MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

1 The structure of the number system

1.4 Simplifying and manipulating expressions, equations and formulae

Guidance document | Key Stage 3

Making connections

The NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of 'core concepts'.

The first of these themes is *The structure of the number system*, which covers the following interconnected core concepts:

1.1 Place value, estimation and rounding

1.2 Properties of number

1.3 Ordering and comparing

1.4 **Simplifying and manipulating expressions, equations and formulae**

This guidance document breaks down core concept *1.4 Simplifying and manipulating expressions, equations and formulae* into five statements of knowledge, skills and understanding:

1.4.1 Understand and use the conventions and vocabulary of algebra, including forming and interpreting algebraic expressions and equations

1.4.2 Simplify algebraic expressions by collecting like terms to maintain equivalence

1.4.3 Manipulate algebraic expressions using the distributive law to maintain equivalence

1.4.4 Find products of binomials

1.4.5 Rearrange formulae to change the subject

Then, for each of these statements of knowledge, skills and understanding we offer a set of key ideas to help guide teacher planning.

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Overview

At the heart of algebra and algebraic thinking is the expression of generality. Algebraic notation and techniques for its manipulation, including conventions governing its use, should naturally arise from exploring the structure of the number system and operations on number. For this reason, algebra is not a separate theme in these materials but is linked to the two themes associated with number: *1 The structure of the number system* and *2 Operating on number*.

In this core concept, students are presented with situations where the structure of numbers can be generalised. Students are introduced to conventions concerning the writing of algebraic symbols and learn techniques for symbolic manipulation. For example, students who know that equivalent subtractions can be formed by adding or subtracting the same quantity from both the subtrahend and the minuend (e.g. 3 476 – 1 998 = 3 478 – 2 000), can be taught to generalise this as $(a + n) - (b + n) = a - b = (a - n) - (b - n)$.

In Year 6, a key focus in relation to algebra is that students 'should be introduced to the use of symbols and letters to represent variables and unknowns in mathematical situations that they already understand' (Department for Education, 2013)[†]. This work continues into Key Stage 3, with the important development that students use algebraic notation to examine and analyse number structure, and to deepen their understanding.

Prior learning

Before beginning to teach *Simplifying and manipulating expressions, equations and formulae* at Key Stage 3, students should already have a secure understanding of the following from previous study:

Key stage	**Learning outcome**
Upper Key Stage 2	• Use their knowledge of the order of operations to carry out calculations involving the four operations • Use simple formulae • Express missing number problems algebraically • Find pairs of numbers that satisfy an equation with two unknowns • Enumerate possibilities of combinations of two variables • Be introduced to the use of symbols and letters to represent variables and unknowns in mathematical situations that they already understand (non-statutory guidance)

You may find it useful to speak to your partner schools to see how the above has been covered and the language used.

[†] Department for Education, 2013, *National curriculum in England: mathematics programmes of study, Key Stages 1 and 2*, Year 6

You can find further details regarding prior learning in the following segments of the NCETM primary mastery professional development materials[1]:

- Year 5: 1.28 Common structures and the part–part–whole relationship
- Year 6: 1.31 Problems with two unknowns

Checking prior learning

The following activities from the NCETM primary assessment materials[2] offer useful ideas for assessment, which you can use in your classes to check whether prior learning is secure:

Reference	Activity
Year 6 page 29	*Which of the following statements do you agree with? Explain your decisions.* • *The value 5 satisfies the symbol sentence* $3 \times \square + 2 = 17$ • *The value 7 satisfies the symbol sentence* $3 + \square \times 2 = 10 + \square$ • *The value 6 solves the equation* $20 - x = 10$ • *The value 5 solves the equation* $20 \div x = x - 1$
Year 6 page 29	*I am going to buy some 10p stamps and some 11p stamps.* *I want to spend exactly 93p. Write this as a symbol sentence and find whole number values that satisfy your sentence.* *Now tell me how many of each stamp I should buy.* *I want to spend exactly £1.93. Write this as a symbol sentence and find whole number values that satisfy your sentence.* *Now tell me how many of each stamp I should buy.*

Key vocabulary

Term	Definition
binomial	An algebraic expression of the sum or difference of two terms.
equation	A mathematical statement showing that two expressions are equal. The expressions are linked with the symbol = Examples: $7 - 2 = 4 + 1$ $4x = 3$ $x^2 - 2x + 1 = 0$
expression	A mathematical form expressed symbolically. Examples: $7 + 3$ $a^2 + b^2$
factorise	To express a number or a polynomial as the product of its factors. Example 1: Factorising 12: $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$ The factors of 12 are 1, 2, 3, 4, 6 and 12. 12 may be expressed as a product of its prime factors: $12 = 2 \times 2 \times 3$

	Example 2: Factorising $x^2 - 4x - 21$: $x^2 - 4x - 21 = (x + 3)(x - 7)$ The factors of $x^2 - 4x - 21$ are $(x + 3)$ and $(x - 7)$.
formula	An equation linking sets of physical variables. Example: $A = \pi r^2$ is the formula for the area of a circle. Plural: formulae.
substitute/ substitution	Numbers can be substituted into an algebraic expression in x to get a value for that expression for a given value of x. Example: When $x = -2$, the value of the expression $5x^2 - 4x + 7$ is $5(-2)^2 - 4(-2) + 7 = 5(4) + 8 + 7 = 35$.
variable	A quantity that can take on a range of values, often denoted by a letter, $x, y, z, t, \ldots$, etc.

Collaborative planning

Below we break down each of the five statements within *Simplifying and manipulating expressions, equations and formulae* into a set of key ideas to support more detailed discussion and planning within your department. You may choose to break them down differently depending on the needs of your students and timetabling; however, we hope that our suggestions help you and your colleagues to focus your teaching on the key points and avoid conflating too many ideas.

Please note: We make no suggestion that each key idea represents a lesson. Rather, the 'fine-grained' distinctions we offer are intended to help you think about the learning journey irrespective of the number of lessons taught. Not all key ideas are equal in length and the amount of classroom time required for them to be mastered will vary, but each is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

The following letters draw attention to particular features:

D Suggested opportunities for **deepening** students' understanding through encouraging mathematical thinking.

L Examples of shared use of **language** that can help students to understand the structure of the mathematics. For example, sentences that all students might say together and be encouraged to use individually in their talk and their thinking to support their understanding (for example, *'The smaller the denominator, the bigger the fraction.'*).

R Suggestions for use of **representations** that support students in developing conceptual understanding as well as procedural fluency.

V Examples of the use of **variation** to draw students' attention to the important points and help them to see the mathematical structures and relationships.

PD Suggestions of questions and prompts that you can use to support a **professional development** session.

For selected key ideas, marked with an asterisk (*), we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches, together with suggestions and prompts to support professional development and collaborative planning. You can find these at the end of the set of key ideas.

Key ideas

1.4.1 Understand and use the conventions and vocabulary of algebra, including forming and interpreting algebraic expressions and equations

The fundamental understanding in this set of key ideas is that a letter can be used to represent a generalised number and that algebraic notation is used to generalise number properties, structures and relationships.

Students will have gained a sense of certain generalities in Key Stage 2 (for example, commutativity of addition and multiplication). They should also have had experience of recording such generalities using symbols (e.g. $a + b = b + a$ and $ab = ba$).

At Key Stage 3, students experience a wide range of examples where generalisations can be made (for example, the sum of three consecutive integers being a multiple of three). Students realise that such generalised statements can become expressions in their own right (for example, $3n$ represents a generalised multiple of three). They also understand that such statements capture an infinity of cases and hence represent, for example, all the multiples of three 'in one go'. All these are examples of working from the particular to the general, and students should have a clear understanding of the particular number relationships before generalising using algebra.

One of the ways in which students interpret algebraic expressions and equations is to work from the general to the particular. For example, to interpret the meaning of an algebraic statement, such as $3x + 5$ or $x^2 – 2$, it is important that students consider the questions:

- *'How does the value of the expression change as the value of x changes?'*
- *'When does the expression take a particular value?'*

Students should realise that there is a difference between situations where a letter represents a variable which can take *any* value across a certain domain and where, because of some restriction being imposed (e.g. $3x + 5 = 7$, $x^2 – 2 = 9$ or $3x + 5 = x^2 – 2$), it has a *particular* value (which may be as yet unknown).

1.4.1.1 Understand that a letter can be used to represent a generalised number

1.4.1.2 Understand that algebraic notation follows particular conventions and that following these aids clear communication

1.4.1.3 Know the meaning of and identify: term, coefficient, factor, product, expression, formula and equation

1.4.1.4* Understand and recognise that a letter can be used to represent a specific unknown value or a variable

1.4.1.5* Understand that relationships can be generalised using algebraic statements

1.4.1.6 Understand that substituting particular values into a generalised algebraic statement gives a sense of how the value of the expression changes

1.4.2 Simplify algebraic expressions by collecting like terms to maintain equivalence

Students should see the process of 'collecting like terms' as essentially about adding things of the same unit. Younger students are often excited by the fact that calculations such as 3 000 000 + 2 000 000 are as easy as 3 + 2. Later, they realise that the same process is at work with equivalent fractions, such as $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$. Students begin to generalise this to 3 (of any number) + 2 (of the same number), and finally to symbolise this as $3a + 2a$.

Teaching approaches that are solely procedural and do not allow students to understand the idea of unitising and the important principle that letters stand for numbers and not objects, should be avoided. For example, to teach that $3a + 2a = 5a$ because 'three apples plus two apples equals five apples' is incorrect and this approach (often termed 'fruit salad algebra') should be avoided.

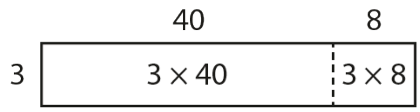
Students should fully appreciate that 'collecting like terms' is not a new idea but a generalisation of something they have previously experienced when unitising in number. They should understand what like terms are and are not, and experience a wide range of standard and non-standard examples (for example, constant terms, terms containing products, indices, fractional terms). Students should come to realise that, when they are simplifying algebraic expressions such as $2xy + 5xy$ as $7xy$, they have obtained an equivalent expression (i.e. one with exactly the same value even though it has a different appearance).

1.4.2.1 Identify like terms in an expression, generalising an understanding of unitising

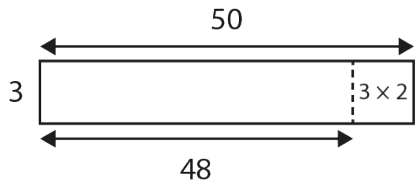
1.4.2.2 Simplify expressions by collecting like terms

1.4.3 Manipulate algebraic expressions using the distributive law to maintain equivalence

Students will have learnt at Key Stage 2 that to calculate an expression such as 3×48 they can think of it as $3 \times (40 + 8)$, which equals $3 \times 40 + 3 \times 8$. Students may know this as the distributive law, although this should not be assumed. What is important at Key Stage 3 is that students come to see this as a general structure that will hold true for all numbers. They should be able to express this general structure symbolically (i.e. $3(a + b) = 3a + 3b$) and pictorially by using, for example, an area model:



Students should also be able to generalise this further to subtraction (i.e. $3(a - b) = 3a - 3b$) by considering a calculation, such as $3 \times 48 = 3(50 - 2) = 3 \times 50 - 3 \times 2$, and an area model, such as this:



It is useful at this stage to draw attention to the 'factor × factor = product' structure of the equivalence $3(a + b) = 3a + 3b$, i.e. two factors, 3 and $(a + b)$, have been multiplied together to give a product equivalent to $3a + 3b$. This will support students' understanding of the inverse process of factorising. For example, *'If the product is 3a + 3b, what might the two factors be?'*.

To gain a deep and secure understanding, students will benefit from experiencing a wide range of standard and non-standard examples (such as negative, decimal and fractional factors, including variables). Careful attention to the use of variation when designing examples will support students to generalise.

1.4.3.1* Understand how to use the distributive law to multiply an expression by a term such as $3(a + 4b)$ and $3p^2(2p + 3b)$

1.4.3.2 Understand how to use the distributive law to factorise expressions where there is a common factor, such as $3a + 12b$ and $6p^3 + 9p^2b$

1.4.3.3 Apply understanding of the distributive law to a range of problem-solving situations and contexts (including collecting like terms, multiplying an expression by a single term and factorising), e.g. $10 - 2(3a + 5)$, $3(a \pm 2b) \pm 4(2ab \pm 6b)$, etc.

1.4.4 Find products of binomials

In 1.4.3, students used the distributive law to expand a single term over a binomial. Here they use the same law to work with pairs of binomials. Students should understand that this expansion is a generalisation of the familiar 'grid method' for multiplication. For example, the layout below (top) representing $(2x + 4)(3x + 6)$ can be seen as a generalisation of the familiar grid layout (below, bottom) for 24×36 or $(20 + 4)(30 + 6)$.

	$2x$	4
$3x$	$6x^2$	$12x$
6	$12x$	24

	20	4
30	600	120
6	120	24

The use of algebra tiles to represent this may help to make the connection with the area model of multiplication more explicit.

The area model will also support students to understand and justify that the product of an expression with, for example, two terms in the first expression and three terms in the second expression, will have six (i.e. 2×3) terms before simplifying. For example, $(2a + 3)(5a + 6y + 4)$ can be represented as:

	$2a$	3
$5a$		
$6y$		
4		

Students need to generalise further to situations where there are more than two binomials and realise that the product of more than two binomials can be reduced to two polynomials by successive multiplication of pairs. For example, the product $(a + b)(a + 3b)(a - b)$ can be reduced to the product of two polynomials by combining any two binomials. It will be important to introduce examples where alternative approaches might be more efficient and/or elegant, and to give students the opportunity to discuss these. For example, $(a + b)(a + 3b)(a - b)$ can be transformed into $(a^2 + 4ab + 3b^2)(a - b)$ and then multiplied out further. Alternatively, it could be transformed into $(a^2 - b^2)(a + 3b)$ by noticing that the first and last factors produce the difference of two squares.

1.4.4.1* Use the distributive law to find the product of two binomials

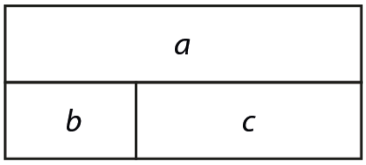
1.4.4.2 Understand and use the special case when the product of two binomials is the difference of two squares

1.4.4.3 Find more complex binomial products

1.4.5 Rearrange formulae to change the subject

At Key Stages 1 and 2, students had experience of expressing number relationships in different ways. So, for example, if students know $3 + 4 = 7$, they should also know the 'three facts for free': $4 + 3 = 7$, $7 - 4 = 3$ and $7 - 3 = 4$. Similarly, students should be aware that $3 \times 4 = 12$ gives rise to $4 \times 3 = 12$, $12 \div 3 = 4$ and $12 \div 4 = 3$. At Key Stage 3, students extend this knowledge to equations, understanding that the same relationship can be expressed in different ways.

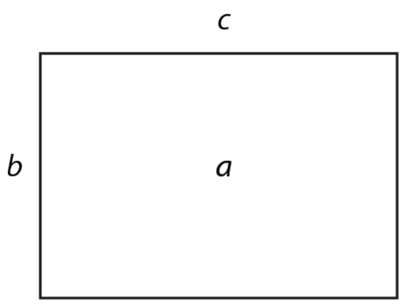
Students should distinguish between additive and multiplicative structures. Additive structures can be shown clearly by a bar model. For example, $a = b + c$ can be represented as:



This gives rise to the following equivalent expressions: $a = b + c$; $a = c + b$; $a - b = c$; $a - c = b$.

Students need to be aware that this additive structure can also be applied to more complex equations. For example, $(x^2 + a) + (x^3 - px + m) = (4 - p)$ can be rewritten as: $(x^2 + a) = (4 - p) - (x^3 - px + m)$, which, because the left-hand side is also an additive expression, can be written as: $a = (4 - p) - (x^3 - px + m) - x^2$ to make a the subject.

When considering multiplicative structures, an area model is helpful to reveal the relationships. For example, $b \times c = a$ can be represented as:



Students can then see the equivalent expressions: $b \times c = a$; $c \times b = a$; $a \div c = b$; $a \div b = c$.

When working with formulae, students should appreciate that, when expressing the relationship between one variable (the subject of the formula) and the rest of the expression, it is possible to evaluate any of the variables, given values for all the others. For example, $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}(F - 32)$ allow for different values to be calculated and offer different perspectives of the relationship between degrees Fahrenheit (*F*) and degrees Celsius (*C*). Students should appreciate that the process of changing the subject of a formula is essentially the same process as solving an equation in one unknown.

1.4.5.1* Understand that an additive relationship between variables can be written in a number of different ways

1.4.5.2 Understand that a multiplicative relationship between variables can be written in a number of different ways

1.4.5.3 Apply an understanding of inverse operations to a formula in order to make a specific variable the subject (in a wide variety of increasingly complex mix of operations)

Exemplified key ideas

1.4.1.4 Understand and recognise that a letter can be used to represent a specific unknown value or a variable

Common difficulties and misconceptions
Dietmar Küchemann (1978)[‡] identified the following six categories of letter usage by students (in hierarchical order): • **Letter evaluated**: the letter is assigned a numerical value from the outset, e.g. $a = 1$. • **Letter not used**: the letter is ignored, or at best acknowledged, but without given meaning, e.g. $3a$ taken to be 3. • **Letter as object**: shorthand for an object or treated as an object in its own right, e.g. a = apple. • **Letter as specific unknown**: regarded as a specific but unknown number and can be operated on directly. • **Letter as generalised number**: seen as being able to take several values rather than just one. • **Letter as variable**: representing a range of unspecified values, and a systematic relationship is seen to exist between two sets of values. The first three offer an indication of the difficulties and misconceptions students might have. The last three outline the progression that students need to make as they develop an increasingly sophisticated view of the way letters are used to represent number.

What students need to understand	Guidance, discussion points and prompts
Understand that unknown quantities can be named and operated on. *Example 1:* *For each of the following statements, use a letter to represent the number Isla is thinking of and write the statement using letters and numbers.* *a) 'I am thinking of a number and I add three.'* *b) 'I am thinking of a number and I multiply by two and add three.'* *c) 'I am thinking of a number and I add three and multiply by two.'* *d) 'I am thinking of a number and I multiply by three and add two.'* *e) 'I am thinking of a number and I add two and multiply by three.'*	**V** In *Example 1*, the numbers are deliberately kept the same in order for students to focus on the order of operations and how algebraic symbolism is used to represent the different order of operations, using brackets where necessary. A key purpose of variation is to support students' awareness of what can change, and it can be useful to ask them to make up some examples like these for themselves. For example, you could ask: *'Using the numbers two and three, make up some different "I am thinking of a number" statements and set them for your partner.'* **D** Students' thinking can be deepened by asking more probing questions at intervals throughout this example. For example, after working on parts b) and c), you could ask: *'Do the two expressions ($2x + 3$ and $2(x + 3)$) mean the same? Do they give the same answers for given values of x?'*

[‡] Dietmar Küchemann, 1978, *Mathematics in School*, Vol. 7, No. 4 (Sep 1978), 23–26

	While this example is a useful precursor to solving equations, the central purpose here is to understand that letters can have a range of values and to get a sense of how the value of expressions can change with these different values. Students should be encouraged to offer a number of possible values for *x*. **L** This is a good opportunity to introduce the language of 'variable' and encourage students to use this term while discussing their answers and their reasoning. For example, *'In the expression 2x + 3, x is a variable because it can take a range of different values.'* **PD** What other ways might there be of helping students to see that unknown quantities can be worked on? You could try this activity with a group of teachers: • Ask two people to each think of a number, one has to think of a two-digit integer, and one has to think of a three-digit integer. • Find the difference between the two numbers, but first ask the two people to add 1 to each of their numbers. What effect will this have on the difference? • What about if they added 1 to one of the numbers and took 1 from the other, etc.?
Example 2: *For each of the following statements, use a letter to represent the number Isla is thinking of, write the statement using letters and numbers, and find the number she is thinking of.* *f) 'I am thinking of a number; I add four and the answer is 12. What number am I thinking of?'* *g) 'I am thinking of a number; I add four, multiply by three and the answer is 12. What number am I thinking of?'* *h) 'I am thinking of a number; I add four, multiply by three, subtract six and the answer is 12. What number am I thinking of?'* *i) 'I am thinking of a number; I add four, multiply by three, divide by two and the answer is 12. What number am I thinking of?'*	The focus of *Example 2* is to make students aware of the fact that, when constraints are put on a situation, the unknown will take a particular value. **V** The numbers have been chosen in *Example 2* to keep the given answer of '12' the same and to build the operations in sequence. The example will best be tackled by offering and discussing each part individually. Students can be encouraged to make up their own examples for their partners. This will support their realisation that, when they put constraints on a situation like this, their partner will always be able to figure out their number. **L** You could encourage students to use the term 'specific unknown' when talking about

	these examples, as in, '*When I am told that $3(x + 4) - 6 = 12$, there is only one value that* will *make this true and so the letter x stands for a specific unknown*'.
Understand that a letter stands for a variable and can take a range of values. *Example 3:* *Which is bigger $3n$ or $n + 3$?*	*Example 3* is a context for exploring how the value of a variable can change. Students may have an intuition about which is bigger and say, for example, '$3n$ because multiplication always makes numbers bigger than addition'. You could then challenge students and encourage them to prove or disprove the statement. **R** You could encourage students to record their explorations using different representations, for example, in a table or a graph. You could also invite students to draw a rectangle with sides $3n$ and $n + 3$ and ask, *'Will this rectangle be short and fat or tall and thin?'*. This may provide another context in which to think about the problem. **D** Exploring a range of examples (e.g. $2n$ or $n + 2$, $4n$ or $n + 4$, $5n$ or $n + 5$) can provide opportunities for discussions about when the two expressions are the same, and help secure a deeper understanding of the relationships between the expressions (i.e. $2n$ and $n + 2$ have the same value when $n = 2$; $4n$ and $n + 4$ have the same value when $n = \frac{4}{3}$, etc.) and why this might be so.
Example 4: *Arrange these cards in order.* $\boxed{x}$ $\boxed{2x}$ $\boxed{x^2}$ $\boxed{x + 3}$ $\boxed{x^2 - 5}$ $\boxed{3x - 2}$	For *Example 4*, you could give one card each to a group of students and ask them to come to the front of the class and line themselves up (holding the card in front of them) in order from smallest to largest. As the statements on the cards are expressions involving variables, it is not possible to agree an order. This activity is intended to bring to the surface the students' current thinking (including misconceptions) and to engage them in discussion about the possible values these expressions can take. **D** To deepen students' thinking and awareness of the nature of the variables, you could ask questions that probe their thinking and prompt them to reason. For example:

	• *'Will x always be smaller than $x + 3$? Why or why not?'* • *'Will x always be smaller than $2x$? Why or why not?'* • *'How are $2x$ and x^2 different? When is one bigger than the other? Could they ever have the same value?'* • *'What is the largest value that each expression could have? What is the smallest?'* • *'When might other expressions have the same value?'*

1.4.1.5 Understand that relationships can be generalised using algebraic statements

Common difficulties and misconceptions

The non-statutory guidance for the *Year 6 Programme of Study* states that 'Students should be introduced to the use of symbols and letters to represent variables and unknowns in mathematical situations that they already understand.' They may have some familiarity with letter symbols recording relationships but will need to further develop and deepen this.

Students often interpret an algebraic formula or equation as a set of instructions to be followed, or as a problem to be solved, rather than understanding the symbols as a representation of a relationship. It can be useful to give them opportunities to work between different representations, including language, symbolic and graphical representations, to compare and identify equality and then to see how this relationship is captured in each representation.

Students may feel uncomfortable leaving their 'answer' as an expression or equation, and so an error such as rewriting $7 + m$ *as* $7m$ might not simply be a lack of understanding of the conventions of algebra, or the relationship being recorded, but that the student has not accepted 'lack of closure' (Collis, 1978) and believes that their answer should be a single number or term.

Students' intuition to use the letter symbol as shorthand may also lead to errors. For example, when asked to write a formula connecting the number of days and the number of weeks, many students may write $7d = w$ (maybe reading this as seven days equals one week) where the correct formula should be $7w = d$ where d represents the number of days and w the number of weeks. Again, using specific language to describe the relationship in words (for example, reading 7d as 'the number of days multiplied by 7') can help raise awareness of this.

What students need to understand

Use letter symbols to represent mathematical relationships.

Example 1:

Describe how each of the following is represented in this bar model.

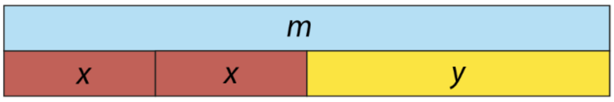
a) $x + 2 = 10$

b) $10 - x = 2$

c) $10 - 2 = x$

Guidance, discussion points and prompts

R *Example 1* uses the familiar representation of a bar model to draw attention to the different relationships that exist and ways that they can be written symbolically. Although students may have seen this sort of image before, the focus here is particularly on the equality between the top and bottom bars.



You might like to follow this task by offering a different relationship represented by a bar model and ask students to write the relationships symbolically.

Or you might like to offer a symbolic representation (such as $2x + y = m$) and ask students to represent this relationship using a pictorial representation.

Example 2:

Write an expression to represent each of these relationships.

a) Two numbers add to 10.

b) Two numbers are 10 apart on the number line.

c) Two numbers are added together to make a third number.

d) Two of the same number are added together to make another number.

L The language here is used as another representation to access the structure and give meaning to the symbols. It is useful to encourage students to work in both directions – from the language to symbolic algebra and to also describe the symbolic algebra verbally.

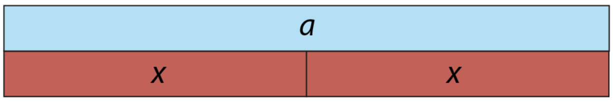
R You could ask students to represent these situations using a number line or bar model.

D Part b) here offers several possible correct solutions ($a + 10 = b$, $m - 10 = n$, $p - q = 10$). You might like to compare these solutions and unpack why this one example has multiple solutions while the others have just one correct answer.

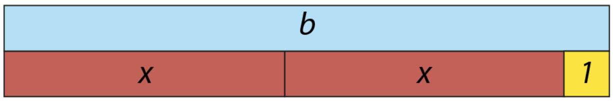
Example 3:

In these bar models, x represents a whole number.

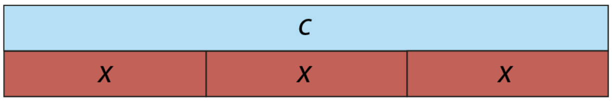
a) Does a represent an odd number or an even number? Explain how you know.



b) Does b represent an odd number or an even number? Explain how you know.



c) Does c represent an odd number or an even number? Explain how you know.



R In *Example 3* the focus is on making explicit that conclusions can be drawn about the properties of a number by interrogating its symbolic representation. The bar model is used to access the structure and to make sense of the symbolic notation. Students should understand that we can state that, for example, $2x$ is an even number and $2x + 1$ is an odd number even though we don't know what particular numbers they are.

V Part c) offers an opportunity to see the limitations of what can be deduced from a general representation of a number. Although we know that c is a multiple of 3, we can't know whether it is odd or even because the set of multiples of 3 includes both odd and even numbers.

Example 4:

What can you say about the value of y in each of these?

a) $a + y = a$

b) $hy = h$

c) $m - y = 0$

d) $3y < 2y$

V *Example 4* continues the thread from *Example 3* and looks at conclusions that can be drawn about the properties of a general number – in this case offered in a symbolic form.

It is important to discuss with students that while in parts a) and c) it is possible to know the value of y with certainty, that isn't the case in parts b) and d). However, conclusions can still be drawn about the values of y and in

	part d) we can determine that *y* **must** be negative and cannot be zero, while in part b) we can state that either y = 1 or *h* = *0*.
Use letter symbols to model situations. *Example 5:* *1. There are seven days in a week. Which of the following shows the relationship between the number of days, d, and the number of weeks, w?* *a) $7d = w$* *b) $7w = d$* *2. There are twelve months in a year. Which of the following shows the relationship between the number of months, m, and the number of years, y?* *a) $12m = y$* *b) $12y = m$* *3. Richard is 36 years older than Matilda. Which of the following correctly shows the relationship between Richard's age, r, and Matilda's age, m?* *a) $r + 36 = m$* *b) $m + 36 = r$*	**L** *Example 5* is designed to draw attention to the problems associated with reading a letter symbol as an abbreviation of a word. Reading question 1 part a) as 'The number of days multiplied by seven gives the number of weeks' raises the inconsistency in the more intuitive formula and draws attention to the letter symbol as representing a quantity rather than an object.
Example 6: *1. Andrew is three years younger than his sister, Sarah. The formula $x + 3 = y$ represents this relationship. What do you think x and y represent?* *2. An amusement park charges an entrance fee and then charges for tokens to be used on each ride. The formula $15 + 1.5x = y$ represents this relationship.* *a) What do you think x and y represent?* *b) What else do you know about the costs of visiting the amusement park?*	**L** *Example 6* challenges students to use the structure of the situation to make sense of the algebraic representation. Students may be familiar with those structures as in *Example 5*, where the letter symbol used connects to the real-life context (for example, *y* represents the *number of years*). However, where there is no obvious connection that *x* represents Andrew's age, students' need to make their own connections between the symbols and the context.
Interpret the impact of changing one variable on another within a generalised relationship. *Example 7:* *Look at the formula $r + 3u = t$.* *a) When r increases by 10, how much does t increase by?*	*Example 7* encourages students to develop their understanding of the letter symbol as a variable and consider the 'shape' of the relationship represented as one element changes. As in *Examples 3* and *4*, students need to understand that although they don't know the particular value, they can draw reasoned conclusions about changes that result from altering one variable.

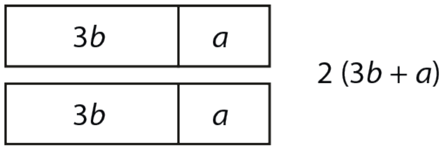
b) *When u increases by 10, how much does t increase by?* c) *Use r and u to write down an expression that will always be less than t.*	In part c) students may offer suggestions such as $r + 2u < t$. Substituting a range of values will help them understand it is not possible to write, with certainty, an expression that **only** uses r and u which is less than t. For example, if $r = 12$ and $u = -2$ then $t = 6$ which leads to $r + 2u = 8$, which is clearly not less than 6. Alternatively, students may offer (or the teacher might suggest) $r + 3u - 10 < t$ and comparing these different solutions provides an opportunity to consider the difference between the use of a letter symbol and a number in the expression. While working with *Example 7* it is important that students are made aware of the range of possible values that r and u can take. Asking students whether their solution is valid when, for example, r is negative, or a fraction, or zero gives insight into the structure but also raises awareness that the letter symbol does not necessarily represent a positive integer. **PD** Note that none of the examples here explicitly ask students to substitute numbers in order to explore the relationship. The reasoning they require to make sense of the structure of the relationship is an initial focus, with substitution of particular values being used to check this reasoning and offer particular examples of the general case. Why do you think 'substitution' is so commonly taught as a topic in Key Stage 3 maths curricula? Assuming that students understand the order of operations, what do they learn by substituting values into expressions

1.4.3.1 Understand how to use the distributive law to multiply an expression by a term such as $3(a + 4b)$ and $3p^2(2p + 3b)$

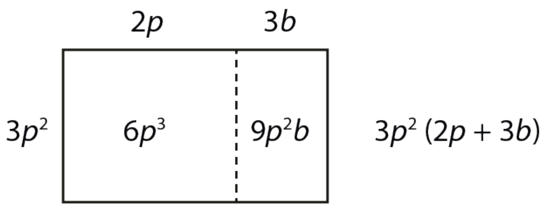
Common difficulties and misconceptions

Students may see processes such as $3(a + 4b) = 3a + 12b$ as purely symbolic exercises with no relationship to a fundamental law (the distributive law) that they are very likely to have experienced and understood at Key Stage 2 in the context of number.

R Bar models and diagrams based on an area model can support students' understanding and help link number and algebra. For example, $2(3b + a)$ can be represented as a bar model:



Similarly, $3p^2(2p + 3b)$ can be represented as an area model:



Students' confidence in using these representations can be developed by asking them to both draw diagrams for given expressions and write expressions for given diagrams. These activities will also support students in seeing the structure behind the mathematical procedure. It will be important that the symbolic representation is used alongside any diagrams to support students to understand how the symbols represent what they know and understand from the diagrams. Once students are familiar with this, you may wish to provide questions for which the use of diagrams is not efficient or appropriate (for example, where negative terms are used). This will encourage students to generalise and not become reliant on the representation.

V Avoid mechanical practice of exclusively standard questions (see *Example 1* below), where the same letter is used for the unknown and the terms are written in the same order throughout, as this can result in students instinctively following a procedure instead of thinking deeply about the mathematical concepts involved. Also, it is useful to use examples of errors or non-examples (see *Examples 4* and *5* below) for students to critique and reason about, as well as asking them to apply skills in different contexts to support the development of deep and sustainable understanding.

What students need to understand	**Guidance, discussion points and prompts**
Understand the structure of the distributive law. *Example 1:* *Calculate as efficiently as possible:* *a)* 16×101 *b)* 25×10010 *c)* $143 \times 100\,001$	**V** In *Example 1*, students could simply answer the questions mechanistically. However, students should be encouraged to notice the additions inherent in the multipliers 101 $(100 + 1)$, 10 010 $(10\,000 + 10)$ and 100 001 $(100\,000 + 1)$, and use these to calculate an answer efficiently.

You could ask students to create their own examples and set them for a partner.

Understand the impact of the multiplier.

Example 2:

For each of these expressions, write another expression without brackets that will always have the same value.

a) $1(3a+5)$

b) $2(3a+5)$

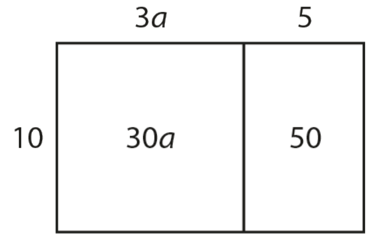
c) $3(3a+5)$

d) $10(3a+5)$

R Students may find it useful to consider a visual representation for each expression. For example, part b) can be represented as a bar model:

a	a	a	$+5$
a	a	a	$+5$

and part d) could be represented using an area model:



V The questions in *Example 2* have been chosen to allow students to notice the impact that the multiplier has on **both** terms inside the brackets.

Students may begin answering parts a), b), and c) by using repeated addition instead of multiplication. It will be important to prompt students to see that multiplication can also be used and to realise that this is a more efficient method, particularly in part d).

Example 3:

Write an equivalent expression without brackets.

a) $10(2xy+z)$

b) $10(a+2b+4c)$

c) $10(p^2+3q)$

V In *Example 3*, students may notice that every term in the equivalent expression is a multiple of ten. Their attention should be drawn to this is to help reinforce the idea that every term inside the brackets is multiplied by the factor.

Understand that the multiplier can be a variable.

Example 4:

Use the distributive law to write equivalent expressions for these expressions.

a) $2(b+7)$

b) $200(b+7)$

c) $2a(b+7)$

d) $2a^2(b+7)$

e) $2a^2b(b+7)$

V The choice of what to keep the same and what to vary in *Example 4* can help students to spot patterns and consider the mathematical structures behind the calculations.

In discussing these questions as a class, it will be helpful to ask students, *'Can you "see" the*

	$b + 7$ in each answer?' You could ask, for example: • *'Can you see the $b + 7$ in $2b + 14$?'* • *'Can you see the $b + 7$ in $2a^2b + 14a^2$?'* **R** You could encourage students to draw diagrams (a bar model or area model) to justify their answers and to critique the answers of other students where they feel there are mistakes. **D** Students could be presented with possible answers and be challenged to find the question, e.g. $2a^2bc + 14a^2c$. Note, it will be important for students to see factorising as the inverse process of multiplying two expressions together. **L** Ensure students can verbalise their method accurately, using key mathematical terms. For example, *'**Every** term inside the brackets is multiplied by the term outside'*.
Understand the importance of the sign (positive or negative) of each term in an expression and how it affects the final result. *Example 5:* *Write an equivalent expression without brackets.* a) $2a(3c + 5b)$ b) $2a(5b + 3c)$ c) $2a(3c - 5b)$ d) $2a(5b - 3c)$ e) $2a(-5b + 3c)$ f) $-2a(5b - 3c)$ g) $-2a(3c - 5b)$	**V** The use of variation in *Example 5* is to draw students' attention to the signs of each term in the expression. You could draw attention to when answers are the same and when they are not by asking, for example, *'Why is $2a(5b - 5c)$ not the same as $2a(5c - 5b)$ but is the same as $2a(-5c + 5b)$?'* **R** The use of an area model diagram (as in *Example 2*) alongside the purely symbolic form will support students' understanding here.
Example 6: *Which expression is correct? Justify your answer.* *Expand $8p(2pq - 3p)$.* a) $16p^2q + 5p$ b) $10p^2q + 5$ c) $16p^2q - 24p^2$ d) $16p^2q + 24p^2$	**V** *Example 6* has been designed to help students clarify the concept by testing 'what it is' as well as 'what it's not'. The options are carefully chosen to address various misconceptions. In choosing part a) students may have incorrectly added the multiplier to the final term and performed $8 + (-3)$ instead of multiplying them.

	The answer in part b) has been reached by adding eight to each number seen rather than multiplying. Part c) is the correct expansion and in part d) students may have ignored the negative sign in the brackets. Asking students to explain why options are incorrect will help them develop their reasoning skills.
Example 7: a) *Samira says that to expand $5e^2f(4g - 3)$ you first do $5e^2f \times 4g$ and then $5e^2f \times 3$. Is she correct?* b) *Jeremiah says that to expand $-5e^2f(4g - 3)$ you first do $-5e^2f \times 4g$ and then $-5e^2f \times 3$. Is he correct?*	**V** *Example 7* has been chosen to test students' understanding of the concept and addresses misconceptions that can arise when students learn a procedure (multiply the terms in the brackets with the term at the front and put the same sign in the middle). While that method might work for part a) it does not work for part b). Showing students examples of both 'what it is' and 'what it's not' will help develop a deeper understanding of the concept. This example also encourages students to verbalise their methods clearly. Part a) would lead to the correct answer if Samira went on to put a negative sign between the terms, but would be improved upon if she had considered she was really multiplying the $5e^2f$ by negative three. **PD** Students will need to be confident and fluent with manipulating negative numbers when tackling these questions. How could you assess this before starting work on this example? Getting students to discuss and explain why statements are incorrect, or asking them to improve upon given answers, are strategies to encourage reasoning (a fundamental aim of the national curriculum).
Understand the impact of a negative multiplier on the result. *Example 8:* *Expand these brackets.* a) $-2a(9d + 4b)$ b) $-2a(9d - 4b)$ c) $-1a(9d + 4b)$ d) $-1a(9d - 4b)$ e) $-a(4b - 9d)$	**V** *Example 8* provides an opportunity to assess students' understanding when dealing with negative numbers. Parts a) and b) and parts c) and d) are paired so students notice what happens to the final term when multiplied by a negative number. Students should notice the difference in how the multiplier is written for part e) compared with part d) and

		understand what this means. Part e) also has a different order of terms within the brackets. Small changes like this are important to encourage students to stop and think deeply about what they are doing. Again, using the same numbers throughout will help students to focus on what is varying in each question.
Example 9: *Expand these brackets. Which expression is the odd one out?* a) $2n(3m + 9p)$ b) $3n(6p + 2m)$ c) $\frac{2}{3}n(9m + 27p)$ d) $-n(-18p - 6m)$	**V**	All parts of *Example 9* have the same answer. By doing these questions, students should appreciate that different expansions (with different multipliers, different terms inside the brackets, different order of terms, different signs) can result in the same expression. Students should be given opportunities to verbalise their mathematical thinking, and questions that do not have an obvious correct answer are good ways to challenge their understanding. This example could prompt conversations about common factors – parts a) and c) could have a factor of three taken out of the brackets, part d) could have a factor of negative six taken out of the brackets. This links to future work on factorising, so be mindful not to progress students onto a new topic. Part d) would normally not be written in this way, so discussions about standard notation may also develop.
Apply knowledge of fractions and decimals when expanding brackets. *Example 10:* *Expand these brackets.* a) $0.1x(80y + 30xy)$ b) $0.2e(5e - 7f)$ c) $\frac{1}{2}v(3u - \frac{1}{3}v)$ d) $-\frac{4}{5}p(5q + 4p)$ e) $-1\frac{3}{4}(4 - \frac{4}{7}d)$	**D** **V**	*Example 10* provides an opportunity to make connections with work on decimals and fractions. It allows students to see that the concept of expanding brackets can be applied to seemingly more complicated numbers, but the structure remains the same. It could be a good strategy for challenging students while still keeping them working on the same topic. Asking students to develop their own questions and answers is another way of promoting deeper thinking. In part a) the multiplier is a decimal, but the terms inside the brackets have been carefully chosen so that students can perform the

multiplications easily and the answers are integers.

In part b) the multiplier is again a decimal, but the linear coefficient is the final term and the numbers are more complicated, so students will need to think carefully about how they express their answer.

Part c) has a fractional multiplier, so students will need to be comfortable with multiplying fractions by integers and other fractions.

In part d) the fractional multiplier is not a unit fraction and terms can be simplified after multiplying. The terms are also not in the 'standard' format.

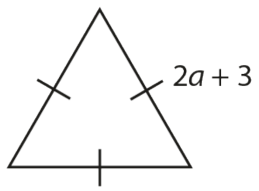
In part e) the multiplier is a negative mixed number and terms can again be simplified. In all questions, a range of letters has been used for the linear unknown so that students become familiar at dealing with letters other than *a* or *x*.

Apply the use of algebra to a different context.

Example 11:

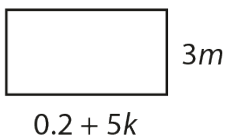
a) *Use brackets to write an expression for the perimeter of these shapes.*

(i)



(ii) A regular polygon with n sides each of length 7 – 3p.

b) *Use brackets to write an expression for the area of this rectangle.*



D *Example 11* is designed to allow students to see how algebra could be applied in different contexts. Students may recognise that there is more than one method for finding the perimeter or area, and discussions could then be had about whether using brackets might provide a more efficient method.

Applying algebra to other topics, such as perimeter and area, will help students to realise that algebra is not a standalone concept but permeates many other areas of mathematics.

PD Do you need to recap perimeter and area with your classes, so that all students can engage in this question?

Can you think of any other contexts where expanding brackets might be used? Consider your Schemes of Learning; this might provide an opportunity to revisit previous topics in different contexts.

1.4.4.1 Use the distributive law to find the product of two binomials

Common difficulties and misconceptions

Students may see 'multiplying two pairs of brackets' as a purely symbolic exercise using a trick such as FOIL, CLAW, smiley face, etc., with no connection to the distributive law. Teaching approaches that are solely procedural and do not help students understand how to find the product of two binomials and make connections with previous learning should be avoided. It is crucial that students see this as an example of 'same value, different appearance' where, although the expression has changed its appearance, the value of it remains unchanged. This is an idea they will already have met in contexts such as equivalent fractions.

R Students should understand that finding the product of two binomials is a generalisation of the familiar 'grid method' for multiplication (which is itself an abstraction of the area model). For example, the layout below representing $(x + 2)(x + 3)$ can be seen as a generalisation of a grid layout for 12×13 or $(10 + 2)(10 + 3)$.

	x	2
x	x^2	$2x$
3	$3x$	6

	10	2
10	100	20
3	30	6

Using the distributive law $(x + 2)(x + 3) = x(x + 3) + 2(x + 3)$, make connections with *1.4.3.1 Multiplying an expression by a term*.

L A binomial expression is an algebraic expression with two terms, such as $(x + 2)$, $(y - 4)$, $(4 - 3p)$, etc. Although in common use, the phrase 'expand these brackets' does not necessarily offer an insight into the mathematical structure and *Example 1* uses both 'find the product' and 'expand' to help students make the connection. Using the word 'product' will also help eliminate the common mistake of students adding rather than multiplying the integers (in the example above, writing 5 in the bottom right-hand cell rather than 6).

V Mechanical practice of exclusively standard questions where the same letter is used for the variable and terms are written as the same throughout should be avoided. (*Example 1* has 'standard' questions whereas other examples include some non-standard questions.) Some students may have difficulties with binomials including negative terms and this is explored in *Example 5*. The questions are designed to foster rich 'what's the same and what's different?' discussions to secure and deepen understanding.

It is also useful to include some non-examples (see *Examples 2* and *3*) for students to critique and reason about and apply skills to solve problems (see *Example 7*).

PD Do we use FOIL, CLAW, smiley face, etc., to find the product of two binomials? Consider the benefits and disadvantages of these approaches.

<table>
<tr><th>What students need to understand</th><th>Guidance, discussion points and prompts</th></tr>
<tr><td>Recognise that the product of two binomials is an expression with four terms.

Example 1:

Find the product of:

a) $(x + 2)(x + 6)$

b) $(x + 3)(x + 4)$

c) $(x + 1)(12 + x)$</td><td>V In Example 1, the binomials have been chosen to give an answer of $x^2 + \square x + \square x + 12$ to support students’ awareness of what can change.</td></tr>
<tr><td>Example 2:

Expand:

a) $(y + 1)(y + 3)$

b) $(2y + 1)(y + 3)$

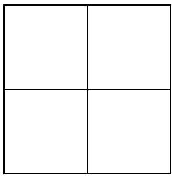
c) $(3y + 1)(y + 3)$

d) $(1 + 4y)(y + 3)$

e) $(4y + 1)(3 + 2y)$</td><td rowspan="3">In Example 2, the binomials have been chosen to support students’ awareness of the impact of the coefficient of the ‘y term’ if one or both coefficients is greater than 1. The binomials in Examples 2–4 have also been deliberately chosen to prevent students thinking that the variable must always be x.

D Students’ thinking can be deepened through more probing questions, such as ‘Find two binomials with a product of $x^2 + x + x + 24$’.

R Using a grid layout helps students see the expansion as a generalisation of the ‘grid method’ for multiplication.



PD Discuss the prompts.

Do we make connections using the distributive law to expand a single term over a binomial when working with pairs of binomials?

Do we make connections with the ‘grid method’ for multiplication when teaching how to find the product of two binomials?

L This is a good opportunity to introduce the language of ‘binomial’ and ‘product’, e.g. ‘The product of two binomials will have four terms.’

PD Example 4 has been chosen to test students’ understanding of $(x + 3)^2 = (x + 3)(x + 3)$. Do we value the importance of asking students to discuss and explain why statements are</td></tr>
<tr><td>Example 3:

Ahmed thinks $(b + 3)(b + 4) = b^2 + 3b + 4b + 7$. Explain why Ahmed is incorrect.</td></tr>
<tr><td>Example 4:

Carol thinks $(a + 3)^2 = a^2 + 9$. Is she correct?</td></tr>
</table>

incorrect, or asking them to improve upon given answers? Is reasoning – a fundamental aim of the national curriculum – a feature of a typical maths lesson in your school?

Appreciate when the product of two binomials can be simplified.

Example 5:

Place a tick (✓) in the cell if the product of the binomials can be simplified to an expression with two, three or four terms.

		Two terms	**Three terms**	**Four terms**
a)	$(x + 1)(x + 3)$			
b)	$(2x + 1)(x + 3)$			
c)	$(y - 3)(y - 3)$			
d)	$(a + b)(a + b)$			
e)	$(p + 4)(p - 4)$			
f)	$(a + b)(c + 2)$			

V The questions in *Example 5* have been chosen to show students the impact of the structure of the binomials when trying to simplify the final expression.

- The questions in parts a) to d) can all be simplified to three terms.
- Part e) can be simplified to two terms.
- Part f) cannot be simplified.

D Students' thinking can be deepened through more probing questions. For example, *'Find two binomials with a product that:*

- *can be simplified to three terms*
- *can be simplified to two terms*
- *cannot be simplified.'*

PD Students need to be confident and fluent with manipulating positive and negative numbers when finding the product of binomials such as $(p + 4)(p - 4)$ and collecting like terms in expressions such as $p^2 + 4p - 4p - 16$. How could you assess this before starting on the example?

Understand that the product of $(x + a)(x - b)$ is an expression of the form $x^2 + cx - d$ or $x^2 - cx - d$.

Example 6:

Write an equivalent expression without brackets:

a) $(x + 5)(x - 2)$

b) $(x - 5)(x + 2)$

V The binomials in *Example 6* have been chosen to help students notice that the product of pairs of binomials of the form $(x + a)(x - b)$ can be simplified to an expression of the form $x^2 + cx - d$ or $x^2 - cx - d$ depending on the values of a and b.

D Investigating the values of a and b that result in the simplified expression changing from $x^2 + cx - d$ to $x^2 - cx - d$ will deepen and challenge students' thinking.

PD Students need to be confident and fluent with manipulating positive and negative numbers when finding the product of binomials such as $(p + 4)(p - 2)$. How could you assess this before starting on this example?

	Consider the benefits and disadvantages of using an 'area' model for finding the product of binomials with negative terms.
Solve problems involving the product of pairs of binomials. *Example 7:* *Which whole numbers could be placed in the box so that the product of two binomials is:* *a)* $x^2 + \square x + 24$ *b)* $y^2 - \square y - 8$ *c)* $p^2 + \square p - 8$ *d)* $a^2 - \square a + 8$	**D** The problems in *Example 7* are empty box problems that have more than one solution. Questions like these encourage students to consider the overall structure of the expansion and simplification of two binomials. Asking students to explain the process they went through to find a solution will also help to refine their mathematical thinking. **PD** How do you manage a question like this with your classes? How long would you give them before you intervene and support? What prompts could you give?

1.4.5.1 Understand that an additive relationship between variables can be written in a number of different ways

Common difficulties and misconceptions

A key misconception for some students is thinking that expressions such as $2x + 3$, $x^2 - 7$ and $x^2 + 2x + 4$ are not 'finished' and another step is required to 'complete' them and get 'an answer'. Consequently, some students will want to combine $2x + 3$ to make 5 or $5x$, or some will try to combine $x^2 + 2x$ by treating the x^2 and x terms as somehow the same.

Students need to understand that algebraic expressions like the ones above cannot be simplified but can be thought of as one term when appropriate. For example, $2x + 3$ can be thought of as the sum of $2x$ and 3, and $x^2 + 2x + 4$ can be thought of as the sum of x^2 and $(2x + 4)$.

What students need to understand

Every addition can be rewritten as a subtraction and every subtraction as an addition.

Example 1:

Identify two addends and their sum in the following equations and show them on a bar model (as below).

<table>
<tr><td colspan="2">Sum</td></tr>
<tr><td>Addend</td><td>Addend</td></tr>
</table>

a) $126 + 437 = 563$

b) $2x + 17 = y$

c) $r = p + q$

d) $x^2 + 6x = 4p^2 + 9$

e) $3m - 2n + r = V$

Guidance, discussion points and prompts

An important awareness in this key idea is that equations of the form A = B + C are examples of additive relationships even though the expressions A, B and C themselves are not. Once students develop this awareness, they are able to transform such equations in a number of different ways, depending on what is required. For example, students could transform $v^2 = u^2 + 2as$ into $v^2 - u^2 = 2as$ (to begin to isolate a or s) or $v^2 - 2as = u^2$ (to begin to isolate u).

V In *Example 1*, the numbers in part a) have been chosen so that students cannot easily calculate the subtraction and check that this gives one of the addends. The emphasis on students' thinking (and in any ensuing discussion) needs to be on the structure of the number sentence (i.e. $A + B = C \Leftrightarrow A = C - B$ and $B = A - C$).

Part c) has the single term on the left, not on the right as in parts a) and b), and students should be familiar with such variability and not be thrown by such changes.

Parts d) and e) introduce extra terms (2 on both sides in part d) and then 3 on one side in part e)). Again, students need to appreciate that this does not change the overall additive structure.

It will be important for discussions to enable students to find more than one way of seeing the additive structure and, therefore, rearranging it.

In part e), students may see the addends as $(3m - 2n)$ and r. It will be useful to ask the question, *'Is there any other way to write this?'* in order to show the additive relationship as the alternative: $(3m + r) - 2n = V$.

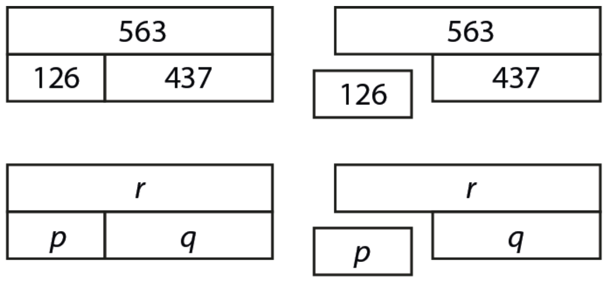
This is also showing the additive relationship A – B = C.

Such flexibility of thinking will support students in working on *Example 3*.

Example 2:

Re-express the equations in Example 1 as subtractions.

R Examining the answers to *Examples 1* and *2* in a bar model formation allows students to see the additive relationship and to manipulate it to reveal the inverse relationship. Parts a) and c) could be represented as:



The right-hand diagram in each case reveals the inverse additive relationship:

563 – 126 = 437 or 563 – 437 = 126

and

$r - p = q$ or $r - q = p$

Example 3:

Identify the additive relationship in the following expressions and rewrite them in as many different ways as you can.

a) $v = u + at$

b) $P = 2w + 2l$

c) $\cos^2\theta = 1 - \sin^2\theta$

d) $s = ut + \frac{1}{2}at^2$

e) $x + 3y - 2p^3 = 5x^2y$

V In *Example 3*, there is a mixture of equations and formulae of the form A + B = C and X – Y = Z. It is important that students see both types as an additive relationship, each of which can be written in three different ways.

D You could ask students to make up some of their own expressions and set them as a challenge for their partner. Alternatively, ask students what formulae they have used in other subjects (as well as in mathematics) and ask them to write these in different ways.

Weblinks

[1] NCETM primary mastery professional development materials
https://www.ncetm.org.uk/resources/50639

[2] NCETM primary assessment materials
https://www.ncetm.org.uk/resources/46689