
\end{array}
$$

$24.5 \times 1.3 = \boxed{}\boxed{} . \boxed{}\boxed{}$

Example 5:

Tick the (✓) the correct calculations:

a) *52.4 × 1.6 = 524 × 16*

b) *52.4 × 1.6 = 524 × 16 ÷ 10*

c) *52.4 × 1.6 = 524 × 16 ÷ 100*

d) *5.24 × 1.6 = 524 × 16 ÷ 1000*

e) *0.524 × 0.6 = 524 × 16 ÷ 1000*

V *Examples 3 and 4* are empty box problems to help students understand how the standard written method for multiplying whole numbers can be used as part of the strategy for multiplying decimals.

In *Example 4* the multiplicand is changed to a decimal number, to prompt students to notice that the equivalent calculation is (245 × 13) ÷ 100 and not (245 × 13) ÷ 10, as is the case in *Example 3*.

Example 5 draws students' attention to transforming a calculation such as 5.24 × 1.6 by multiplying and dividing by the correct power of 10. They should do this by noticing the number of decimal places in both the multiplicand and the multiplier.

D Deepen students' thinking by asking them to predict and explain the number of decimal places in the products below:

a) *2.45 × 1.3*

b) *2.45 × 0.13*

c) *0.245 × 0.13*

Solve problems where there is more than one answer and there are elements of experimentation, investigation, checking, reasoning, proof, etc.

Example 6:

Which of the following statements are always true, which are sometimes true, and which are never true? Explain your thinking.

a) *The product of two decimals is a decimal number.*

b) *The product of two decimal numbers with two decimal numbers has four decimal places.*

c) *The product of two decimal numbers is smaller than the multiplicand.*

D Statements like those in *Example 6* provide opportunities for students to 'play' and conjecture when multiplying decimals and explore the structure to greater depth.

Students should be encouraged to think about simpler calculations like 0.2×0.4 or 0.25×1 as well as more complex calculations that need a written method.

Statement c) provides students with an opportunity to explore multiplying by a number between 0 and 1.

PD How do you manage a task like this with your classes? How long would you give them before you intervene and support? What prompts could you give?

2.1.4.1 Understand the mathematical structures that underpin the multiplication of fractions

Common difficulties and misconceptions
When multiplying fractions, students' awareness can be directed to the idea (possibly made explicit for the first time) that the product of two numbers can be smaller than either of those numbers. Students who see multiplication *only* as repeated addition (and, therefore, as always making something bigger) will find this difficult.

For students to make sense of this, they need to deepen their understanding of multiplication to include scaling and the idea that multiplying any number by a fraction between one and zero makes that number smaller.

For example, when considering a calculation such as $3 \times \frac{3}{4}$, it will be important to encourage students to think of this in two ways:

- the number $\frac{3}{4}$ taken three times – i.e. $\frac{3}{4} + \frac{3}{4} + \frac{3}{4}$
- the number 3 multiplied by $\frac{3}{4}$ – i.e. scaling the number 3 to three-quarters of its size

and to see that the answer is the same.

The first reinforces the idea that multiplication 'makes bigger' as the $\frac{3}{4}$ has, indeed, got bigger.

However, in the second, the 3 has been made smaller when multiplied by the fraction.

| What students need to understand	Guidance, discussion points and prompts
Think of multiplication as scaling/reduction when both numbers are fractions and that either fraction can be thought of as the scale factor. *Example 1:* a) $1 \times \dfrac{1}{3} = \square$ $\dfrac{1}{3} \times 1 = \square$ b) $2 \times \dfrac{1}{3} = \square$ $\dfrac{1}{3} \times 2 = \square$ c) $4 \times \dfrac{1}{3} = \square$ $\dfrac{1}{3} \times 4 = \square$ d) $7 \times \dfrac{1}{3} = \square$ $\dfrac{1}{3} \times 7 = \square$	**V** In *Example 1*, the fraction remains the same while the integer changes. Each part also has two versions of the same calculation. It is important to draw students' attention to the different ways we can think of multiplying. While working through this example, you could ask, *'What does each statement mean?'* and *'Why do they give the same answer?'* If the integer is the multiplier, then it is helpful to think of multiplication as 'groups of' (as in 'two groups of one third is two-thirds'). However, if the fraction is the multiplier, students are forced to re-think multiplication as scaling (as in 'one third of two'). This is important because when both numbers are fractions, multiplication cannot be thought of as 'groups of'. **R** Students can be encouraged to draw diagrams to show such calculations. For example, to show that $4 \times \frac{1}{3}$ and $\frac{1}{3} \times 4$ are the same:

| $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | | | |

| 1 | 1 | 1 | 1 |
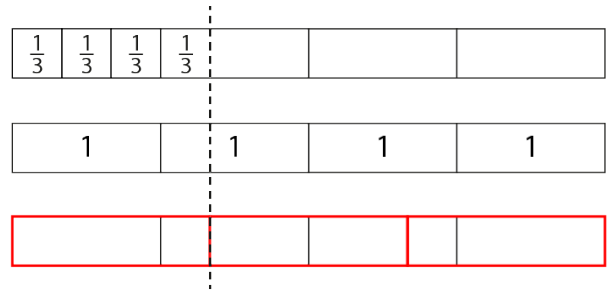
The first bar model shows $\frac{1}{3}$ replicated four times (i.e. four groups of $\frac{1}{3}$). The third bar model shows how four (depicted in the second bar model) has been multiplied by one third (i.e. reduced to one third).

Students could also be encouraged to use number lines to show this equivalence:
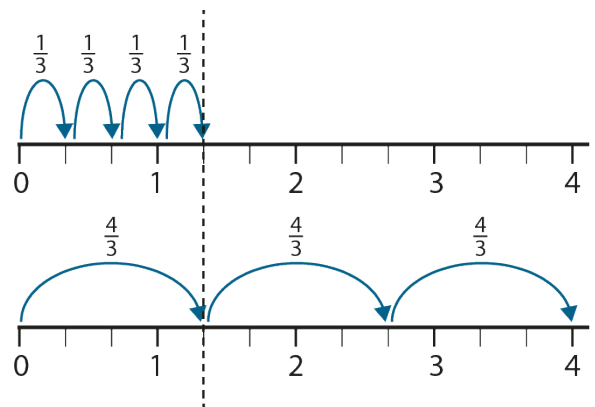

Example 2:

a) *Which has the larger value:*

$\frac{1}{2}$ *of* $\frac{1}{3}$ *or* $\frac{1}{3}$ *of* $\frac{1}{2}$?

$\frac{1}{4}$ *of* $\frac{1}{3}$ *or* $\frac{1}{3}$ *of* $\frac{1}{4}$?

b) *Can you make up an example of your own and show how the two answers compare?*

PD *Example 2* has been posed in this way (i.e. not including *'or are they the same?'*) because it is anticipated that students may be surprised by the fact that the values are the same. Try this with your students and see how they react. Is the possible surprise element important?
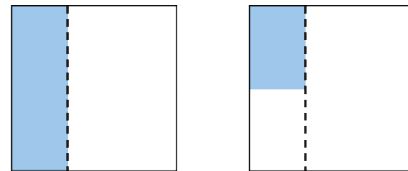
R For this example, students could be encouraged to draw a dashed line across a square to represent $\frac{1}{2}$, for example:
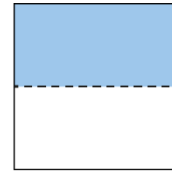
and then to use other dashed lines like this to show $\frac{1}{3}$ of $\frac{1}{2}$:
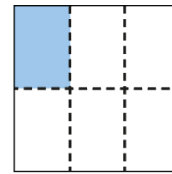
They could then contrast the above with the following diagrams showing $\frac{1}{3}$ and $\frac{1}{2}$ of $\frac{1}{3}$:

PD Students should see how the above diagrams represent the multiplications $\frac{1}{2} \times \frac{1}{3}$ and $\frac{1}{3} \times \frac{1}{2}$ and begin to make sense of the rule 'multiply the denominators'. They should appreciate that the commutative law (with which they will be familiar for integers) also applies to fractions. What questions might you ask to help students do this after having worked through these examples together?

Example 3:

Fill in the gaps with <, > or =.

a) $\frac{1}{2} \times 4$ ◯ $\frac{1}{2}$ d) $\frac{1}{2} \times \frac{1}{2}$ ◯ $\frac{1}{2}$

b) $\frac{1}{2} \times 2$ ◯ $\frac{1}{2}$ e) $\frac{1}{2} \times \frac{1}{4}$ ◯ $\frac{1}{2}$

c) $\frac{1}{2} \times 1$ ◯ $\frac{1}{2}$

V By comparing the relative sizes of one of the factors and the resulting product in *Example 3*, students' attention can be drawn to the point where the product becomes equal to and then less than $\frac{1}{2}$, and so develop their understanding of when the multiplier makes the product smaller. This example offers a context in which to explore the misconception that 'multiplication makes bigger'.
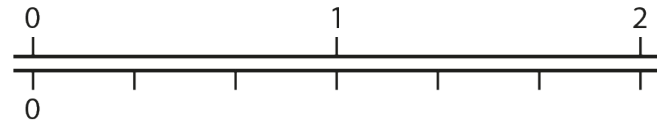
Example 4:

Here are two identical number lines.

0

0
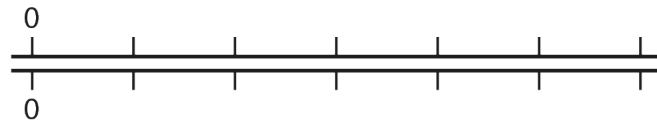
Now the top number line has been stretched out so that it is three times longer than before.

The bottom number line has not changed at all.

0   1   2

0

a) *Think about the top number line.*

   (i) *Which number is directly below 2?*

   (ii) *Which number is directly below where 9 is?*

   (iii) *Which number is directly below $\frac{1}{3}$?*

   (iv) *Which number is directly below $\frac{2}{3}$?*

b) *Write a number sentence to describe each of your answers to part a).*

R The representation here is not intended as a tool to calculate an answer; rather it is to provide an image of multiplication as stretching or scaling. Aligning values on the top of the resulting double number line (DNL; also known as *stacked number lines*) with those on the bottom allows for multiplication by three (since it is 'three times' larger).

The values in part a) have been chosen so that, initially, students can use the DNL to see the answer (and so make more sense of the DNL) and then move to a number further up the lines.

It is worth noting that some students might try to 'scale' values along the line, observing that since 2 is above 6, 4 will be above 12 and 8 above 24. Students who notice that 1 is above 3 will then deduce that 9 is above 27. It is important to make sure at this point that students notice that any value on the bottom line is three times the value above it on the top line.

Parts (iii) and (iv) can be seen on the diagram. The calculations are relatively straightforward ($\frac{1}{3}\times3$ and $\frac{2}{3}\times3$) in order that the focus can be on the scaling image that results from the diagram, rather than the solution to the calculations.

It is important that time is given to part b), in which students connect the DNL representation with the more familiar symbolic representation.

PD You might like to consider the following situation yourself and with colleagues as a way of thinking more deeply about the double number line.

*The bottom line is now also stretched out. It is two times longer than previously.*
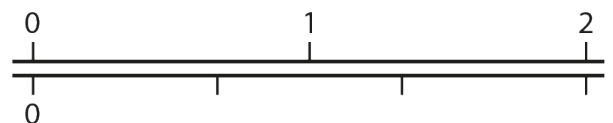
0   1   2

0

a) *Think about the top number line.*

   (i) *Which number is directly below 2?*

   (ii) *Which number is directly below 1?*

   (iii) *Which number is directly below 3?*

   (iv) *Which number is directly below $\frac{1}{2}$?*

b) *Write a number sentence to describe each of your answers to part a).*

Discuss and reflect on the ways that you tackled this task. How does this fit in with your understanding of multiplication, and particularly multiplication of fractions?

Understand how an area model can represent the multiplication of fractions.

Example 5:

This is a picture for 4 × 3:
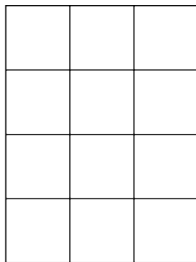
a)	4×2	*f)*	3×3	*k)*	1×1
b)	4×1	*g)*	2×3	*l)*	$\frac{1}{2} \times \frac{1}{3}$
c)	$4 \times \frac{1}{2}$	*h)*	1×3	*m)*	$\frac{1}{3} \times \frac{1}{5}$
d)	$4 \times \frac{1}{3}$	*i)*	$\frac{1}{2} \times 3$	*n)*	$\frac{1}{7} \times \frac{1}{5}$
e)	$4 \times \frac{1}{7}$	*j)*	$\frac{1}{5} \times 3$		

V In *Example 5*, the numbers have been chosen to draw attention to the idea of reducing one side of the rectangle (while keeping the other side constant), and so reducing the area (product), and to focus on the fraction as the multiplier.

It will be important to pause after a few of these (during the first set of five and then again during the second set of five) and ask students what is the same and what is different about successive calculations, and what they notice about the product.

Part k) could be done orally, with a student coming to the board and drawing a unit square. It will be important, at this stage, to introduce the idea of seeing each side as a unit that can be further reduced, or as a number line from 0 to 1 where fractions can be placed.

R This important image and associated awareness will support students in seeing the multiplication of any two fractions as an area within the unit square, and hence in answering parts l), m) and n):

$$\frac{1}{2} \times \frac{1}{3} \qquad \frac{1}{3} \times \frac{1}{5} \qquad \frac{1}{7} \times \frac{1}{5}$$
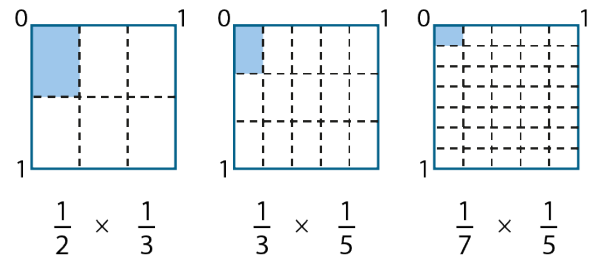
D Such diagrams may help students to make sense of the 'multiply the denominators' rule. It will be important to ask questions, such as, *'Why do you get sixths when you multiply halves and thirds?'* and *'Why do you get fifteenths when you multiply thirds and fifths?'* These might be generalised to consider what the denominator would be if fractions with denominator of *a* and denominator of *b* were to be multiplied.


Example 6:

What multiplication might these diagrams represent?

a)   *Write a number sentence for each.*

b)   *Which of these diagrams show the answer to the multiplication clearly, and which do you need to think about more?*

(i)
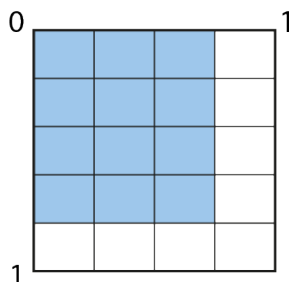
V The representation is varied in *Example 6* to deepen students' understanding of the area model of multiplication and to connect multiplication of fractions and integers.

In part (i), a unit square is used to model a multiplication. In part (ii), a different multiplication is being modelled, although the visual structure looks the same as part (i).

It will be helpful to ask *'What is the same and what is different about diagrams (i) and (ii) and the calculations they represent?'* in order to draw students' attention to the similarities in structure between 3×4 and $\frac{3}{4} \times \frac{4}{5}$.

In part (iii), students need to think more deeply about what the dimensions of the shaded area are – what the '3' and the '4' represent (the diagram is representing three halves multiplied by four ones).

Similarly, in part (iv) students will need to give careful thought to what the '3 by 4' shaded rectangle represents.

Discussion of these diagrams and their respective calculations should lead to the awareness that while the numerator is 12 each time, the denominator is different.
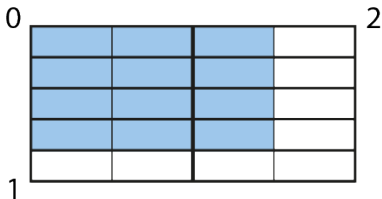
(ii)
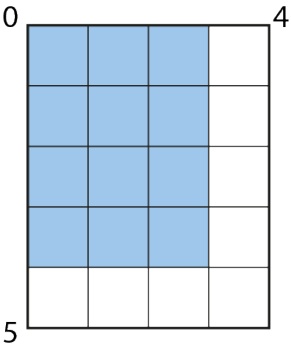
(iii)
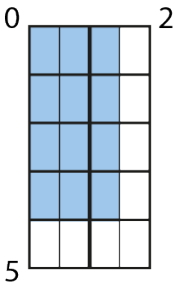
(iv)

2.1.4.2 Understand how to multiply unit, non-unit and improper fractions

Common difficulties and misconceptions

One common difficulty with multiplication of fractions is that the rule for doing this is very simple (multiply the numerators and multiply the denominators) and students often apply this rule without any understanding.

It will be important to use representations (for example, the area model) alongside the symbolic calculation in order to get a sense of what is happening when two fractions are multiplied together. In this way, students can make sense of the rule rather than just learning it without justification.

Students may also use inefficient methods due to simply following the algorithm, e.g. $\frac{4}{5} \times \frac{3}{4} = \frac{12}{20}$, and

then need to cancel down without recognising that, in the original calculation, the 4 and the $\frac{1}{4}$ (i.e. the

4s in the first numerator and the second denominator) reduce to one and the answer must be $\frac{3}{5}$.

What students need to understand	Guidance, discussion points and prompts
Appreciate that non-unit fractions are integer multiples of unit fractions and that, by considering the commutative and associative laws, the multiplication of non-unit fractions can be derived. *Example 1:* *Complete these calculations.* a) $\dfrac{1}{3} \times \dfrac{1}{5} = \boxed{}$ b) $2 \times \dfrac{1}{3} \times \dfrac{1}{5} = \boxed{}$ c) $\dfrac{2}{3} \times \dfrac{1}{5} = \boxed{}$ d) $\dfrac{1}{3} \times 2 \times \dfrac{1}{5} = \boxed{}$ e) $\dfrac{1}{3} \times \dfrac{2}{5} = \boxed{}$ f) $\dfrac{2}{3} \times \dfrac{2}{5} = \boxed{}$ g) $\dfrac{3}{4} \times \dfrac{3}{5} = \boxed{}$	**V** For *Example 1* part a), you could ask questions such as: • *'Can you draw a picture to show this?'* • *'Can you see this as $\frac{1}{3}$ of $\frac{1}{5}$ and as $\frac{1}{5}$ of $\frac{1}{3}$?'* and ask similar 'commutativity' questions for parts c) and e). For parts b) and d), it will be important to help students connect the answers to those for parts c) and e). Some discussion of part d) could include how the calculation might be read: Is it $\left(\frac{1}{3} \times 2\right) \times \frac{1}{5}$ [i.e. the same as $\frac{2}{3} \times \frac{1}{5}$]? or $\frac{1}{3} \times \left(2 \times \frac{1}{5}\right)$ [i.e. the same as $\frac{1}{3} \times \frac{2}{5}$]? and to realise that these calculations can be seen as equivalent. Parts f) and g) can be used as an opportunity to practise these ideas. The prompt *'In how many ways can you think of this calculation?'* can support students in seeing them in different ways to support fluency: $\frac{2}{3}$ of $\frac{2}{5}$; $\frac{2}{5}$ of $\frac{2}{3}$; $2 \times \frac{1}{3} \times \frac{2}{5}$; $\frac{2}{3} \times 2 \times \frac{1}{5}$; $2 \times \frac{1}{3} \times 2 \times \frac{1}{5}$; $4 \times \frac{1}{3} \times \frac{1}{5}$, etc.

h) $\dfrac{3}{4} \times \dfrac{4}{5} = \boxed{}$

i) $\dfrac{2}{3} \times \dfrac{3}{5} \times \dfrac{5}{7} = \boxed{}$

You can use this sequence of different versions of the same calculation to discuss and make sense of the 'multiply the numerators' rule. It will be important to encourage students to reason why this is true, using examples like the one above.

Asking questions, such as the following, will support students' understanding:

- *'What is the same and what is different in these two calculations: $\frac{1}{3} \times \frac{1}{5}$ and $\frac{2}{3} \times \frac{2}{5}$?'*

- *'Why is the answer to the second calculation four times bigger than the answer to the first?'*

Parts h) and i) provide opportunities for students to practise these ideas but with some added challenge (i.e. the possibility of cancelling before multiplying in part h) and the extension to three fractions in part i)).
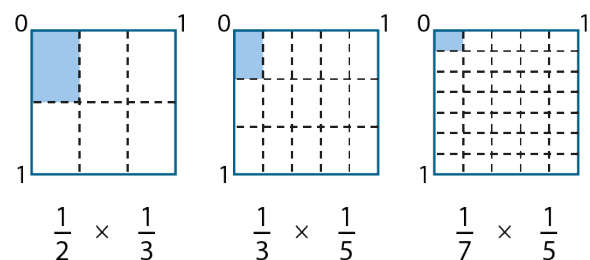
The strategy of rewriting $\frac{2}{3} \times \frac{3}{5} \times \frac{5}{7}$ as

$2 \times \frac{1}{3} \times 3 \times \frac{1}{5} \times 5 \times \frac{1}{7}$ and bracketing – i.e.

$2 \times \left(\frac{1}{3} \times 3 \right) \times \left(\frac{1}{5} \times 5 \right) \times \frac{1}{7}$ – will help to reveal the

structure and support students in understanding the process of 'cancelling' to simplify before calculating.

L From these exercises and the ensuing discussions, it will be useful for students to articulate (both as a whole class and to themselves) the sentence *'When multiplying two fractions, multiply the numerators and multiply the denominators.'*

D Students can be challenged to modify the diagrams they have seen before for unit fractions, for example:

$\dfrac{1}{2} \times \dfrac{1}{3}$ $\dfrac{1}{3} \times \dfrac{1}{5}$ $\dfrac{1}{7} \times \dfrac{1}{5}$

to show the following related multiplications:

	$\frac{1}{2} \times \frac{2}{3}$, $\frac{2}{3} \times \frac{3}{5}$, $\frac{5}{7} \times \frac{3}{5}$, etc. and so generalise beyond unit fractions.
Example 2: Given that $\frac{8}{15} \times 465 = 248$, find: a) $\frac{4}{15} \times 465$ b) $\frac{16}{15} \times 465$ c) $\frac{8}{3} \times 465$	**V** In *Example 2*, students are invited to manipulate a given expression to find the answers to other calculations, further developing their understanding of the use of one fraction (such as a unit fraction) to find the product of two fractions. On completion of the three given calculations, and having discussed their reasoning, you could then ask students to give some other results that they can find using the original product.
Example 3: Which of these gives the greater product? $\frac{20}{9} \times \frac{27}{45}$ *or* $\frac{20}{45} \times \frac{27}{9}$ *Explain how you calculated your answer.*	Students could consider cancelling fractions in order to simplify multiplication calculations with fractions, but this may not be the most efficient approach. **V** In *Example 3*, the numbers have been carefully chosen to draw attention to how the commutative law can be used to simplify when multiplying fractions. Initially, students may calculate the product without simplifying, and the numbers used in the first multiplication calculation are chosen to make this reasonably straightforward. Students may be more likely to notice that the second calculation can be simplified before multiplying (by simplifying $\frac{27}{9}$ to 3). On finding that the products are equal, students' attention should be drawn to the common structure of each calculation and the way that the commutative law can be used to simplify.

Example 4:
 	
a) Which gives the greater result, calculation A or calculation B?
 	
   Explain how you know.
 	
   A: $\frac{12}{13} \times \frac{14}{15}$
 	
   B: $\frac{14}{15} \times \frac{15}{16}$
	D A key aspect of fluency with calculations of this type is to look for simplifications before multiplying numerators and denominators.
	
	In *Example 4*, students should be encouraged to reason about the relative size of the answers by looking at the constituent parts of the product rather than calculating the product.

b) *Which gives the greater result, calculation C or calculation D?*

Explain how you know.

C: $\frac{12}{13} \times \frac{14}{15}$

D: $\frac{12}{14} \times \frac{13}{15}$

Do students notice that *A* and *B* share a factor of $\frac{14}{15}$ and reason about the relative size of the other factors? Students can use similar reasoning to identify whether *C* or *D* gives the greater product.

You might like to ask students to investigate further examples, similar to those in part b), deciding what arrangement of four consecutive numbers in a frame (such as the one below) will give the greatest product and the least product, and considering whether their decision is the same for all sets of four consecutive numbers.

$$\frac{\square}{\square} \times \frac{\square}{\square}$$

2.1.5.5 Use the associative, distributive and commutative laws to flexibly and efficiently solve problems

Common difficulties and misconceptions
Students' understanding of the laws of arithmetic is crucial if they are to be able to work flexibly to evaluate calculations.

A key idea here is that students are able to identify known facts, connections and relationships and use them to strategically simplify calculations. For some students, whose experience of mathematics may be that there is only one correct process that should be followed, this may prove challenging.

The strategy of inviting students to solve problems 'in as many different ways as you can' can help to develop the skill of making sensible choices based on the numbers involved and the relationships between them. It is also helpful to choose examples that draw students' attention to certain, useful connections and asking them, *'What do you notice?'*, e.g. $9\,999 + 999 + 99 + 9 + 5$ or $2.75 \times 5.4 + 27.5 \times 0.46$.

What students need to understand	Guidance, discussion points and prompts
Identify relationships so that known facts can be used to simplify calculations.	**R** In *Example 1*, the area model for multiplication is used as a context to give students an opportunity to consider different calculation strategies.

Identify relationships so that known facts can be used to simplify calculations.

Example 1:

The area of the shape below can be found by calculating $2 \times 14 + 8 \times 5 + 9 \times 8$.
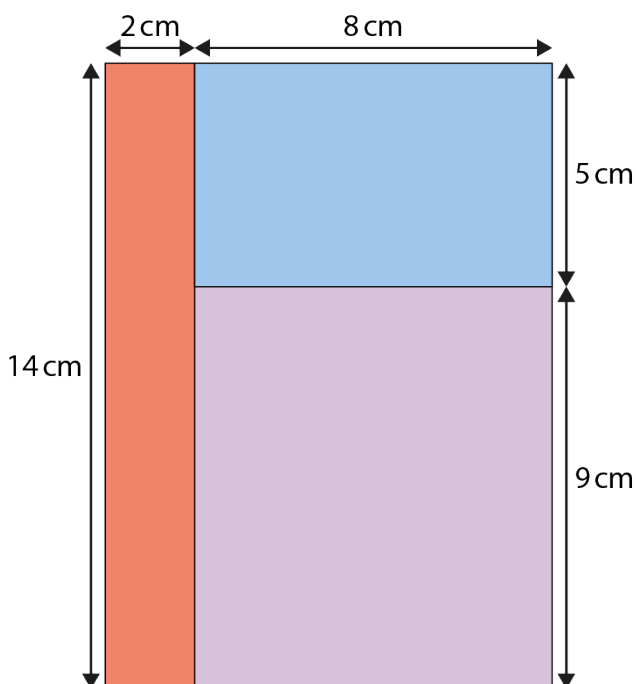
a) *Find the area of the shape in cm².*

b) *Write down some other calculations that represent the area of the shape.*

R In *Example 1*, the area model for multiplication is used as a context to give students an opportunity to consider different calculation strategies.

You might like to ask students which area of the shape is being calculated by each of the products in the number sentence, giving them an opportunity to make sense of the representation and connect it to the symbols. Then, discuss the different strategies students have found to simplify their calculation.

If it does not arise during the discussion, draw students' attention to the way that the commutative and associative laws can be used to identify that $8 \times 5 + 9 \times 8 = 8 \times 14$ and that the distributive law can be used to simplify the overall expression to 10×14.

Example 2:

a) *Which of these is correct?*

$13 \times 99 = 10 \times 90 + 3 \times 9$

$13 \times 99 = 13 \times 100 - 13 \times 1$

$13 \times 99 = 13 \times 90 + 13 \times 9$

$13 \times 99 = 10 \times 99 + 3 \times 99$

$13 \times 99 = 15 \times 99 - 2 \times 99$

Which method do you prefer to calculate this product? Why?

b) *Which of these is correct?*

$19 \times 99 = 25 \times 99 - 6 \times 99$

$19 \times 99 = 20 \times 99 - 1 \times 99$

$19 \times 99 = 19 \times 100 - 19 \times 1$

$19 \times 99 = 10 \times 90 + 9 \times 9$

$19 \times 99 = 10 \times 90 + 9 \times 90$

$19 \times 99 = 19 \times 90 + 19 \times 9$

Which method do you prefer to calculate this product? Why?

c) *Which of these is correct?*

$77\,068 \div 5 \div 2 = (77\,068 \div 5) \div 2$

$77\,068 \div 5 \div 2 = 77\,068 \div (5 \div 2)$

$77\,068 \div 5 \div 2 = 77\,068 \div 2 \div 5$

$77\,068 \div 5 \div 2 = 77\,068 \div (5 \times 2)$

Which method do you prefer to calculate this quotient? Why?

d) *Which of these is correct?*

$7\,742 \div 14 = (7\,742 \div 7) \div 2$

$7\,742 \div 14 = (7\,742 \div 2) \div 7$

$7\,742 \div 14 = (7\,742 \div 10) \div 4$

Which method do you prefer to calculate this quotient? Why?

V In *Example 2*, students are required to think flexibly to evaluate whether the different methods are correct or not, before choosing their preferred method for calculating the product or quotient.

The invalid methods have each been chosen to highlight and draw out a particular misconception, while the valid ones show a range of different possibilities.

It is important that students understand that while each of the valid methods gives a correct solution, some may be more efficient at reaching it than others. It is also worth noting that the most efficient method for one student may not be the most efficient for another. Students' preferences will depend on what facts they are able to recall fluently.

Example 3:

Are the following statements true or false?

Explain how you know.

a) $5 \times 3.2 + 3.2 \times 3 = 2.5 \times 4 \times 3.2$

b) $2 \times (17 \times 3.2 + 1.8 \times 17) = 17^2 - (7 \times 17)$

c) $7 \times (3 + 0.2) - 3.2 \times 2 + -3.2(1.75 + 3\frac{1}{4}) + 1.8 = 3 \times 1.8 \div 3$

In *Example 3*, students are required to bring together the commutative, distributive and associative laws to simplify complex calculations.

A key awareness for students here is that some calculations can be simplified. Students should not automatically reach for their calculator; instead, they should consider each calculation as a whole in order to identify relationships and possible known facts, so reducing the amount of calculation necessary. Rather than focus on the final result of each calculation, it will be more helpful to emphasise the laws of arithmetic that have been used to simplify the calculations.

PD Consider the prompts and scaffolds that you might use in your class to support students who find these questions challenging. What strategies might you use to help them move forward?

Weblinks

1   NCETM primary mastery professional development materials
     https://www.ncetm.org.uk/resources/50639

2   NCETM primary assessment materials
     https://www.ncetm.org.uk/resources/46689

3   Standards & Testing Agency past mathematics papers
     https://www.gov.uk/government/collections/national-curriculum-assessments-practice-materials#key-stage-2-past-papers

4   NRICH: Adding and Subtracting Positive and Negative Numbers
     https://nrich.maths.org/5947