Interpreting the Mathematics curriculum

Developing reasoning through algebra and geometry
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Introduction

This booklet for mathematics departments is based on materials developed by six working groups as part of QCA's algebra and geometry project. The project has focused on the place of algebra and geometry in developing concepts and skills in reasoning. The booklet brings out key themes from this work that are central to the intentions of the Mathematics national curriculum; these are illustrated by reference to ideas developed by the working groups and approaches they have taken in classrooms.

The booklet will help mathematics departments judge the extent to which their schemes of work reflect these intentions; in the Curriculum there are new emphases in both algebra and geometry and the approaches described in this booklet were developed to help bring out the implications of these emphases more clearly. Appendix A contains a checklist for mathematics departments to use in assessing their teaching approaches. The booklet will also provide some ideas for approaches to teaching in algebra and geometry that readers may wish to develop for use in their own classrooms; it does not, however, attempt to give complete lesson plans or a scheme of work.

The algebra and geometry project

QCA's algebra and geometry project began in 2000, shortly after the revision of the National Curriculum. Its remit was:

‘To develop and consider recommendations for strengthening the place of algebra and geometry in the mathematics national curriculum to raise pupils’ standards of attainment particularly at key stages 3 and 4.’

The initial work of the project was to review and commission research with an international perspective in order to see what could be learnt from the experience of other countries. This first phase resulted in two international curriculum comparisons, one in algebra and one in geometry, which looked at the curriculum documents for the UK side by side with those for other countries. These studies were published in 2001, along with a review of research into the teaching and learning of algebra. Also in 2001, an international seminar was held in Cambridge and the proceedings of this were published under the title *Reasoning, explanation and proof in school mathematics and their place in the intended curriculum*. Details of these publications can be found in Appendix B.

Reasoning, explanation and proof

The two central themes of the project are (i) to clarify the role that algebra and geometry have in developing mathematical reasoning, explanation and proof, and (ii) to propose teaching strategies that may be used to give these skills a higher profile in these areas of the curriculum.

Reasoning is an important strand within mathematics and it is also important as a wider ‘thinking
skill’ that can be developed through learning mathematics. Each section of the mathematics programmes of study begins with a subsection in which the skills of reasoning are specified alongside those of problem solving and communicating. In this context reasoning includes:

- recognising patterns and opportunities to generalise
- explaining why an answer must be correct
- constructing chains of deductions
- understanding the difference between a practical demonstration and a proof
- appreciating assumptions and constraints and knowing how varying these would affect the results.

The National Curriculum handbooks contain a list of cross-curricular ‘thinking skills’ grouped under five headings (see Appendix C), and mathematical reasoning (in a broad sense) contributes to all aspects of these.

Activities that require pupils to reason carefully are essential. But to demonstrate a complete grasp of a suitable activity requires also that pupils develop the language to discuss and explain their thinking and, finally, that they use this language to consider the truth of the conjectures they express.

The development groups

In order to explore approaches that might be helpful in developing reasoning through teaching algebra and geometry, QCA established six working groups to develop appropriate approaches and activities, and pilot these in classrooms. These development groups were regionally based and had an innovative structure inspired by the French government-sponsored *Instituts de Recherche sur l’Enseignement des Mathématiques* (Institutes for Research in Mathematical Education). Each group included teachers and university academics, with the university representation coming equally from those involved in mathematics education and mathematics itself. Some groups also included a Local Education Authority (LEA) mathematics adviser. We have set out details of the groups’ membership in Appendix D.

The role of the development groups was to use the national curriculum as a starting point to develop approaches and materials for use in schools. The group structure enabled a fruitful combination of research perspectives and classroom experience to be brought to bear and participants, from both schools and higher education, found it exciting and stimulating to work with others whose background and expertise were different.

One of the findings of the development groups was that, in order to foster the development of reasoning through the teaching of algebra and geometry, it is necessary to consider not just the curriculum content – what you teach – but also how you teach. A sense of ownership is important when trying to alter teaching styles or teacher behaviour, and curriculum development, undertaken collaboratively by schools and universities together, seemed to be a good model for promoting this feeling of collective purpose. We hope that schools, LEAs and their local university departments may consider using this approach as a model for future work in developing the school curriculum.

Each development group produced a report for QCA, and this booklet draws on their findings, giving examples of approaches and activities used.
Emerging themes from the project

Developing ways of embedding ideas of generality in all mathematics lessons

Generality is a fundamental concept in algebra. However, the power of algebra lies largely in the fact that its symbols can be manipulated as if they were numbers, and the generality implicit in them can be allowed to lie dormant until it is required. Many mathematical techniques rely on implied generalisation, but often pupils do not notice or appreciate the generality that lies behind them. Pupils are particularly likely to come to regard algebraic techniques as merely a set of behaviours to memorise and copy.

In order to encourage a feel for this implicit generality, it is necessary to draw pupils’ attention to it, to make it explicit, and to do so not just in ‘algebra lessons’ but whenever possible. One opportunity to do this arises through the exercises commonly used in mathematics to build an appreciation of structure and technique. The questions are usually designed to conform to a particular type and so a notion of generality is embedded in them. Making the generality explicit by discussing the design of the questions with pupils will help them to retain an awareness of it and may make them more likely to appreciate the generalisations implicit in other pieces of mathematics.

There are many similar examples where the opportunities to generalise, which might otherwise be overlooked, can be recognised and exploited.

Example – An opportunity to generalise

A teacher gave a year 6 class (levels 3 and 4) an example of four statements linking 3, 12 and 4:

\[
3 \times 4 = 12 \quad 4 \times 3 = 12 \quad 12 \div 3 = 4 \quad 12 \div 4 = 3
\]

She then asked them to give four statements connecting 5, 35 and 7.

This proved to be easy, so she then asked for four statements connecting 14, 168 and 12 (something that the pupils were unlikely to know as part of their tables). This required them to apply knowledge from one situation to another and starts to open up opportunities to generalise.

Results from the development groups suggest that there are benefits from getting started on generalising as early as possible, even before year 7. Learners can be very responsive to opportunities to generalise.

One development group experimented with strategies for getting learners to express generality through the use of questions and prompts such as the following:

- ‘What can you change and still it is... (the same type of object, the same type of question)?’
- ‘In what ways may that aspect change, and in what ways may it not change?’
‘Will it always work/happen?’
‘Find as many ways as you can.’
Opening up questions by switching answer and question.
Characterising all possible... (objects meeting certain constraints, questions with a given answer).

Example – ‘What can you change...?’
Year 9: top set
A teacher introduced the topic of compound interest with some numerical examples. In the examples, the teacher gave the initial amount invested, the percentage interest per annum and the number of years of investment. Using this information the final value of the investment was calculated.

The teacher then asked the pupils to generalise the methods to arrive at expressions such as \( a(1 + p /100)^y \) where \( a \) is the initial amount invested, \( p \) is the percentage interest per annum and \( y \) is the number of years. After discussing how \( a, p \) and \( y \) could change depending on the particular question, the teacher then asked “What else could change?”

This disconcerted the pupils at first, but gradually they started to engage with variations to the question such as ‘How many years to go from £120 to £160 at 5% per annum?’ or ‘What percentage rate is needed to enable growth from £120 to £160 over four years?’. The process of expanding the original problem and exploring the implications of doing this made the topic more interesting for the pupils and helped to deepen their understanding of the underlying concepts.

Example – ‘In what ways may that aspect change?’
Year 9: simplifying and factorising using perimeters
The lesson began with a mental starter asking what different quadrilaterals could be made with a perimeter of 24. A natural part of this was to discuss whether the sides had to have integer length or not and to clarify what was meant by ‘different’ in this context. This discussion was important because in the main part of the lesson pupils considered perimeters of \( 6x, 6x + 3, 8x + 4 \) and \( 15x + 3 \). They worked in pairs or small groups, finding both regular and irregular polygons with those given perimeters. They were thus required to focus on different expressions for the side lengths as denoting different polygons, and to ignore considerations such as the angles that might be in the shapes. Artificially restricting the meaning of ‘different’ was a key part of this activity. Discussing the full range of permissible variation at the initial stage made clear to pupils the arbitrary nature of this restriction.

Example – ‘Will it always work/happen?’
Many pupils believe that multiplying by ten involves ‘adding a nought’. Their understanding can be broadened by using a range of numbers to explore whether this always works, and then looking at when it does and doesn’t work. Would something else work better – is there a way of saying what it is in general that covers all cases?
Example – ‘Find as many ways as you can.’
A year 8 class worked on areas of compound shapes. Initially, they were given an L-shape

and were asked ‘In how many ways can the area be found?’ Pupils wanted numbers inserted on the edges, but the teacher kept the discussion at a general level.

Three ways were found quickly, and eventually a fourth way, using trapezia.

These then led into a discussion of what the different formula for each method would be like, and on into discussion of different formulae leading to the same numerical answer. The pupils appreciated the use of algebra in this context: their comment was “letters let us generalise”.

Example – Opening up questions by switching answer and question.
Year 9: lower set
The teacher set a task for pupils to build up their own equations. For example starting from $x = 4$, generating the following equations:

$$3x = 12$$
$$3x + 2 = 14$$
$$3x + 2 + 5x = 14 + 5x$$

and so

$$8x + 2 = 5x + 14$$

The methods used were then extended to bring in brackets and then division.

Pupils initially worked individually. They then swapped equations with a neighbour and tried to solve the other person’s equation. Their understanding of algebraic manipulation had improved as a result of generating their own equations and they were better able to solve equations than they had been previously.
The use of prompts such as those above can serve to draw out implicit generality for discussion and reflection. Used repeatedly, over a series of lessons they can become part of what pupils expect in their mathematics lessons. Once they are a normal part of classroom practice then pupils will be familiar with working towards generality and may spontaneously start to look for opportunities to generalise.

Example – Characterising all possible...

These are the first three questions from an exercise on solving quadratic equations by factorization:

\[ x^2 + 7x + 6 = 0 \]
\[ x^2 – 8x + 15 = 0 \]
\[ x^2 – 9x + 14 = 0 \]

What do these equations have in common?

Find some more equations like these (eg quadratic in this form, quadratic but in another form, quadratic with integer roots...).

Describe what these equations are like in general (a quadratic equation is of the form....., or a quadratic equation has integer roots if and only if....).
Using discussions to explore and extend limits of understanding

Pupils do not come to lessons as ‘blank slates’ but as active learners. They have their own intuitive, perhaps incomplete, understandings of the mathematics. The role of discussion is to recognise and make explicit these understandings so that they can then be modified and refined. ‘Cognitive conflict’ occurs when pupils realise that there are inconsistencies between their existing beliefs and observed events. Such cognitive conflict, when resolved through reflective discussion, leads to more permanent learning than teaching methods which discourage pupils from making ‘mistakes’.

The teacher’s role in discussion is to encourage the articulation of intuitive viewpoints and to challenge with alternative viewpoints. By fostering cognitive conflict, the teacher can help learners to re-evaluate their ideas. Discussion will help pupils to expose their beliefs and then to reconstruct and develop their own concepts – to learn from their mistakes.

Typical work in mathematics lessons involves large numbers of short questions, starting with simple problems and gradually moving towards more complex questions. However, pupils tend to solve simple problems by intuitive methods that may not generalise, and when the teacher insists that they use more generalisable methods, pupils may not understand why they should do so. Simpler tasks often do not motivate a need to learn; tasks that force pupils to recognise the limitations of their intuitive methods are more likely to develop an enhanced understanding.
One development group concentrated on using a smaller number of more substantial tasks as the basis for discussion of methods. Activities were designed around three generic activity types:

- ‘Always, sometimes, never true’
  Pupils are given a number of statements or solutions and have to decide about the validity of each. They justify their decisions with examples, counter-examples and explanations.

- ‘Multiple representations’
  Symbolic formulae, verbal expressions, tables of values and graphs form the basis for card-sorting activities. Different forms of the same relationship have to be shared, interpreted, compared and classified.

- ‘Creating and solving problems’
  One pupil does something (e.g., creates an equation, expands an expression) and the other then undoes this operation (solves the equation, factorises the expression). The doer assists the undoer when he or she becomes stuck. Thus they resolve difficulties collaboratively.

Example – Always, sometimes, never true
In this activity the pupils were invited to consider whether 12 algebraic statements were always, sometimes or never true, discussing their views with each other and then displaying their (agreed) reasoning as a poster.

The work of these pupils raises some interesting questions for subsequent discussion. The reasoning in the first example of the ‘Never’ column is faulty and seems to indicate a belief that $a - b$ and $b - a$ are never equal. The reasoning in the first column suggests that ‘How do you know that something is always true?’ would be worth discussing.
Example – Multiple representations

In one lesson, pupils were asked to match situations, expressions, tables and diagrams such as these:

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

\[
y = x + 6 \quad x = y - 6
\]

<table>
<thead>
<tr>
<th>(y)</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(y = x + 6)</td>
</tr>
</tbody>
</table>

There are six chairs for every table in a restaurant:

\[
x = \text{the number of chairs} \quad y = \text{the number of tables}
\]

\[
x = 6y \quad y = \frac{x}{6}
\]

Translating between verbal statements and algebraic expressions is found difficult by many pupils who consider letters to represent ‘objects’ rather than numerical variables (thus matching ‘each table \(y\) needs 6 chairs \(x\)’ with \(y = 6x\) rather than with \(x = 6y\)).

Mistakes such as this can be used to illustrate the value of substituting numbers in the equation to resolve any uncertainties.

This activity can lead to a variety of opportunities to generalise, including a consideration of the equivalence of different forms of an expression, for example the general addition and multiplication triads:

\[
z = x + y \quad x = z - y \quad y = z - x \quad \text{and} \quad z = xy \quad x = z/y \quad y = z/x.
\]

Example – Creating and solving problems

Below are two typical constructions made by pupils in a year 10 class. While pupil 1 shows that she can use the notation for all four operations correctly, she has made a mistake in the last step where she has divided the left hand side by three and multiplied the right hand side by three. Pupil 2 combines both division notations. Both pupils appear aware of the correct use of brackets.

One of the effects of giving pupils the experience of constructing equations is that they begin to view equations as human constructions rather than as abstract entities. These pupils could be overheard saying “The last thing they did was add 7, so I must take away 7 first”. They were beginning to consider how each equation was constructed before trying to reverse this process.
There is a deliberate intention in these activities to focus on *methods* rather than *answers*. Pupils work on fewer problems than they did in the past, but they often tackle a task using more than one method. Discussion is used to consider different approaches to a task.
Providing opportunities to develop chains of reasoning through geometric activities

Geometry is particularly suitable for developing skills in mathematical reasoning. Reasoning can build on, and be reinforced by, mental imagery and visualisation. Geometric statements are often less abstract than those in other branches of mathematics. Furthermore, geometric deduction often involves drawing conclusions from several statements at the same time. It thus supports important aspects of wider thinking skills.

Development groups adopted one of two approaches to encourage reasoning. In the first approach, activities were chosen to be ‘rich’ in the sense that they should contain a variety of geometric possibilities and offer both a challenge and the possibility of some element of surprise. Such activities are structured to encourage local deduction, where pupils can utilise any geometrical properties that they know to deduce or explain other facts or results. This approach is intended to provide the foundation on which successfully to build a more formal concept of definition and proof at a later stage.

Example – Rich activity, local deduction
Diagonals of a quadrilateral
Fold a piece of A4 paper in half vertically and horizontally.
Put a point on each half of each of the folds.
Join the 4 points to form a quadrilateral.

Investigate which quadrilaterals can be made and which ones can’t.
Discussion points:
- What shapes can be made?
- Are the diagonals always lines of symmetry?
- What happens if all the dots are the same distance from the central point?
- What happens if a pair of dots are the same distance from the central point?

Fold a second piece of paper in a different way and repeat the task to investigate which quadrilaterals can now be made and which can’t.

- With the second pair of diagonals, if a rectangle can be obtained why can’t a square be obtained?
- Why can a kite be made with the first pair of diagonals but not the second?
- Why are the diagonals in the first example sometimes lines of symmetry, but the diagonals in the second example never lines of symmetry?
The second approach is more direct and involves making explicit at each stage which properties can be used in a ‘proof’. Initially, a small number of simple results are assumed. These are used to prove further simple results, which can then be called upon in further deductions. The deductive character of mathematics is thus constantly reiterated and individual results are always placed in the context of the system as a whole.

Example – Emphasising the structure
One group used ‘Statement Cards’ on which were written geometrical results. A display was constructed which initially contained four statements – the starting assumptions. These were chosen to be a reasonably convenient starting point rather than as a minimal set of ‘axioms’.

**Angles on a straight line**
Angles on a straight line add up to 180°.

![Angles on a straight line](image)

(true by definition)

**Congruent triangles: side-angle-side**
If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.

![Congruent triangles: side-angle-side](image)

**Corresponding angles**
If a straight line cuts two parallel straight lines, corresponding angles are equal.

![Corresponding angles](image)

**Congruent triangles: side-side-side**
If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.

![Congruent triangles: side-side-side](image)

Further results were also written on statement cards, and as they were proved, their cards were added to the display.

**Alternate angles**
If a straight line cuts two parallel straight lines, alternate angles are equal.

![Alternate angles](image)

**Exterior angle of a triangle**
An exterior angle of a triangle is equal to the sum of the two interior opposite angles.

![Exterior angle of a triangle](image)
At every stage, only statements already on display could be used in a proof. A diagram was also on display showing one possible network of logical connections between the statements.
Encouraging the generalisation of particular results and lines of reasoning

There is a progression in learning geometry from the particular to the general, from describing to defining and from explaining to proving.

<table>
<thead>
<tr>
<th>Particular examples</th>
<th>Particular results</th>
<th>Generalisations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognising</td>
<td>Describing</td>
<td>Defining</td>
</tr>
<tr>
<td>Observing</td>
<td>Explaining</td>
<td>Proving</td>
</tr>
</tbody>
</table>

Pupils sometimes fail to see that geometric statements are not just about the particular triangle, rectangle or diagram that is on the page. Once over this hurdle, misconceptions can still arise through ‘false generalisations’ where experience of a large number of similar examples can lead learners to believe that incidental features of diagrams are in fact properties of the shape. The perception that a square must have sides that are horizontal and vertical, so that a square drawn at 45° is seen as a ‘diamond’, is a familiar phenomenon and there are many other examples.
Misconceptions such as these can be discouraged, and understanding deepened, by exploring the extent to which what can be said about *this square* is also true for *squares in general*, *quadrilaterals in general* etc. Dynamic geometry software is particularly helpful for seeing which properties of a shape are retained as it is changed by dragging. An appreciation of this *invariance in the face of change* will help pupils gradually to see particular results as possible special cases of more general properties, or to put it another way to see the opportunities for generalising in geometry.

One development group used a strategy that begins with a familiar method for finding unknown angles. Having done this several times with different numbers, pupils are encouraged to generalise what they have done. The pupils then have a general result, which they have proved and which can be used in the future.

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**Examples – Misconceptions**

<table>
<thead>
<tr>
<th>A square</th>
<th>Not a square</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Square" /></td>
<td><img src="image2.png" alt="Not a Square" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parallel lines</th>
<th>Not parallel lines</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image3.png" alt="Parallel Lines" /></td>
<td><img src="image4.png" alt="Not Parallel Lines" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A hexagon</th>
<th>Not a hexagon</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image5.png" alt="Hexagon" /></td>
<td><img src="image6.png" alt="Not a Hexagon" /></td>
</tr>
</tbody>
</table>

**Example – Is it still true?**

<table>
<thead>
<tr>
<th>Diagonals perpendicular</th>
<th>Diagonals perpendicular?</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image7.png" alt="Diagonals perpendicular" /></td>
<td><img src="image8.png" alt="Diagonals perpendicular?" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Diagonals perpendicular?</th>
<th>Diagonals perpendicular?</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image9.png" alt="Diagonals perpendicular?" /></td>
<td><img src="image10.png" alt="Diagonals perpendicular?" /></td>
</tr>
</tbody>
</table>
Example – From the particular to the general
Given only these two facts:

*Angles in a triangle total 180°*
*Angles on a straight line total 180°.*

Work out the value of $x$ in this diagram, giving reasons:

Now do the same for this diagram. What do you notice?

Use the same method in this diagram to prove that $x = y + z$
Another group used a similar strategy, but within a context of ‘nine-point circles’ in which the initial assumptions are less well defined.

Example – Nine point circles
This sequence of activities uses wooden boards with nine pins or nails equally spaced around the circumference of a circle. The pins can be joined with elastic bands to make shapes.

An initial activity is to ask “How many different triangles can you make?”. One effect of this counting activity is to indicate to the pupils that they are to work with ideas of congruence and to forget literal, exact measurement. There is much scope for reasoning in discussing the question “How do you know that there are only seven different triangles?”.

The next main activity is to ask the pupils to find all the angles in the seven different triangles they have found. The choice of nine pins makes the arithmetic straightforward as all the angles are multiples of 20°. A starting hint is to construct three isosceles triangles with bands to the centre pin.

The reasoning used can then be extended to work out the angles in any quadrilateral made by joining four of the circumference pins. A further extension is to consider the angles of quadrilaterals made by overlapping two triangles:

Finally the reasoning used can be extended and generalised to similar problems with different numbers of points around the circle.
Encouraging and facilitating the translation of oral reasoning into written and symbolic forms

The first attempts to explain or justify a finding are usually spoken and the concreteness of geometrical objects and the scope to bring visualisation and measurement to bear on them makes geometry a particularly fruitful area in which to develop informal reasoning through discussion and explanation. However, informal, oral reasoning must gradually give way to increasingly formal written deductions.

Oral explanations are often accompanied by ‘signalling’ (hand gestures or pointing with a pen) and this can be thought of as corresponding to labelling features of diagrams in written explanations. But despite this correspondence we need to acknowledge that the two modes of communication (oral and written) have significantly different characteristics, and that the transition between them needs careful management.

Initially pupils frequently have little idea of what a mathematical explanation should look like. Unsurprisingly therefore, they may be reluctant to move from informal, oral explanations to a more formal written mode. So initially, as well as support and guidance on the form that a geometrical explanation should take, pupils must be given explicit reasons to produce written explanations.

Here are some strategies used by development groups to help the move from spoken to written justifications:

- pupils work in pairs, honing their arguments in conversation before writing them down
- pupils prepare and present their reasoning to the class using OHP slides
- pupils discuss strengths and weaknesses of anonymous extracts from pupils’ written reasoning
- pupils produce a ‘storyboard’ showing the steps of their solution
- pupils draw a flowchart to show how a solution was arrived at
- pupils produce a proof based on a writing frame.

These bullet points represent a hierarchy of progression. The first two strategies encourage some written explanation supported by speaking and gesturing. The storyboard allows an informal style of reasoning that is wholly written, and a flowchart may be designed to impose more structure and require some notation. A writing frame may be used to indicate what a formal proof should look like and to help with its production.
Example – Storyboard
This year 8 pupil’s work relates to the nine point circle activity described above. She is describing how to find the angles of a quadrilateral made by joining points on the circumference.

Some steps are not explained clearly, either because they lack sufficient detail or due to use of incorrect terminology. However the written reasoning is beginning to take shape, which is the purpose of this kind of activity. Having made an attempt at a written explanation, this pupil should be given opportunities to consider where the clarity or detail of it needs to be improved. This might profitably be done with a partner.

Example – Writing frame
The diagram shows three intersecting straight lines. For each ‘proof statement’ below, state your reasoning.

(a) If \( p = q \), prove that \( a = b \)

\[
p = \ldots \quad (\text{vertically opposite angles})
\]

\[
q = \ldots \quad (\ldots \ldots \ldots \ldots \ldots)
\]

\[
p = q \quad (\text{given in the question})
\]

So \( \ldots = \ldots \) as required

(b) If \( b = c \), prove that \( s = t \)

(c) If \( a = c \), prove that \( q + t = 180^\circ \)
Finally, one group further emphasized the significance and structure of written proofs by producing exemplar ‘proof sheets’ giving both informal and formal proofs of a given result.

Example – Proof sheet

Exterior angle of a triangle

An exterior angle of a triangle is equal to the sum of the two interior opposite angles.

GOT | WANT
---|---
\(\angle ACD = \angle CAB + \angle CBA\)

Proof

\[\angle ACD + \angle ACB = 180^\circ \text{ (angles on a straight line).}\]
\[\angle ACD = 180^\circ - \angle ACB \quad (1)\]
\[\angle CAB + \angle CBA + \angle ACB = 180^\circ \text{ (angle sum of triangle)}\]
\[\angle CAB + \angle CBA = 180^\circ - \angle ACB \quad (2)\]

From (1) and (2),

\[\angle ACD = \angle CAB + \angle CBA\]

Commentary

\[z = x + y, \text{ because each is } 180^\circ \text{ minus the slice . . . }\]
Exploring situations in geometry and algebra and the links between them

The most obvious feature of dynamic geometry software is that geometrical objects can be moved and transformed. Shapes can be manipulated and in the process the limits of applicability of definitions and properties can be explored.

In effect, the sense of generality which we hope to develop in the minds of pupils can be made manifest on the screen.

Apparently identical dynamic geometrical shapes can have different properties when they are dragged, depending on the construction underlying them. This provides a powerful analogy to the role of definition in geometry. In the example below, three apparently identical shapes have their properties ‘defined’ by their underlying constructions. Dragging the points of each shape reveals that despite appearances they have different properties, and that one of them is ‘not really a square’ because its construction is not sufficient to give it the properties of a square.

Example – The underlying construction

Figure 1 below shows a dynamic geometry screen. The three figures all appear to be square, and partial confirmation is provided by the read-out of the side lengths.

Figure 1
Point A in each figure is now dragged in a similar way. Figure 2 shows the results. It appears that there are different constructions underlying the three shapes which gives them different properties when dragged, and that the construction for the green shape is not sufficient to make it a square.

To explore the properties of these shapes further the hidden construction elements are revealed (Figure 3). At the same time the read-out of the side lengths is removed, so that the emphasis moves from arguments based on measurement to more purely geometrical reasoning.

A possible task for pupils from this point on might be to reproduce the construction for the blue and orange shapes themselves (decisions have to be made about the order in which the elements of the constructions are produced) and having done that to ‘prove’ that the blue and orange shapes must be squares.
Dynamic geometry software can allow the user to switch seamlessly between a numerical analysis based on measurements and a more purely geometric line of reasoning based on congruence, and this is one of its strengths. However, an understanding of the construction used to obtain a figure is in most cases a necessary prerequisite before its properties can be reasoned about and therefore familiarity with, and practice in, constructing figures is needed at an earlier stage.

One development group focused on ways of exploiting the power of dynamic geometry to provide a link between algebra and geometry. Making connections between different mathematical concepts is important for developing understanding. In addition, current post-16 mathematics courses such as A-levels and Free Standing Mathematics Qualifications tend to integrate the application of algebraic and geometric concepts, so introducing these ideas earlier may ease the post-16 transition for some pupils.

One way of linking algebra and geometry is to exploit the capacity of dynamic geometry to provide novel ways of visualising algebraic relationships.

**Example – Dynamic number lines**

A dynamic number line is a construction using dynamic geometry which gives a way of visualising the relationship of a variable to various functions of it. In the diagram below, as the point representing the variable $n$ is dragged along the line the points representing $n - 2$, $n/2$, $n^2$, $2n$ and $2/n$ move accordingly.

![Dynamic number line diagram](https://example.com/dynamic-number-line.png)

From the starting position shown, as $n$ increases – moves to the right – all the other points do likewise with the exception of $2/n$ which decreases – moves to the left. Likewise, as $n$ decreases, so initially do all the other points except $2/n$. However, as $n$ passes through 0, the direction of movement of $n^2$ reverses; it seems to ‘bounce off’ the 0 position; in addition, $n^2$ ‘overtakes’ $n$ as they pass together through 1. Similarly, as $n/2$, $n$, and $2n$ pass through 0, their relative positions reverse; and these three variables move along the number line at different, but clearly related, ‘speeds’. By contrast, $n - 2$ never exchanges position with $n$; rather, both move at identical ‘speed’, and maintain a fixed ‘distance’.
A second approach to linking algebra and geometry is to use different approaches to tackle the same problem. Dynamic geometry allows diagrams to be analysed using arguments based on congruence, transformations or coordinate geometry – or all three.

**Example – The midpoint of a sliding ladder**

This example outlines the problem of the locus of the middle rung of a ladder as one end slides down a wall and the other along the ground. Figure 1 shows the results of a basic construction for the problem. Dragging G drives the sliding motion of the ladder; the path of the midpoint M is traced accordingly. It appears that the locus of M is a quarter-circle, centred at O and such a curve has been constructed so that its fit to the locus can be tested visually. This hypothesis can then be proved in a variety of ways.

Figure 2 shows the first stage in developing a classical analysis by constructing the point P and the midpoint M' of the line OP.

Two alternative approaches are shown below. In figure 3 a transformation argument is used based on constructing the point W' which is the reflection of W in a vertical mirror line through M. Finally, Figure 4 has a coordinate grid superimposed on the system in order to explore the x, y equation of the locus.
Finally, dynamic geometry software packages are particularly powerful in their ability to analyse graphs and the more recent versions include facilities to import visual images such as photographs or video and impose coordinate systems and function graphs on them.

**Example – Modelling a suspension bridge**

Bridges provide a valuable source of lines and curves for geometric and algebraic modelling. Figure 1 shows a photograph of the Clifton suspension bridge which has been imported as a fixed background. A curve is being fitted by pasting the parabola tool into the window, and then dragging focus and directrix so as to fit the parabola to the main cable of the bridge. Some preliminary lines have been drawn to guide positioning of the focus and directrix.

![Figure 1](image1.png)

Similarly, the constructed lines indicate one suitable position for coordinate axes, allowing a corresponding algebraic expression to be fitted to the main cable as shown in Figure 2. While this task can be approached through trial and improvement, it is also possible to capitalise on algebraic analysis greatly to simplify the problem.

![Figure 2](image2.png)

In the approach shown, the half-span of the bridge has been taken as the unit measure, so that the tops of the bridge towers lie at positions (-1, 0) and (1, 0). A quadratic polynomial passing through these points must take the form \( c(x + 1)(x - 1) \). Reading off the minimum value of the polynomial at around (0, -0.22) indicates the value of \( c \) required, allowing the expression to be deduced directly without any trialling.
Appendices

Appendix A: checklist of teaching approaches

This checklist may be used individually or collectively in a variety of ways, for example:

- Use numbers to indicate for each approach whether its use is up to the individual teacher or built in to the department’s schemes of work
- Use a grading system to indicate the frequency of use of each approach
- Use a simple yes/no indication.

Do you plan lessons where pupils...

... are encouraged to generalise, not just in algebra?
  - for example to consider whether a property of squares also applies to rectangles

... identify what sets of problems or questions have in common?
  - for example the equations in a typical textbook exercise

... discuss mathematics and work in groups?
  - for example, agree how to match up a set of equations with a set of graphs

... work through the same problem in more than one way – and then evaluate the methods?
  - for example, find the area of a composite shape by counting squares and then by splitting it into simpler shapes

... develop chains of reasoning in geometry, making explicit the facts used?
  - for example, work out the size of an angle giving reasons for each step

... reason from a given set of facts in geometry?
  - for example, given that the angles of a triangle sum to 180° work out the angle sum of a pentagon

... generalise from particular examples and then prove the generalisation?
  - for example, state and prove that the sum of the first $n$ odd numbers is $n^2$

... explore the constraints on a generalisation – when it is true/not true – in both algebra and geometry?
  - for example, investigate for what range of values $n’ > n$

... use a range of formats and media to present their reasoning?
  - for example verbal report to class, storyboard, OHT

... are encouraged to move from oral reasoning to written explanations?
  - for example, by using writing frames for proofs

... use dynamic geometry software in geometry and algebra?
  - for example, to investigate which trapezia have perpendicular diagonals

... tackle problems that link algebra and geometry?
  - for example, to fit an equation to a locus
Appendix B – QCA’s algebra and geometry publications

A comparative study of geometry curricula
Celia Hoyles, Derek Foxman and Dietmar Küchemann
This report looks at similarities and differences in the intended geometry curriculum and its assessment for pupils up to age 16 in England and other countries throughout the world.
Order ref: QCA/02/915
Price: £20

A comparative study of algebra curricula
Rosamund Sutherland
This report looks at how the nature of the schooling system (comprehensive or not) seems to influence the way in which algebra is introduced. The differences and similarities in aspects of algebra teaching, such as introduction of graphs, emphasis on algebra as a study of systems of equations and introduction of algebra within the context of problem situations, are also examined.
Order ref: QCA/02/914
Price: £20

Key aspects of teaching algebra in schools
John Mason and Rosamund Sutherland
This volume surveys important international research on the teaching and learning of algebra in schools. The work is collected into five ‘assemblages’, reflecting different schools of thought from around the world.
Order ref: QCA/02/913
Price: £20

Reasoning, explanation and proof in school mathematics and their place in the intended curriculum
Jack Abramsky(editor)
In October 2001, the Qualifications and Curriculum Authority held a special invitation seminar in Cambridge to explore the role of reasoning, explanation and proof in the intended school mathematics curriculum. Leading experts from around the world presented papers and discussed the issues that emerged.
This publication presents all the seminar papers and reproduces the discussions that took place. The seminar proceedings make an important contribution to the debate on the importance of mathematics within the school curriculum and the significance of the skills of reasoning and logic that mathematics helps to develop.
Order ref: QCA/02/916
Price: £30
Appendix C: thinking skills

The following extract is from the National Curriculum for England secondary teachers’ handbook (pages 23-24) and the National Curriculum for England primary teachers’ handbook (page 22).

Thinking skills

By using thinking skills pupils can focus on ‘knowing how’ as well as ‘knowing what’ – learning how to learn. The following thinking skills complement the key skills and are embedded in the National Curriculum.

Information-processing skills

These enable pupils to locate and collect relevant information, to sort, classify, sequence, compare and contrast, and to analyse part/whole relationships.

Reasoning skills

These enable pupils to give reasons for opinions and actions, to draw inferences and make deductions, to use precise language to explain what they think, and to make judgements and decisions informed by reasons or evidence.

Enquiry skills

These enable pupils to ask relevant questions, to pose and define problems, to plan what to do and how to research, to predict outcomes and anticipate consequences, and to test conclusions and improve ideas.

Creative thinking skills

These enable pupils to generate and extend ideas, to suggest hypotheses, to apply imagination, and to look for alternative innovative outcomes.

Evaluation skills

These enable pupils to evaluate information, to judge the value of what they read, hear and do, to develop criteria for judging the value of their own and others’ work or ideas, and to have confidence in their judgements.
Appendix D: algebra and geometry development groups

Professor John Mason, group leader, Centre for Mathematics Education, Mathematics and Computing Faculty, Open University
Nicole Adams, mathematics teacher, Arnold Hill Comprehensive School, Notts.
Dr Johnston Anderson, senior lecturer, School of Mathematical Sciences, University of Nottingham
Martin Brown, mathematics teacher, Thomas Telford School, Telford
Josephine Burgess, mathematics teacher, Burgoyn Middle School, Beds.
Jackie Fairchild, mathematics teacher, Wheatley Park School, Oxon.
Sheila Hirst, mathematics teacher, Magdalen College School, Northants.
Andrew Rogers, mathematics teacher, King Edward VI Camp Hill Boy's School, Birmingham
Dr Chris Sangwin, lecturer, School of Mathematics and Statistics, University of Birmingham
Paul Smith, mathematics teacher, Chilwell Comprehensive School, Notts.

Geoff Faux, group leader, Independent Consultant
Professor Amanda Chetwynd, Department of Mathematics and Statistics, University of Lancaster
Julienne Ellison, mathematics teacher, St Bede's Catholic Comprehensive, Peterlee, Durham
Shirley Fall, mathematics inspector, Durham LEA
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Paul Hampshire, mathematics teacher, Millom School, Cumbria
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Mike Ollerton, mathematics lecturer, St Martin's College, Lancaster
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John Stanton, mathematics teacher, Millom School, Cumbria

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Keith Jones, lecturer, Department of Education, University of Southampton
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Paul Morris, mathematics teacher, Brune Park Community School, Gosport, Hants.
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Nick Lord, mathematics teacher, Tonbridge School, Kent
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Jill Barton, mathematics teacher, Cams Hill School, Fareham, Hants.
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Seema Patel, mathematics teacher, Friesland School, Derbyshire
Margaret Swan, research support, MARS, School of Education, University of Nottingham
Karen Tawn, mathematics teacher, Friesland School, Derbyshire
Stephen Wren, mathematics teacher, Friesland School, Derbyshire
# Curriculum and Standards

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<tr>
<td>Descriptions</td>
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<td>Cross ref</td>
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**Price and order ref** £6 QCA/04/1289