National Centre
for Excellence in the
Teaching of Mathematics

Mathematics Matters

Deriving practices from what constitutes effective learning of mathematics

A year-long review of current attitudes, practices and influences across today's mathematics education.

This is the first progress report. It includes a first-draft copy of the discussion paper assembled by leading researchers, trainers and teachers at the initial conference in London on May 23rd 2007

Contents

Appendices – papers that were used to prompt the initial conference

The Rationale

The mathematics community celebrates two significant anniversaries in 2007.

It is 20 years since the publication of 'Better Mathematics' (HMSO 1987) and 25 years since the publication of 'Mathematics Counts' (HMSO 1982).

Each of these documents articulated views on what constituted effective learning of mathematics – informed by accumulated research findings and interpreted through the prevailing culture and values.

In 'Better Mathematics' we find the memorable Statement 4:

"Mathematics is effectively learned only by experimenting, questioning, reflecting, discovering, inventing and discussing. Thus, for children, mathematics should be a kind of learning which requires a minimum of factual knowledge and a great deal of experience in dealing with situations using particular kinds of thinking skills."

And in 'Mathematics Counts' we find the eponymous Cockcroft Paragraphs 242 & 243:

"242 We wish now to discuss the implications of the previous sections for work in the classroom. We are aware that there are some teachers who would wish us to indicate a definitive style of teaching mathematics, but we do not believe that this is either desirable or possible.
Approaches to the teaching of a particular piece of mathematics need to be related to the topic itself and to the abilities of both teachers and pupils. Because of the differences of personality and circumstance, methods which may be extremely successful with one teacher and one group of pupils will not necessarily be suitable for use by another teacher or with a different group of pupils. Nevertheless, we believe that there are certain elements which need to be present in successful mathematics teaching to pupils of all ages."

"243 Mathematics teaching at all levels should include opportunities for

- exposition by the teacher;
- discussion between teacher and pupils and between pupils themselves;
- appropriate practical work;
- consolidation and practice of fundamental skills and routines;
- problem solving, including the application of mathematics to everyday situations;
- investigational work.

In setting out this list we are aware that we are not saying anything which has not already been said many times over many years. The list which we have given has appeared, by implication if not explicitly, in official reports, DES publications, HMI discussion papers and the journals and publications of the professional mathematics associations. Yet we are aware that although there are some classrooms in which the teaching includes, as a matter of course, all the elements we have listed, there are still many in which the mathematics teaching does not include even a majority of these elements."

The National Centre for Excellence in the Teaching of Mathematics (NCETM) has framed the key question, "what constitutes the effective learning of mathematics?" on the understanding that the answers to this question are **fundamental**. Methods of teaching, the design of the curriculum, the use of assessment (both formative and summative), the organisation of learning groups, the selection and use of supporting resources and materials, the initial and continued professional development of teachers are all considerations that are **consequential**.

It is quite proper that the question is asked by each generation and the answers refreshed in the light of experience and latest research findings. Equally, the answers need to be interpreted through the prevailing culture and values.

Thus, the NCETM, with its clear brief to provide coordination and leadership on all aspects of professional development for teachers of mathematics, takes responsibility for stimulating and undertaking the debate for this generation.

The exercise

Stage 1

Initiating (May 23rd 2007 National Conference)

The purpose of the conference is to initiate the national debate by:

- identifying, confirming and agreeing core values and principles that underpin the effective learning of mathematics;
- illustrating, through examples, how practice may reflect and interpret these core values and principles;
- exploring the factors that inhibit or modify the implementation of these values and principles.

The outcome of the conference is a paper, structured under the three sections above, that scopes and prompts the next step in the process.

Stage 2

Inferring (September 2007 – February 2008)

A series of five regional colloquia running between September 2007 and February 2008 hosted by the NCETM and chaired by Malcolm Swan.

The purposes of the colloquia are to reflect on the issues raised in the initial conference paper and articulate implications for practice.

The outcome of the colloquia is a compilation that forms the basis for national strategy and development.

Stage 3

Engaging (April 2008 – June 2008)

The published compilation becomes the basis of a series of nationwide conferences and workshops that inform and influence the work of the Centre and the wider community of mathematics educators. This, in turn sets the agenda for the continuing professional development of teachers that will be promoted and supported by the Centre.

Throughout all stages of the process, the NCETM will be hosting the debate on its portal: www.ncetm.org.uk

Programme for the initial conference

1030 Refreshments and registration

1100 Introductory remarks
Background Purpose and scope

1120 Principles and values that underpin the learning of mathematics.
Introduction and group activity

1245 Lunch

1345 Examples of practices that reflect and interpret the principles and values
Introduction and group activity

1520 Factors that inhibit or modify practice
Group Activity

1620 Concluding remarks

1630 Refreshments and depart

Participants

The programme was introduced by Jane Imrie and chaired by Malcolm Swan

61 delegates from all phases of mathematics education attended the initial conference in London.

59 delegates contributed to the writing of the summary report through their written responses during and immediately after the conference.

Conference people

Dr Malcolm Swan Lecturer Mathematics Education and Co-Director of the MARS/Shell Centre, Nottingham University.

Malcolm has been at the University of Nottingham since 1979 where he has taken a central role in numerous research and development projects. He conducts “design research” into the theory, development and evaluation of teaching situations in mathematics education. This includes the design of situations which foster reflection, discussion and metacognitive activity; the design of situations in which children construct mathematical concepts and develop problem solving strategies; and the design of formative and evaluative assessment. His recent work has included professional development, resources such as "Improving Learning in Mathematics" (for the DfES) and "Thinking through Mathematics" for the NRDC. He has recently published a book on his research: "Collaborative Learning in Mathematics: A Challenge to our Beliefs and Practices", published by NIACE/NRDC.

Jane Imrie Acting Director of the National Centre for Excellence in the Teaching of Mathematics (NCETM)

Jane Imrie is an Executive Director of the National Centre for Excellence in the Teaching of Mathematics (NCETM). Prior to joining the NCETM she was a National Subject Lead for Mathematics in the DfES Standards Unit, where she led the development of the post 16 mathematics teaching and learning framework ‘Improving learning in mathematics’. Jane taught and managed mathematics in schools and further education for 23 years and currently serves on the Mathematical Association Council and as an observer to the NANAMIC committee. She has acted as a consultant in post-16 mathematics for organisations such as DfES, LSDA, QCA, and others.

How the initial conference paper was compiled

In order to report the outcomes from the first discussion, “Principles and values that underpin the learning of mathematics”, we used the design of the forms presented at the conference as a template on which we recorded each and every delegate’s response. This composite response sheet allowed scanning, searching and classifying so that common themes as well as ‘outlying’ or ‘exceptional’ responses’ could be identified.

We were mindful of our commitment to construct a synthesis where possible rather than to seek consensus.

Synthesis (n):	The putting together and building up of conceptions or propositions or facts into a connected whole or theory; A complex whole made up of a number of parts united. (OED)
Consensus (n)	Agreement; Majority view. (OED)

In this regard, our task was to judge whether different responses were different perspectives of a common idea or descriptions of different ideas. To illustrate this point: a group of five people may record their different observations of a set of shapes as, variously, a square, a rectangle, a hexagon, a pentagon and a circle. A synthetic report would notice that the first four observers may be looking at the same shape, likely to be a cube, from different viewpoints, whilst the fifth observer is looking at something else. A consensus report would suggest that the majority of observers were looking at rectilinear shapes.

Our approach to reporting the outcomes from the second discussion, “Examples of practices that reflect and interpret the principles and values”, was different. Having identified the purposes and principles in the report of the first discussion, we searched for and reported on tasks that illustrated those purposes and principles. Given the high degree of agreement with the principles in the tabled papers we used these as the criteria for sorting the tasks offered by delegates.

Our approach to reporting the outcomes from the third discussion, “Factors that inhibit or modify practice”, was similar to that of the first discussion.

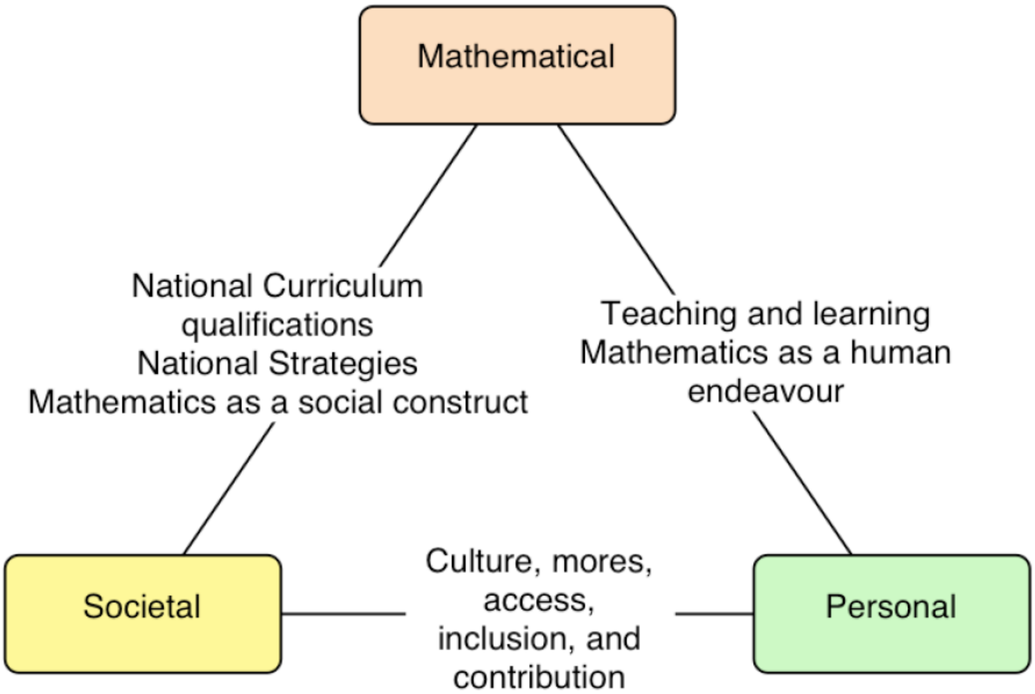
The three reports make up the initial conference paper. They are presented here **in first draft form**. Through further discussion with the initial conference delegates and feedback through open discussion on the portal, this draft paper will be refined to the point at which it is ready to use as the basis for the second stage of the exercise in working with the mathematics education community to pursue the answers to the key question: “What constitutes the effective learning of mathematics?”

Report on the first discussion

1. Principles and values that underpin the learning of mathematics

1.1 What values and purposes do you think should underpin mathematics education?

A synthesis of delegates' responses suggests a triad of related foci, namely: mathematical, personal and societal. The diagram below shows these three foci or areas for consideration and illustrates some of the factors that may reflect or inform the relationship between each.



In considering the values and purposes that underpin mathematics education, delegates suggested a relationship across all three of these elements. For example:

> "Equipping students to sustain the planet through their humanity" (Professor Afzal Ahmed, The Mathematics Centre, University of Chichester);
>
> "An appreciation of the power of mathematics in society" (Non-attributable);
>
> "Interpreting the world around us" (Ros Hyde, University of Southampton);
>
> "Explaining and analysing real world events (Ruth Tanner, Lodge Park Technology College);
>
> "The demystification of mathematics through an awareness of it as a cultural artefact and an understanding of how mathematics is created" (David Wright, Newcastle University)
>
> "Empowering thinking – showing that issues in life, whether scientific, sociological, psychological etc can be made sense of" (Dr. Els De Geest, University of Oxford)

The relationship between mathematical and societal, and mathematical and personal have been recently articulated by Adrian Smith and these were echoed and agreed in the discussion.

> Mathematics is of central importance to modern society. It provides the language and analytical tools underpinning much of our scientific and industrial research and development. Mathematical concepts, models and techniques are also key to many vital areas of the knowledge economy, including the finance and ICT industries. Mathematics is crucially important too, for the employment opportunities and achievements of individual citizens.
> (Smith, 2004 foreword, page v)

Delegates drew attention to the relationship between pairs of these elements. For example:

> “Connecting with life and work” and “linked to industry, economy, citizenry” (Pat Drake, University of Sussex);
>
> “Links to money management and occupational routes” (Carolyn Brooks, Anglia Ruskin University);
>
> “Becoming aware of and developing social skills for effective collaboration with and appreciation of the different perspectives of others” (Professor John Mason, The Open University);
>
> “Participation in and awareness of historical-cultural-social origins and role of mathematics in modern society” (Professor John Mason, The Open University);
>
> “To help society function”; “To share society’s history and use of mathematics” (Rose Griffiths, University of Leicester School of Education);
>
> “Mathematics for critical citizenship” (Non-attributable);
>
> Supporting active participation in an increasingly technological society” (Professor Dave Pratt, institute of Education, University of London);
>
> “Enable groups and societies to function effectively and creatively” (Peter Hough, NCETM);
>
> “Enculturation” (Jenny Golding, Mathematical Association);
>
> “To participate in life both at work and at home” (Dr Jenni Back, Middlesex University);
>
> “Enabling the learner to function effectively in society” (Joy Garvey, Royal Borough of Kingston on Themes).

all point to a purpose of mathematics education that links the personal to the societal in both directions.

A purpose that links the personal to the mathematical was frequently expressed. For example:

> "Learners seeing themselves as mathematicians" (Steve Feller, Edge Hill University);
>
> "Pupils appreciating the beauty of mathematics" (Sue Cronin, Liverpool Hope University);
>
> "Achieving mental empowerment through cycles of abstraction and experience" (Dr Ann Watson, University of Oxford);
>
> "To enable every student to make the most of their mathematical ability and have the confidence to use it" (Sid Tyrrell, Coventry University);
>
> "Opportunities to develop their thinking and to enjoy mathematics for its own sake" (Dr Howard Tanner, Swansea School of Education)
>
> "Being aware of the nature of mathematics" (Bernard Murphy, MEI);
>
> "Showing mathematics as a human activity" (Professor Afzal Ahmed, The Mathematics Centre, The University of Chichester) (Jones) (Alison Clark-Wilson, University of Chichester);
>
> "Mathematics learning contributing to spiritual, moral and cultural development"" (Kath Cross, Retired HMI) (Julia Croft, University of Bedfordshire);
>
> "Mathematics as a potential to change own and others' world view" (Non-Attributable);
>
> "Confidence in strategies to approach and solve problems" (Jenny Golding, Mathematical Association) (Bernard Murphy, MEI) (Nicholson);
>
> "I experience mathematics as the 'mother of all languages', a kind of grammatology of thinking. Therefore mathematics is very useful to get to know" (Dr. Els De Geest, University of Oxford)

Behind purposes that related to pairs or triples of elements were those that related specifically to the elements themselves. For example, in relation to the personal, delegates reported:

> "Empowering individuals to help them control their own lives" (Rose Griffiths, University of Leicester School of Education);
>
> "To develop confidence, enjoyment, creativity and enquiry" (Holly Isherwood, The Royal Institution);
>
> "To develop logical thinking" (Dr Matt Homer, School of Education, University of Leeds);
>
> "To develop critical thinking, reasoning and justification" (Dr Julie-Ann Edwards, University of Southampton);
>
> "To enable information to be converted into knowledge and insight" (Peter Lacey, ECARDA);
>
> "Strategies and thinking skills" (Michael Ling, The Royal Statistical Society);
>
> "To think (for themselves, i.e. independently) logically, systematically and analytically" (Pete Griffin, NCETM);

"Acting independently and effectively" (Dr Jenni Back, Middlesex University);

"Offering autonomy" (Bob Ansell, University of Northampton) (Tanner);

"Developing personal powers of sense-making" (Professor John Mason, The Open University);

"To engender an enjoyment and desire to continue learning" (Joy Garvey, Royal Borough of Kingston on Thames);

"A fostering of self-awareness of the learner as a learner" (David Wright, Newcastle University)

Some comments focused specifically on the element of the mathematical. For example:

"Mathematics as a set of tools to understand and change things" (Kath Cross, Retired HMi);

"Tools with which to think, communicate, analyse, argue and solve problems" (John Rickwood, Burlington Junior School);

"Mathematics as about making meanings" (Pat Drake, University of Sussex);

"The big ideas in mathematics and its history" (Peter Griffin, NCETM) (Professor Geoff Wake, University of Manchester);

"Mathematics to organise and sort information, to solve problems and explore the world" (Ros Hyde, University of Southampton);

"A way of viewing and describing the world" (Peter Lacey, ECARDA);

"Concepts to understand and strategies to be applied" (Dr David Martin, NANAMIC);

"Fluency and problem solving are core aims" (Nicholson);

"A combination of fluency with and application of skills" (Taylor);

"Variance and invariance are central things to observe at every level of mathematical work" (Dr Anne Watson, University of Oxford).

1.2 What principles for teaching and learning do you think should underpin mathematics education?

This part of the discussion homes in on the line on the triangle that joins the element of personal with that of mathematical and considers the purposes and principles that underpin teaching and learning.

There was widespread agreement with the set of values and principles proposed as the basis for this discussion, namely:

- **Fluency in recalling facts and performing skills**
- **Interpretations for concepts and representations**
- **Strategies for investigation and problem solving**
- **Awareness of the nature and values of the educational system**
- **Appreciation of the power of mathematics in society**

and

Teaching is more effective when it ...	
▪ **builds on the knowledge learners already have;**	This means developing formative assessment techniques and adapting our teaching to accommodate individual learning needs (Black & Wiliam, 1998).
▪ **exposes and discusses common misconceptions**	Learning activities should exposing current thinking, create 'tensions' by confronting learners with inconsistencies, and allow opportunities for resolution through discussion (Askew & Wiliam, 1995).
▪ **uses higher-order questions**	Questioning is more effective when it promotes explanation, application and synthesis rather than mere recall (Askew & Wiliam, 1995).
▪ **uses cooperative small group work**	Activities are more effective when they encourage critical, constructive discussion, rather than argumentation or uncritical acceptance (Mercer, 2000). Shared goals and group accountability are important (Askew & Wiliam, 1995).
▪ **encourages reasoning rather than 'answer getting'**	Often, students are more concerned with what they have 'done' than with what they have learned. It is better to aim for depth than for superficial 'coverage'.
▪ **uses rich, collaborative tasks**	The tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage 'what if' and 'what if not?' questions (Ahmed, 1987).
▪ **creates connections between topics**	Learners often find it difficult to generalise and transfer their learning to other topics and contexts. Related concepts (such as division, fraction and ratio) remain unconnected. Effective teachers build bridges between ideas (Askew et al., 1997).
▪ **uses technology in appropriate ways**	Computers and interactive whiteboards allow us to present concepts in visual dynamic and exciting ways that motivate learners.

Additions and/or refinements to this list of principles include:

- Teachers recognise themselves as learners too (Alison Clark-Wilson, University of Chichester) (Jones) (Thorpe) (David Wright, Newcastle University) (Pat Drake, University of Sussex);

- Time is set aside for learning (PL) (Non-Attributable) (Professor Dave Pratt, Institute of Education, University of London) (John Rickwood, Burlington Junior School) (Taylor) (Dr Anne Watson, University of Oxford);

- Enjoyment of mathematics is made apparent (Julia Croft, University of Bedfordshire) (Rose Griffiths, University of Leicester School of Education) (Non-Attributable) (Tanner) (David Wright, Newcastle University);

- Learner groupings should be fit for purpose rather than pre-determined (Dr Anne Watson, University of Oxford) (Jenny Golding, Mathematical Association);

- Develops awareness of the bigger picture of mathematics (Dr Matt Horner, School of Education, University of Leeds) (Ros Hyde, University of Southampton) (Joy Garvey, Royal Borough of Kingston on Thames) (Jones) (Peter Lacey, ECARDA) (Taylor) (Professor Geoff Wake, University of Manchester);

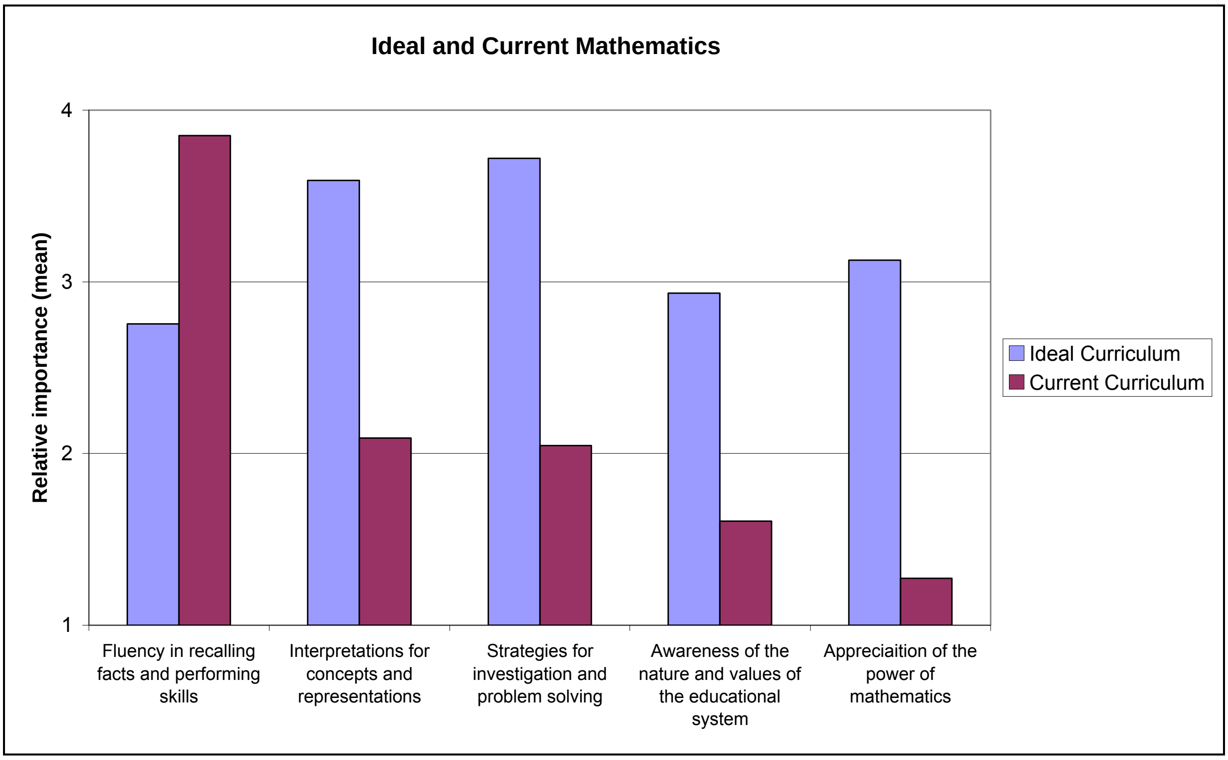
- Focuses on the exactness of the reasoning and language of mathematics (Bernard Murphy, MEI) (Sidney Tyrrell, Coventr University);

- What is learnt and how it is learnt is made explicit (Professor John Mason, The Open University) (David Wright, Newcastle University) (Professor Dave Pratt, Institute of Education, University of London) (Holly Isherwood, The Royal Institution)

The gap between ideal and implemented values

Delegates were asked to rate the relative importance of five values for teaching mathematics on a four point scale (1= least important, 4 = most important). This was done twice: firstly to show their vision for an ideal curriculum, and secondly their views on the values implied by the current curriculum in most schools and other settings. The mean rating for each value was calculated and the results are shown in the figure below.

This shows that delegates felt that fluency was overvalued in the current curriculum, while the remaining values were underemphasised. The most undervalued aspect was seen as being the appreciation of the power of mathematics in society.



(The views of 51 participants are represented here.)

Report on the second discussion

Examples of practices that reflect and interpret the purposes and values

Worthwhile tasks are likely to reflect more than one of the purposes set out in the conference paper and agreed by delegates. The examples below, which were offered by delegates, are selected not because they exclusively relate to a particular purpose but because a particular purpose can clearly be seen behind it.

Purpose 1: Fluency in recalling facts and performing skills

Example 1 1 (Dr Anne Watson)

Constructing examples introduces sense of structure, characteristics, properties of new objects. Also includes practice and fluency. Also 'big' idea of conjugation is being introduced. Also it builds on successful past tool use of the grid. Also ownership, mystery, intrigue. Also it feels do-able, but the difficulty adds to the motivation. Also, you don't have to tell students everything – or show them.

(a) What was the mathematical task(s)?

Having been just introduced to the idea that there are numbers of the form $a + \sqrt{b}$, and reminded of grid multiplication, find two such numbers which, when multiplied, you 'lose' the root part. On your own or in pairs. All rough working was kept and it is clear that most students shifted from some sort of 'testing' various integers' approach to a 'structural' approach, e.g.: trying $(2+\sqrt{3})(3+\sqrt{2})$ or $(2+\sqrt{2})(2+\sqrt{2})$ etc. Many started by using calculator, some abandoned this as they focused on structure.

(b) What learning culture was created? How was this achieved?

Students trusted the teacher because of having done similar tasks in the past. They were allowed to choose starting numbers and methods of working, except for grid multiplication. Lots of talk. Some 'gossip' method (distributed knowledge!). Teacher says 'why choose this, why choose that?' and reminds them of purpose. Some of the students carried on after class together or separately and the teacher was interested in their work!

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

Saw the written work. Some aspects of purpose achieved by all – $(a+\sqrt{b})(a-\sqrt{b})$ = integer found by some.

(e) Is this example available to see/read about?

It will be (special issue of 'Educational Studies in Mathematics 2008)

(f) Can you say why you chose this example? What criteria were in your mind?

To illustrate use of learner-generated examples to promote shifts between empirical and structural reasoning.

Example 1 2 (Dr Anne Watson)

Practice leads to fluency; multiplication facts are important; learning habits can be retrieved; all students are entitled to work in conducive environments; students can self-monitor progress.

(a) What was the mathematical task(s)?

Students work individually on a list of 120 multiplications (up to 12x12) in random order in silence. They do the same sheet every lesson for 15 mins. At the end the teacher reads out answers once, only once in a quiet voice, then asks for hands up re: how many right.
The next part of the lesson is a 'discuss in pairs' task and teacher rushes round getting public contributions from each pair and thanking them. Strangely, I cannot recall the topic! All pairs were asked before teacher commented on them.

(b) What learning culture was created? How was this achieved?

Teacher's aim is to re-teach Year 9 how achieving adolescents that they can sit still, concentrate, get better at maths, listen to quiet voices etc.. They aim to beat their previous personal best; they also get fluent with x answers. This task provides a 'buffer' from the other events in their lives.

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

Students 'came back' to full participation in lessons. Evidence was the next part of the lesson in which all pairs contributed answers, suggestions when asked and volunteered questions too. This was an on-task, engaged class which has previously been negative and disruptive.

(e) Is this example available to see/read about?

This was one of the lessons video-filmed for a research project which led to the publication of 'Deep Progress in Mathematics'. The lesson is briefly reported in that publication, but the above description is more detailed. The interpretations are my own

(f) Can you say why you chose this example? What criteria were in your mind?

To show that some lesson bits which sound horrid in themselves can be part of overall meaningful empowering strategy.

Example 1.3 (Jane Jones)

(a) What was the mathematical task(s)?

Continuous provision in a nursery class – focus on positional language. All sorts of activities but 2 stick in my mind:
Picture from a book 'Dinosaur's day out' with sunglasses, map, a beach etc replicated in classroom. Rich discussion about positioning of items referring to picture and artefacts. Later, outside, children 'painting' hollow shapes with water; seeking hidden dinosaurs; playing in pay house etc. Adults circulated with really high quality talk about position – 'have you painted inside/under that?' 'Where have you painted?; 'where have you looked for the dinosaur?; where did you find it?; who's inside the house/tunnel?

(b) What learning culture was created? How was this achieved?

Children engrossed in play. Well planned range of tasks outside and in. Adults moved to where children were and talked with them.

- Learning through play.
- Rich collaborative tasks.
- Emphasis on key ideas, embedded throughout learning activities.

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

Children talking confidently, understanding and using positional language, aided by adults' questioning (developing a fluency with positional language)

(e) Is this example available to see/read about?

No

(f) Can you say why you chose this example? What criteria were in your mind?

Reading the Ofsted report 'The Foundation Stage' which identified the teaching of positional language was weaker than most other areas. (Calculation was the weakest). This conference made me recall observing the lesson which impressed me so much.

Purpose 2: Interpretations for concepts and representations

Example 2.1 (Peter Lacey)

Extrapolating new knowledge from old is an engine for exploring and charting the territory of mathematics. In this exercise learners are expected to apply their understanding of fundamental concepts and principles (inverse, equivalence etc) in order to “map” and extend their knowledge.

(a) What was the mathematical task(s)?

What else do you know (*and why*) if you know that 5 + 3 = 8? The statement is written in the middle of a board visible to a whole class. Initially whole class responses. For example: 500 + 300 = 800. This is written on the board with an arrow to it from the original statement. A ‘mind-map’ is generated with answers to the *why* bit of the question determining the connections. Lead on to groups/pairs with eventual ‘composite’ mind map. Discussing the *why* proves productive.

(b) What learning culture was created? How was this achieved?

Learners in control. No perceived limit. Genuine sharing of personal understandings.

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

I think the activity actually altered the views of some of the learners on what mathematics actually is. Certainly challenged the ‘quantum’ view of mathematics as isolated facts.

(e) Is this example available to see/read about?

Reported in ‘Mathematics Teaching’ 187 June 2004 as a special conference insert, after being included in an ATM annual conference presentation

(f) Can you say why you chose this example? What criteria were in your mind?

Simple and accessible start – almost trivial; but deep in its engagement. Explicit and shared discussions on personal maps of understanding have a sense of deep learning.

Example 2.2 (Ruth Tanner)

(a) What was the mathematical task(s)?

Car racing game to demonstrate probability of getting different totals on 2 dice. (Done with special needs Y9).
Cars labelled 1-12 on board. Roll 2 dice. Car with the total moves forwards. Teacher chooses 6 and allows pupils to bet on others.

(b) What learning culture was created? How was this achieved?

Game and competition generated desire to understand probability. Game is played and teacher discusses what they have learned. Game is then played again. Offer comments while class discuss amongst themselves. The need to sort out thinking leads to generation of a sample space. Which number would you like to choose next time and why?

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

Buzz in the classroom and desire to bet on 7. Clarity of pupils final explanations.

(e) Is this example available to see/read about? (Give reference)

Mathematics Teaching 199, (Nov 06) pages 28-30

(f) Can you say why you chose this example? What criteria were in your mind?

Enjoyment and cognition conflict. Class thought 7 would ALWAYS win and the game demonstrated otherwise. The game also hinted as to why 7 was the best bet through practice.

Building a feel for the shape of a probability distribution before defining it arithmetically roots the concept in a practical experience (less likely to be forgotten) rather than in a theoretical algorithm (more likely to be forgotten). The latter can be developed from the former.

Example 2.3 (Dr Howard Tanner)

(a) What was the mathematical task(s)?

Horse racing game to demonstrate probability of getting different totals on 2 dice. (Done with special needs Y8). Horses labelled 1-12 down the side of the board. 6 “fences”, drawn as vertical lines down the board.

Teacher chooses Horse number 6 and invites pupils to choose other horses to bet on.

Teacher invites a pupil to roll 2 dice.

The horse with the total from the dice moves forward one fence.

(b) What learning culture was created? How was this achieved?

Atmosphere of playing a competitive game was generated. Game is played and as it progresses, pupils quickly realise that some horses are a bad bet!

After the game the teacher asks the class to think about what they have learned and to discuss with their friends which horse they should bet on next. After a few minutes the teacher takes suggestions and the game is played again.

The need to sort out thinking leads to the class beginning to generate a sample space. After a couple of iterations, the teacher begins to formalise the learning and the creation of a formal sample space.

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

Buzz in the classroom and desire to bet on 6. Clarity of pupils’ final explanations.

(e) Is this example available to see/read about? (Give reference)

Stahl (2002) *101 Red Hot Maths Starters,* London: Letts ISBN: 1-84085-696-3

(f) Can you say why you chose this example? What criteria were in your mind?

Enjoyment of the game led to cognitive conflict between initial naive ideas and experience of the game. Initially the class thought all values equally likely but the game demonstrated otherwise. Playing the game also hinted as to why 6 was the best bet through experience with examples.

The task helps pupils to build a feel for the shape of a probability distribution before defining it formally. The approach roots the concept in a practical experience rather than in a theoretical algorithm. Children are encouraged to build their own abstract mathematical concepts from a solid concrete base, rather than merely being presented with the ideas of other people.

The emphasis on pupils articulating their own thinking, discussion and prediction allowed the teacher access to the pupils’ ideas and misconceptions and provided opportunities for the scaffolding of learning, leading eventually to an understanding of formal, abstract mathematics.

Purpose 3: Strategies for investigation and problem solving

Example 3.1 (Dr Julie-Ann Edwards)

(a) What was the mathematical task(s)?

Set 10 of 10, group of 3 girls. Task: This is the final score in a hockey match, what are the possible half-time scores? Explain.
In the previous lesson, they have generated enough data to generalise and after considerable debate, arrive at the number of possible half-time scores being $(x+1)(y+1)$ where x and y are the final scores. When challenged to explain <u>why</u> this expression 'worked', they argue, challenge, justify amongst themselves for half an hour. Eventually, they are able to describe why they need to add one to each score. Value of this is in: (a) being able to comfortably challenge each other; (b) achieving 'hard' maths (for them); (c) being aware that mathematics can describe situations d) being able to interpret a situation mathematically.

(b) What learning culture was created? How was this achieved?

A background of feminist epistemology.
The 10 school 'classroom rules' were abandoned in favour of 3 mathematics classroom rules:
1. everyone does mathematics
2. everyone does mathematics in a a way that enables others to do mathematics
3. everyone shares their mathematics (collaborates)

Small groups based on friendship, ethos of collaboration, feminist principles of connectedness,. Understanding that a mathematics classroom is where thinking takes place.

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

Being able to communicate the outcomes and process both verbally and in writing. Pleasure expressed by the pupils about their achievement. (Despite being in set 10 of 10, this 'diet' of mathematics learning resulted in GCSE grades D,E,E for this group of girls)

(e) Is this example available to see/read about?

Graded Assessment in Mathematics (GAIM) Macmillan Education 1988 p68

(f) Can you say why you chose this example? What criteria were in your mind?

- Level of enjoyment for pupils;
- Level of challenge;
- Potential for awareness that mathematics can describe 'real' situations;
- Potential to develop confidence – 'doing quadratics' is for 'good' people.

Example 3.2 (Bernard Murphy)

(a) What was the mathematical task(s)?

Context – Medium-high ability Y8, two lessons after being introduced to, use and proved, Pythagoras' Theorem.
Given approximately 15 diagrams, each containing 2 sizes of circles, asked to find radius of larger, given radius of smaller = 1. Tackle in any order.
For example:

(b) What learning culture was created? How was this achieved?

Initially competitive. When individual problems solved, student demonstrated to class (if they wanted to listen). Occasional applause for some. I saw others applying techniques they'd seen others use. Debate over one or two – presenter had made assumptions which weren't true, for example, about symmetry Impressive creative thinking (eg adding lines to the diagram). Real sense of students appreciating each others ideas

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

All students were building on the work of others. Those who said it was 'hard' also said they followed others' work and had a sense of awe: "How did he think to do that?" Genuine joy in other students' ideas

(e) Is this example available to see/read about?

No - but the 15 diagrams are taken from a 'Mathematics in School' Article January 1999

(f) Can you say why you chose this example? What criteria were in your mind?

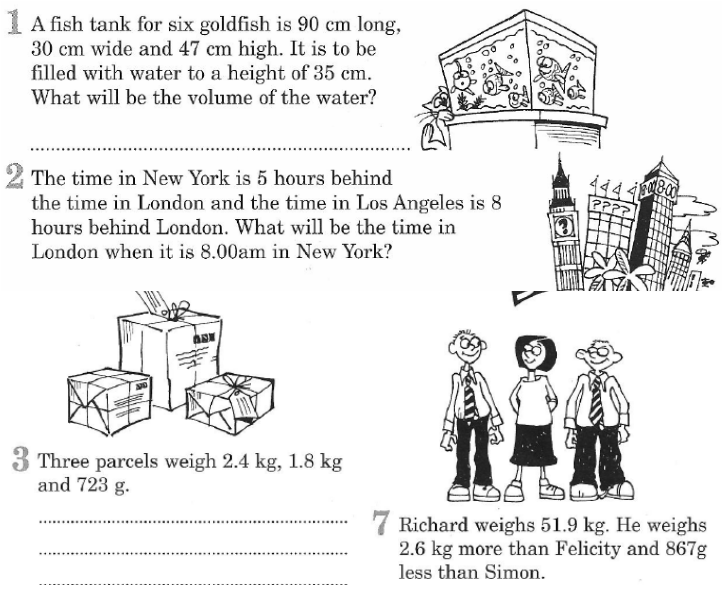
First and foremost the appreciation students showed for each others work; a sense of achievement and desire to prove; enjoyment of running with one idea (Pythagoras) and creative thinking.

Purpose 4: Awareness of the nature and values of the educational system

Example 4.1 (Professor Afzal Ahmed)

(a) What was the mathematical task(s)?

A set of given problems that include more information than needed in order to solve them. Pupils are asked to cross out the information not needed and to then solve the problems.
In the associated task pupils are given incomplete problems and asked what else they need in order to solve them; pupils propose the additional information and then solve the problems.


1 A fish tank for six goldfish is 90 cm long, 30 cm wide and 47 cm high. It is to be filled with water to a height of 35 cm. What will be the volume of the water?

2 The time in New York is 5 hours behind the time in London and the time in Los Angeles is 8 hours behind London. What will be the time in London when it is 8.00am in New York?

3 Three parcels weigh 2.4 kg, 1.8 kg and 723 g.

7 Richard weighs 51.9 kg. He weighs 2.6 kg more than Felicity and 867g less than Simon.

(b) What learning culture was created? How was this achieved?

The unusual nature of these problems generates interest and discussion. The nature of the problem prompts a more considered response to its solution

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

After working on these problems, questions presented in the form associated with end of Key Stage 3 tests became more straightforward and were more confidently tackled.

(e) Is this example available to see/read about?

Numeracy Activities Key Stage 3 Mathematics – Resource File Series: pages 10-14
Ahmed & Williams   Network Educational Press Ltd

(f) Can you say why you chose this example? What criteria were in your mind?

On the face of it these problems appear to be about developing fluency and preparing to take end-of-key-stage tests. (and in part they are) However they bring an element of surprise to pupils in their deficit or superfluity of information. The surprise acts as a motivator. Pupils develop skills in setting their own problems and become discerning respondents to questions set in end-of-key-stage tests.

Example 4.2 (Malcolm Swan)

This task was designed to help students become aware of what it feels like to teach something, and in so doing to review and reorganise their own understanding of a topic.

(a) What was the mathematical task(s)?

Two parallel year 7 classes were taught different topics by different teachers. One class was taught "Area" and the other was taught "Bearings". At the end of this period, the teachers told their students that they would now be given two lessons to prepare to *teach* the topic they had just learned to members of the other class.

Students were given some advice on how to go about this process. They were asked to decide exactly what the big ideas in the topic were, the order these should be presented in, to identify difficult ideas and prepare resources help to explain them (e.g a worksheet). They were also asked to think how they would find out if their teaching had been successful.

When both classes were ready, half of the pupils moved to the other classroom and sat alongside a pupil from the other group (who they did not know) and taught them the topic. In a second lesson the roles were swapped.

(b) What learning culture was created? How was this achieved?

Students became appeared nervous but very motivated by this challenge. They took a great deal of trouble in creating attractive and informative worksheets and some even had them typed out and checked beforehand. During the teaching phase, students adopted teachers roles with some enthusiasm. One girl (teaching two boys due to uneven class sizes), adopted a very confident style, making statements like:

> "How do you find the area from these two numbers? You can draw centimetre squares if you want."
>
> " Talk then or you won't learn anything!"
>
> "Pretend you are the teacher, how would you explain it to me?"
>
> "Put your hands up if you don't know."

The boys, it must be said, couldn't wait for their turn to teach!

This positive culture was achieved because these teachers had high expectations for their classes and wanted their students to take active roles in lessons.

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

We didn't collect evidence on the mathematics learned, but the reflective thought that went into the review and design of the teaching sessions was considerable. Pupils began to show evidence of appreciating that teaching is not just about 'telling'. When, for example, the girl above was asked about the experience, she responded as follows: (Int = interviewer):

> Int: What was hard about being a teacher?
>
> Girl: After teaching for some time, I find they don't understand.
>
> Int: How did the experience help you?

Girl: It gave me more confidence.

Int: Would you like to try teaching again?

Girl: Yes, so what I've learned now I must try on other pupils so I get better at teaching. I can't understand how to teach well.

(e) Is this example available to see/read about?

This is referred to in (Bell, A; Crust, R; Shannon, A; Swan, M; "Awareness of Learning, Reflection and Transfer in School Mathematics, ESRC Project report: R000-23-2329, Shell Centre for Mathematical Education, University of Nottingham)

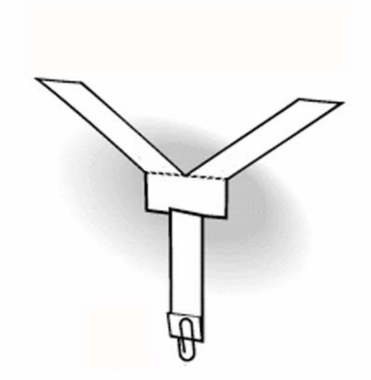
(f) Can you say why you chose this example? What criteria were in your mind?

To illustrate how changing classroom roles can begin to develop pupils' awareness of the nature of teaching and learning itself. This can be done alongside the development of mathematical skills and concepts.

Purpose 5: Appreciation of the power of mathematics in society

Example 5.1 (Professor Dave Pratt)

(a) What was the mathematical task(s)?



What makes the best design for a paper "helicopter"?
(1) Make a spinner and try it out.
(2) Compare 2 spinners.
(3) Why did you like that one?
(4) What variables affect the flight (usually time of flight)
(5) In groups, explore the variable (e.g. length of wings) comparing it to the dependent variable (time).
(6) Perhaps use Active Graphing:
    (a) Enter a few ordinate points into a spreadsheet
    (b) Generate a scatter-graph
    (c) Decide what to do next in the experiment
    (d) Go to (a)

(b) What learning culture was created? How was this achieved?

No one task generates a culture. Indeed a task can create discipline problems of culture clashes with expectations of behaviour in task.
Aiming for a culture in which mathematics is used to explore an interesting problem, resulting in the learning of new mathematics.
Students need to work in small groups and work together, sharing roles in task.
The mathematics should be seen as powerful.

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

Students
- used graphs to make decision about where to take the experiment.
- used data analysis methods to increase accuracy of experiment.
- worked productively together.
- could describe relationship between independent and dependent variables in question

(e) Is this example available to see/read about?

Ainley, Pratt and Nardi: article in Educational Studies in Mathematics (ESM)
Ainley: article in Micromath.
Ainley, Pratt and Hansen: article in British Educational Research Journal (BERJ)

(f) Can you say why you chose this example? What criteria were in your mind?

Engaging students (purpose); Engagement leading to focus (utility); Problem solving; Awareness of the power of mathematics (graphing as an analytical tool; average for smoothing errors. I have used the example many times and at many levels. It seems to provide an opportunity for thinking of graphs as analytical tools, rather than as presentational tools. Chosen out of frustration at narrow use of graphing in schools.

Example 5.2 (Bob Ansell)

(a) What was the mathematical task(s)?

To determine where an object rolling along a path in the shape of a quadrant, leaves the surface.
The students Upper sixth Mechanics 'A' level, had access to track and small toy cars. They modelled this and recorded results – at what angle and above did the vehicle leave the surface.
This was also tackled mathematically. The results were compared to theory.

(b) What learning culture was created? How was this achieved?

Experimentation. Problem solving at all levels.

1. how do we make the track the right shape?
2. how do we record?
3. how do we interpret results?

the essential link between a model and reality

(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?

Student engagement and, later, their understanding with related tasks.

(e) Is this example available to see/read about?

No

(f) Can you say why you chose this example? What criteria were in your mind?

It was fun, engaging and allowed me to promote mathematical modelling alongside simulation and practical problems.

Report on the third discussion

Factors that inhibit or modify practice

In response the question, "what are the major obstacles to progress and how do these obstacles function?" delegates, writing could be broadly categorised into six overlapping categories, namely:

- A taught curriculum that is defined by its assessment;
- An emphasis on quantities rather than qualities;
- A language in use at odds with professional values
- Different values, principles and expectations held by different constituencies;
- Creativity sacrificed on the altars of conformity and adherence;
- Textbooks geared to examination / test passing rather than mathematics learning.

A taught curriculum that is defined by its assessment

"Teach what will be in the test" (Dr Els De Geest, University of Oxford);

"The aim is seen as passing tests rather than learning mathematics" (Pete Griffin, NCETM);

"Assessment strategies that do not reflect effective pedagogy" (Keith Jones, University of Southampton);

"Assessment that leads to curriculum coverage" (Carol Knights, University of Chichester);

"If it is not in the examination then don't teach it" (Michael Ling, The Royal Statistical Society);

"Assessment in its current form is often an obstacle because teachers perceive a need to focus on procedural competence almost to the exclusion of broader (*and deeper*) understanding (Nicholson);

"High stakes external testing leads teachers to aim at short-term instrumental understanding" (Tanner);

"Teaching for the next test takes away the richness of the subject" (Taylor);

"Assessment that focuses on a narrow range of content" (David Wright, Newcastle University).

An emphasis on quantities rather than qualities

"Curriculum coverage at the expense of learning" (Non-Attributable);

"Rushing to get things done before the examinations" (Norma Honey, NCETM);

"The emphasis in examination and test results" (Jane Jones, Specialist Adviser for Mathematics);

"Curriculum coverage relates to the teacher rather than the learner. (Dr David Martin, NANAMIC);

“Raising headline examination figures leads to short-term initiatives being imposed on teachers” (Ruth Tanner, Lodge Park Technology College);

“Meeting targets set by others does not necessarily promote good teaching” (Sidney Tyrrell, Coventry University);

“Students get grades to meet school targets rather than for themselves” (Dr Anne Watson, University of Oxford);

“Functioning through performance data and league tables” (David Wright, Newcastle University).

A language in use at odds with professional values

“Delivery” (Jim Thorpe, NCETM);

“Coverage” (Several).

Different perceived values, principles and expectations held by different constituencies

Public and politicians’ “views of mathematics” (Professor Afzal Ahmed, The Mathematics Centre, University of Chichester);

“Have to teach in a particular way because of OFSTED and/or strategies” (Dr Jenni Back, Middlesex University);

“What is the game of teaching mathematics really about?” (Alison Clark-Wilson, University of Chichester);

“Cutting allotted timetable time cuts the time needed for learning” (Dr Julie-Anne Edwards, University of Southampton);

“A cultural lack of value of education in particular and mathematics in particular” (Dr Matt Homer, School of Education, University of Leeds);

“High value on assessment results on the part of learners, parents, managers and assessors” (Steve Feller, Edge Hill University);

“Lack of confidence about expectations from those in positions of power” (Norma Honey, NCETM);

“It will be impossible to make meaningful improvements to teaching mathematics if assessment continues to be uninspiring –demanding the regurgitation of facts and ‘teaching to the test” (Holly Isherwood, The Royal Institution).

“Excessive emphasis on assessment is at odds with principles and values” (Mann);

“There is a climate of distrust and excessively detailed accountability” (Professor John Mason, The Open University);

"Who decides what and what not A level students should be taught?" (Peter Saunders, Kings College, London);

"A common view of mathematics shared by teachers and pupils is that it is a dead weight of predetermined knowledge to be absorbed" (John Rickwood, Burlington Junior School);

"Lack of understanding about effective mathematics teaching by senior managers" (Ruth Tanner, Lodge Park Technology College);

"A lack of agreement about what mathematics is" (Professor Geoff Wake, University of Manchester);

"An inspection system that expects adherence to 3-part lesson, lesson objectives that define end points, certain amounts of written work etc. (Dr Anne Watson, University of Oxford);.

Creativity sacrificed on the altars of conformity and adherence

"Teachers have misconceptions about what they are 'allowed' to do" (Kath Cross, Retired HMI);

"A national strategy that drives teaching and learning along wee defined paths with scant acknowledgement of more creative approaches" (Joy Garvey, Royal Borough of Kingston on Thames);

"A culture of unthinking teacher conformity rather than thoughtful experimentation" (Pete Griffin, NCETM);

"Strategies over interpreted by teachers who fear stepping out of line" (Rose Griffiths, University of Leicester School of Education);

"Teachers being told exactly what they should be doing and how" (Carol Knights, University of Chichester);

Non-professional attitudes are amplified by excessive constraints including a shortage of time" (Professor John Mason, The Open University);

"A culture of conformity within a 'managerialist' culture where few people are prepared to take risks" (Non-Attributable);

"Teachers feel they have to toe the line (3-part lessons etc) in order to keep up with colleagues" (Bernard Murphy, MEI);

"Hegemonic 'best practice' that homogenises" (Dr Andrew Noyes, University of Nottingham);

"Emphasis on compliance" (Dr Andrew Noyes, University of Nottingham);

"Fear of testing leads teachers to teach to the test; fear of authority keeps lessons safe and dull" (John Rickwood, Burlington Junior School);

"Teachers have a perception on how they are being judged (3-part lesson etc) (Terea Smart, NCETM);

"Too many prescriptive structures and strategies take control away from the teacher" (Dr Howard Tanner, Swansea School of Education).

Textbooks geared to examination / test passing rather than to mathematics learning

“Text books, associated with awarding bodies, atomise the curriculum and promote teaching to the tests”. (Non-Attributable);

“Whilst some textbooks are excellent, not all of them are all that good” (Roger Porkess, Mathematics in Education and Industry);

“Revision books condense content that learners have never had chance to explore before condensing” (Peter Lacey, ECARDA).

Teacher confidence in mathematics

“Lack of subject knowledge as in not knowing what is possible” (Dr Els De Geest, University of Oxfod);

“Teachers’ lack of confidence in mathematics can hinder ‘letting go of the content’” (Pat Drake, University of Sussex);

“Teachers not having sufficient knowledge to depart from the script” (Non-Attributable);

“Fear – teachers who lack confidence in mathematics” (Rose Griffiths, University of Leicester School of Education);

“Lack of subject specific training in schools” (Jane Jones, Specialist Adviser for Mathematics);

“The curriculum has to be limited if it has in mind the under-qualified teacher” (Pete Saunders, Kings College, London).

In answer to the question “What practical steps can we take to help ourselves and others to overcome these obstacles? There were high levels of agreement. Recommendations centred around six themes, namely:

i. Rather than invest in more initiatives, invest in empowering teaches as educators, researchers and agents for change; Support their development by, for example, coaching and mentoring;

ii. Give every support and encouragement to teachers working on their own mathematics;

iii. Alter the ‘high stakes’ basis of ‘league tables’ to replace competition between schools with collaboration amongst teachers;

iv. Encourage collaboration to exploit and tap into the combined experience, wisdom and expertise of teachers of mathematics in order to support each other’s development and to develop rich tasks and useful resources.

v. Use standards related to collaborative professional learning that exemplify values and practices to alter the perceptions of managers in schools (and others) and to inform performance management;

vi. Reduce and change end-of-key-stage-assessments so that they project more faithfully what mathematics is about ,reflect better its learning and create more time for learning and teaching.

Appendix (i)

Some extracts from writing over the last 100 years that describe characteristics of learning and teaching mathematics.
(sent out with the invitation to the initial conference in order to set an historical context)

….The booliness of the method depends essentially on my not making any statement as to the nature of the connection between two groups of facts. The method is to set brains vibrating with the simultaneous consciousness of the two groups of facts, *free from any hamper, from any opinions as to the nature of the connection between the two groups*, ad start them investigating the nature of the connection, (circa 1903)
”A Boolean Anthology – Selected writings of Mary Boole” Compiled by D. G. Tahta
Association of Teachers of Mathematics 1972

Mathematics is a difficult subject, but not as difficult as it is often made out to be. Most people have a greater capacity for mathematical understanding than they are aware of, and a large reservoir of undeveloped mathematical competence certainly exists amongst youngsters of ordinary ability which good teaching and an enlightened approach could reveal. Few, if any, of our pupils are ever likely to become mathematicians, but some well come to find satisfaction in mathematical work if its purpose has first been clearly seen and confidence established through the successful use of mathematics as a tool.

The basis of all practical mathematics is a sound knowledge of the “facts” of elementary arithmetic - ……..

“Half our Future” A report of the Central Advisory Council for Education (England)
HMSO (*often known as the Newsom Report)* 1963

1. Children learn mathematical concepts more slowly than we realised. They learn by their own activities.
2. Although children think and reason in different ways they all pass through certain stages depending on their chronological and mental ages and their experience.
3. We can accelerate their learning by providing suitable experiences, particularly if we introduce the appropriate language simultaneously.
4. Practice is necessary to fix a concept once it has been understood, therefore practice should follow, and not precede, discovery.

“Mathematics in Primary Schools”, Schools Council curriculum Bulletin No. 1
HMSO 1965
And quoted in:
“Children and their Primary Schools” A report of the Central Advisory Council for Education (England) (*often known as the Plowden Report)* HMSO 1967

Running through all the work is the central notion that the children must be set free to make their own discoveries and think for themselves, and so achieve understanding, instead of learning off mysterious drills.

"Mathematics Begins" Nuffield Project 1967

If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly.

"Educational Psychology: A Cognitive View" D. P. Ausubel
Holt, Rinehart & Winston 1968

How effective an intrinsic motivation for learning mathematics can be is something many teachers do not yet appreciate. On a number of occasions, teachers finding that children actually enjoy mathematics when it is intelligently taught and learnt have reported this to me with a mixture of surprise and pleasure, but also of doubt, as if something must be wrong with an approach to mathematics which children enjoyed. But until this intrinsic motivation is better comprehended and put to work, mathematics will remain for many a subject to be endured, not enjoyed; and dropped as soon as the necessary exam results have been achieved.

The Psychology of Learning Mathematics Richard R Skemp
Penguin Books 1971

Though the first principles of the learning of mathematics are straightforward, it is the communicator of the mathematical ideas, and not the recipient, who most needs to know them. And though they are simple enough in themselves, their mathematical applications involve much harder thinking. The first of these principles (is)

1. Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples.

   The second follows directly from it:

2. Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner.

The first of these principles is broken by the vast majority of text books, past ands present. Nearly everywhere we see new topics introduced, not by examples, but by definitions: of the most admirable brevity and exactitude for the teacher (who already has the concepts to which they refer), but intelligible to the student.

The Psychology of Learning Mathematics Richard R Skemp
Penguin Books 1971

His mathematical experience, like all his experience, must progress through this sequence of abstraction. We shall categorize this experience by:

E experience with physical objects;
L spoken language that describes this experience;
P pictures that represent the experience;
S written symbols that generalize the experience;

The Psychology of Learning Mathematics — Richard R Skemp
Penguin Books — 1971

Mathematics is a universe, always growing, always needing to be recast by those who inhabit it. Our students can be invited to share this kind of life from the beginning of their study. We no longer need to burden their memory with many isolated relationships but can ask them to look and tell us what they see. The majestic edifice transmitted by tradition was too imposing, too intimidating; it could only lead for most to a feeling of inadequacy and total dependence on others. A human outlook is found by looking at mathematization rather than mathematics, at the activity of minds, those of mathematicians, and seeing them as they often are – fumbling, hesitating, abandoning attempts, trying others and accepting as their production what they found, not the ultimate word on the matter.

"On Being Freer" — Caleb Gattegno
Educational Solutions, New York — 1975

Two such words can be identified in the context of mathematics; and it is the alternative meanings attached to these words, each by a large following, which in my belief are at the root of the difficulties in mathematics education today.

One of these is 'understanding'. It was brought to my attention some years ago by Stieg Mellin-Olsen, of Bergen University, that there are in current use two meanings of the word. These he distinguishes by calling them 'relational understanding' and 'instrumental understanding'. By the former is meant what I have always meant by understanding, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I in the past described as 'rules without reasons' without realising that for many pupils *and their teachers* the possession of such a rule, and ability to use it, was what they meant by 'understanding'.

…. A person with a set of fixed plans can find his way from a certain set of starting points to a certain set of goals. The characteristic of a plan is that it tells him what to do at each choice point: turn right out of the door, go straight on past the church, and so on. But if at any stage he makes a mistake, he will be lost; and he will stay lost if he is not able to retrace his steps and get back on the right path.

In contrast, a person with a mental map of the town has something from which he can produce, when needed, an almost infinite number of plans by which he can guide his steps from any starting point to any finishing point, provided only that both can be imagined on his mental map. And if he does take a wrong turn, he will still know where he is, and thereby be able to correct his mistake without getting lost; even perhaps to learn from it.

The analogy between the foregoing and the learning of mathematics is close.

"Relational Understanding and Instrumental Understanding" — Richard R Skemp
In "Mathematics Teaching" 77
The Journal of the Association of Teachers of Mathematics — December 1976

We agree that all children need to succeed; but do we mean the same thing? My own feeling is that success should not be quick or easy, and should not come all the time. Success implies overcoming an obstacle, including, perhaps, the thought in our minds that we may not succeed. It is turning "I can't" into "I can, and I did."

Knowledge, learning, understanding are not linear. They are not little bits of facts lined up in rows or piled up one on top of the other. A field of knowledge (such as mathematics) is a territory, and knowing it is not just a matter of knowing all the items in the territory, but of knowing how they relate to, compare with, and fit in with each other. It is the difference between being able to say that a room in your house has so many tables, so many chairs, so many lamps, and being able to close your eyes and see that this chair goes here and that table there. It is the difference between knowing the names of all the streets in a city and being able to get from any place, by any desired route, to any other place.

"How Children Fail" John Holt Pelican Books 1984

Debate about how mathematics is learned has continued throughout the recorded history of mathematics teaching, yet the process is still not founded on a universally accepted theory. Shulman (1970) points out that '…mathematics instruction has been quite sensitive to shifts in psychological theories', but also that '…mathematics educators have shown themselves especially adept at taking hold of conveniently available psychological theories to buttress previously held instructional proclivities'.

Taken from:
Learning Mathematics: Issues, Theory & Classroom Practice Anthony Orton
Cassell Education 1986

The components of mathematical ability were seen by Krutetskii to be:

1. An ability to extract the formal structure from the content of a mathematical problem and to operate with that formal structure;
2. An ability to generalise from mathematical results;
3. an ability to operate with symbols, including numbers;
4. and ability for spatial concepts, required in certain branches of mathematics;
5. a logical reasoning ability;
6. an ability to shorten the process of reasoning;
7. an ability to be flexible in switching from one approach to another, including both the avoidance of fixations and the ability to reverse trains of thought;
8. An ability to achieve clarity, simplicity, economy and rationality in mathematical argument and proof;
9. a good memory for mathematical knowledge and ideas.

Taken from:
Learning Mathematics: Issues, Theory & Classroom Practice Anthony Orton
Cassell Education 1986

We must acknowledge that we hold many preconceptions or even prejudices about learning in general, learning by the young, and learning mathematics. The preconceptions are justified only because we have not devoted ourselves to the study of either our own learning or the learning of others, the young especially. But once we study learning seriously, we find that there is no reason on earth why we should not do things differently in our schools and do them primarily in conformity with the actual learning taking place in front of us. As teachers, we should link what we do to what students spontaneously do well already, to operations that they mastered quite early in their lives and used to learn to walk, for example, or to speak (and indeed serves them forever). This general approach I call "the subordination of teaching to learning."

"The Generation of Wealth" Caleb Gattegno
Educational Solutions, New York 1986

Mathematics provides a way of viewing and making sense of the real world. It is also a means of creating new and imaginative worlds to explore. An approach to mathematics which includes just those aspects that relate to knowledge, skills and understanding and their application to problems in the 'real' world, is deficient. It fails to provide pupils with insights into the unique character of mathematics, the opportunities it gives for intellectual excitement and an appreciation of the essential creativity of mathematics. Moreover, this aspect of mathematics which encourages pupils to explore and explain the structure, patterns and relationships within mathematics is an important factor in enabling them to recognise and utilise the power of mathematics in solving problems and to develop their own mathematical thinking.

Mathematics Non-Statutory Guidance National Curriculum Council 1989

The overall design and balance of a scheme of work should take into account the following guidelines:

- Activities should bring together the different areas of mathematics;
- The order of activities should be flexible;
- Activities should be balanced between tasks which develop knowledge, skills and understanding, and those which develop the ability to tackle practical problems;
- Activities should be balanced between the application of mathematics and ideas which are purely mathematical;
- Activities should be balanced between those which are short in duration and those which have scope for development over an extended period;
- Activities should, where appropriate, use pupils' own interests or questions either as starting points or as further lines of development;
- Activities should, where appropriate, involve both independent and co-operative work;
- Tasks should be both of the kind that have an exact result or answer and those which may have many possible outcomes;
- Activities should be balanced between different modes of learning: doing; observing; talking; listening; discussing with other pupils; reflecting; drafting; reading and writing etc.

- Activities should encourage pupils to use mental arithmetic and to become confident in the use of a range of mathematical tools;
- Activities should enable pupils to communicate their mathematics;
- Activities should enable pupils to develop their personal qualities;
- Activities should enable pupils to develop a positive attitude to mathematics.

Mathematics Non-Statutory Guidance National Curriculum Council 1989

Roughly speaking, the art of teaching as generally practised is assumed to consist in reducing the mastery of difficult tasks to a succession of more manageable steps, 'improving' the riskily rugged, if beautiful, track up the mountain of Mathematics by cutting small steps in the rock all the way up. Now this is a useful strategy when programming a machine (though its limitations are becoming apparent even then). But human beings are not machines. Even if many individual human skills are best taught in this way, Mathematics is much more than a collection of isolated skills. The power of Mathematics and of human beings derives from the way they can resolve genuinely challenging problems by *coordinating* isolated ideas and skills into effective strategies. A healthy mathematical diet must therefore combine stepwise training with regular challenges that knock pupils off balance and force them to *think*.

"What kind of Mathematics do our able students need ?" A. Gardiner
Mathematics in Schools 19(1) 1990

I like pupils to see mathematics as a network of ideas and learning it as a kind of adventure game in which they are finding out as much about ways of thinking and working as they are about mathematics. Ideally, each fact, method or theory met and learned should lead on to something else. The same method or process can be applied to another area of mathematics and the same bit of knowledge can be used in a different context or viewed from another angle. Frequently it is the pupil who, by choosing a particular way to work, decides to move in the direction of something new.

Although some mathematical ideas are building blocks for other ideas I do not believe that mathematics can or should be learned in a linear fashion. It is often by working with a complex idea that a simpler one suddenly makes sense or a technique begins to feel comfortable.

"What I do in my Classroom" Anne Watson
In "Teaching Mathematics" edited by Michelle Selinger, Routledge/OU 1994

Therefore finding ways of causing the learner to make connections and provide opportunities for the transfer of skills is essential if teachers are going to promote the effective learning of mathematics. I want to find wholesome ways of developing my students' abilities to think mathematically.

"Contexts and Strategies for Learning Mathematics" Mike Ollerton
In "Teaching Mathematics" edited by Michelle Selinger, Routledge/OU 1994

Once you know what mathematics is really about, and once you see how our brains create language, you should find it far less surprising that thinking mathematically is just a specialised form of using our language facility.

What does it take to make a mathematical mind?

- Number sense;
- Numerical ability;
- Algorithmic ability
- The ability to handle abstraction
- A sense of cause and effect;;
- The ability to construct and follow a causal chain of facts or events;
- Logical reasoning ability
- Rational reasoning ability
- Spatial reasoning ability

These then are the mental abilities that combine to give us the ability to do mathematics. Our quest for the origins of mathematical ability reduces in large part to a search for the origins of each of the abilities we have just considered.

"The Maths Gene" Keith Devlin 2000

The quality of teaching was the key factor influencing students' achievement. The majority of the teaching seen was at least satisfactory in preparing students for examinations. However, in promoting a really secure understanding of mathematical ideas, in stimulating students to think for themselves and to apply their knowledge and skills in unfamiliar situations, the picture was less encouraging. In approximately half of the lessons observed, the teaching did not sufficiently encourage these important aspects of learning in mathematics.

The best teaching gave a strong sense of the coherence of mathematical ideas; it focused on understanding mathematical concepts and developed critical thinking and reasoning. Careful questioning identified misconceptions and helped to resolve them, and positive use was made of incorrect answers to develop understanding and to encourage students to contribute. Students were challenged to think for themselves, encouraged to discuss problems and to work collaboratively. Effective use was made of information and communication technology (ICT). In contrast, teaching which presented mathematics as a collection of arbitrary rules and provided a narrow range of learning activities did not motivate students and limited their achievement. Focusing heavily on examination questions enabled students to pass examinations, but did not necessarily enable them to apply their knowledge independently in different contexts.

"Evaluating Mathematics Provision for 14 – 19 year-olds" (HMI 2611)
OFSTED 2006

The survey found that the factors which made the most significant contributions to high achievement in 14–19 mathematics were:

- Secure subject knowledge on the part of the teacher, underpinning an approach to mathematics in which all topics are seen as part of a coherent set of related ideas, with clear progression and links to previous and future learning.
- Teaching that focuses on developing students' understanding of mathematical concepts and enhances their critical thinking and reasoning, together with a spirit of collaborative enquiry that promotes mathematical discussion and debate.
- Assessment that informs teaching, and questioning skills that stimulate learning.

- Well paced lessons that set high expectations and challenge students to apply their own ideas, and that create a positive atmosphere to build students' confidence.
- The effective use of ICT and other high quality learning resources, including the new resources devised by the Standards Unit in the DfES to enhance learning and develop functional skills in mathematics.
- Professional development for teachers which focuses on effective teaching and learning and promotes the sharing of good practice.
- A range of learning programmes that promote wider access to mathematics for students at all levels.

"Evaluating Mathematics Provision for 14 – 19 year-olds" (HMI 2611)
OFSTED 2006

Appendix (ii)

A collection of recent ideas relating to learning and teaching mathematics. (sent out to those who accepted the invitation to the initial conference in order to start to focus thinking)

EXTRACT 1

Views of mathematics, its learning and teaching

A ‘Transmission’ view		**‘Connected’, ‘challenging’ view**
A given body of knowledge and standard procedures that has to be‘covered’.	**Mathematics is**	An interconnected body of ideas and reasoning processes.
An individual activity based on watching, listening and imitating until fluency is attained.	**Learning is**	A collaborative activity in which learners are challenged and arrive at understanding through discussion.
Structuring a linear curriculum for learners. Giving explanations before problems. Checking that these have been understood through practice exercises. Correcting misunderstandings.	**Teaching is**	Exploring meaning and connections through non-linear dialogue between teacher and learners. Presenting problems before offering explanations. Making misunderstandings explicit and learning from them.

EXTRACT 2

Some underlying principles

(i) Build on the knowledge learners bring to sessions

Effective teaching assumes that learners do not arrive at sessions as ‘blank slates’, but as actively thinking people with a wide variety of skills and conceptions. Research shows that teaching is more effective when it assesses and uses prior learning to adapt to the needs of learners. This prior learning may be uncovered through any activity that offers learners opportunities to express their understanding. It does not require more testing. For example, it can take the form of a single written question given at the beginning of a session to set the agenda for that session and elicit a range of explanations. These responses may then be used as a starting point for discussion.

(ii) Expose and discuss common misconceptions

Research has shown that teaching becomes more effective when common mistakes and misconceptions are systematically exposed, challenged and discussed. The sessions described here typically begin with a challenge that exposes learners’ existing ways of thinking. Cognitive conflicts occur when the learner recognises inconsistencies between existing beliefs and observed events. This happens, for example, when a learner completes a task using more than one method and arrives at conflicting answers. Activities are carefully designed so that such conflicts are likely to occur. Research has shown that such conflicts, when resolved through reflective discussion, lead to more permanent learning than conventional, incremental teaching methods, which seek to avoid learners making ‘mistakes’.

(iii) Develop effective questioning

There have been many studies of teachers’ questioning. Typically, most questions are low-level, testing the ability of learners to recall facts and procedures. Such questions are also called ‘closed’, meaning that they permit just one single correct response. Fewer questions promote higher-level reflective thinking, such as the ability to apply, synthesise or explain. Such questions are called ‘open’ because they invite a range of responses. The research evidence shows that a variety of lower-level and higher-level open questions is much more beneficial than a continuous diet of closed recall questions.

A second finding is the importance of allowing time for learners to think before offering help or moving on to ask a second learner. Studies have shown that many teachers wait for less than one second. Longer ‘wait times’ are associated with significantly improved achievement.

(iv) Use cooperative small group work

Many learners think that learning mathematics is a private activity. They frequently enter post-16 education under-confident and reluctant to discuss difficulties. It is therefore essential that a supportive and encouraging atmosphere is created in the learning environment. It is the teacher’s responsibility to ensure that everyone feels able to participate in discussions and this is often easier in small group situations. It is interesting to consider why small group activities are used less often in mathematics than in other subject areas, where they are commonly used to good effect. One possible reason might be the lack of suitable resources. We hope that this resource will help to fulfil this need.

There is now general agreement in research that cooperative small group work has positive effects on learning, but that this is dependent on the existence of shared goals for the group and individual accountability for the attainment of these goals. It has also been seen to have a positive effect on social skills and self-esteem.

(v) Emphasise methods rather than answers

Often we find that learners focus more on obtaining a correct answer than on learning a powerful method. They often see their task as 'getting through' an exercise rather than working on an idea. Completion is seen as more important than comprehension. In these resources, we do not concern ourselves with whether or not learners complete every task, but instead we try to increase their power to explain and use mathematical ideas. Learners may work on fewer problems than in conventional texts, but they come to understand them more deeply as they tackle them using more than one method.

(vi) Use rich collaborative tasks

Rich tasks:

- are accessible and extendable;
- allow learners to make decisions;
- involve learners in testing, proving, explaining, reflecting, interpreting;
- promote discussion and communication;
- encourage originality and invention;
- encourage 'what if?' and 'what if not?' questions;
- are enjoyable and contain the opportunity for surprise.

Textbooks often assume that we should begin topics by solving simple questions and then gradually move towards more complex questions. While this may appear natural, we find that learners tend to solve simple questions by intuitive methods that do not generalise to more complex problems. When the teacher insists that they use more generalisable methods, learners do not understand why they should do so when intuitive methods work so well. Simple tasks do not motivate a **need** to learn.

Rich tasks also allow **all** learners to find something challenging and at an appropriate level to work on.

(vii) Create connections between mathematical topics

A common complaint of teachers is that learners find it difficult to transfer what they learn to similar situations. Learning appears compartmentalised and closely related concepts and notations (such as division, fraction and ratio) remain unconnected in learners' minds. In this resource, we have therefore included 'linking activities' that are particularly designed to draw out connections across mathematical topics. The index refers to sessions as 'mostly number' or 'mostly algebra' in order to reflect these connections.

(viii) Use technology in appropriate ways

While new technologies have transformed our lives in many ways, they have had less impact inside most mathematics classrooms. They do offer us the opportunity to present mathematical concepts in dynamic, visually exciting ways that engage and motivate learners. In the sessions that follow, we have sought to illustrate some of this potential through the provision of a few computer ‘applets’; these are small pieces of purpose-built software that are designed to be very easy to use.

EXTRACT 3

Purposes of learning mathematics and related learning activities

Fluency in recalling facts and performing skills	Interpretations for concepts and representations	strategies for investigation and problem solving	Awareness of the nature and values of the educational system	Appreciation of the power of mathematics in society
• Memorising names and notations	• Discriminating between examples and nor-examples of concepts	• Formulating situations and problems for investigation	• Recognising different purposes of learning mathematics	• Appreciating mathematics as human creativity [+ historical aspects]
• Practising algorithms and procedures for fluency and 'mastery'	• Generating representations of concepts	• Constructing, sharing, refining, and comparing strategies for exploration and solution	• Developing appropriate strategies for learning/ reviewing mathematics	• Creating and critiquing 'mathematical models' of situations
	• Constructing networks of relationships between mathematical concepts	• Monitoring one's own progress during problem solving and investigation	• Appreciating aspects of performance valued by the examination system	• Appreciating uses/abuses of mathematics in social contexts
	• Interpreting and translating between representations of concepts	• Interpreting, evaluating solutions and communicating results		• Using mathematics to gain power over problems in one's own life

The lists in the table above offer an intimidating design challenge. Different purposes may be emphasised within a lesson or series of lessons. Thus, when considering the topic of 'fractions and decimals' an imaginative teacher may include a combination of:

- exercises to develop fluency with multiplication or division algorithms (a skill focus);
- discussions concerning the meaning of place value and its links with fractional notation (a concept focus);
- 'rich' calculator-based, investigative activities or real problems to solve (a strategy focus);
- discussions on the uses and abuses of fractions and decimals in the media (a social context focus);
- discussions on the types and purposes of the learning activities used (for awareness of the nature and values of the educational system).

EXTRACT 4

How teachers describe their practices

Statements are rank ordered from most common to least common.

	Mean
Learners start with easy questions and work up to harder questions	4.26
I tell learners which questions to tackle	4.02
I teach the whole group at once	3.90
I know exactly what maths the lesson will contain	3.80
Learners learn through doing exercises	3.67
I try to cover everything in a topic	3.56
I avoid learners making mistakes by explaining things carefully first	3.31
Learners work on their own, consulting a neighbour from time to time.	3.30
I teach each topic from the beginning, assuming they know nothing.	3.29
I tend to teach each topic separately	3.19
Learners use only the methods I teach them	3.18
I draw links between topics and move back and forth between topics	3.03
I tend to follow the textbook or worksheets closely	2.99
I only go through one method for doing each question	2.95
I encourage learners to make and discuss mistakes	2.63
Learners work collaboratively in pairs or small groups	2.57
Learners learn through discussing their ideas	2.53
I jump between topics as the need arises	2.51
I find out which parts learners already understand and don't teach those parts	2.44
I teach each learner differently according to individual needs.	2.43
Learners compare different methods for doing questions	2.24
I am surprised by the ideas that come up in a lesson	2.08
I encourage learners to work more slowly	2.03
Learners choose which questions they tackle	1.98
Learners invent their own methods	1.73

The sample consists of 120 teachers and trainers from more than 60 providers. Each statement was rated as follows:
1 = almost never, 2 = occasionally, 3 = half the time, 4= most of the time; 5 = almost always.

EXTRACT 5

How learners describe their learning strategies

Statements are rank ordered from most common to least common.

	Mean
I listen while the teacher explains	4.28
I copy down the method from the board or textbook	4.15
I only do questions I am told to do	3.88
I work on my own	3.72
I try to follow all the steps of a lesson	3.71
I do easy problems first to increase my confidence	3.58
I copy out questions before doing them	3.57
I practise the same method repeatedly on many questions.	3.42
I ask the teacher questions	3.40
I try to solve difficult problems in order to test my ability	3.32
When work is hard I don't give up or do simple things	3.32
I discuss my ideas in a group or with a partner	3.25
I try to connect new ideas with things I already know	3.20
I am silent when the teacher asks a question	3.16
I memorise rules and properties	3.15
I look for different ways of doing a question	3.14
My partner asks me to explain something. . .	3.05
I explain while the teacher listens	2.97
I choose which questions to do or which ideas to discuss. . .	2.54
I make up my own questions and methods	2.03

The sample consists of 779 16—21 year old learners attending **44** different FE and sixth form colleges.
Each statement was rated as follows:
1 = almost never, 2 = occasionally, 3 = half the time, 4 = most of the time; 5 = almost always.

EXTRACT 6

How teachers describe the learning behaviours of their pupils

When learning mathematics pupils are . . . . . .

Questioning	Demonstrating	Persisting
Listening	Organising	Concentrating
Discussing	Assessing	Watching
Justifying	Relating	Risking
Reasoning	Remembering	Engaging
Visualizing	Reviewing	Puzzling
Imagining	Practising	Collaborating
Reflecting	Explaining	Patterning
Analysing	Inventing	Checking
Experimenting	Hypothesising	Refining
Testing	Articulating	Proving
Evaluating	Applying	Predicting

This is a selection of the most popular responses by more than 3000 primary and secondary school teachers' to the prompt above.

EXTRACT 7

Principles for the design of teaching

In this book, I attempt to design learning situations that focus on the development of mathematical concepts and strategies rather than on procedural knowledge, and I therefore draw most heavily on constructivist theories of learning when eliciting design principles. From the foregoing analysis, I therefore draw the following principles:

1. Students do not learn from passively ‘receiving’ information, but through their active participation in social practices, their reflection on these practices and through the internalisation and reorganisation of their own experience.

2. Students do not arrive in classrooms as ‘blank slates’ but as active learning participants who continually construct extensive conceptual frameworks. These pre-existing frameworks should be recognised and made explicit, not ignored. Pre-requisite knowledge must be activated before new learning can take place.

3. Conceptual frameworks do not develop along predetermined linear hierarchies. Activities must be designed so as to provide opportunities for students to create their own multiple connections. This will not happen in the same way for all students.

4. The designer’s/teacher’s role is to find/deploy adidactical situations and problems that stimulate vivid ‘perturbations’ or ‘conflicts’ with students’ conceptual frameworks to promote reinterpretation, reformulation and accommodation.

5. Students must appropriate the situations and problems, taking ownership over them, so that they can freely apply and direct their actions and thoughts. Situations must be ‘devolved’ to students. While this happens, the teacher must refrain from suggesting the knowledge he or she wants to appear.

6. In order for experiences to promote vivid perturbations/conflicts, learning situations must be so designed to encourage students to recognise surprises and inconsistencies that result from using their own intuitive methods and concepts. Current methods and concepts must be brought to a state of consciousness. If students are not given this opportunity, then ‘foreign’ methods and concepts may fail to be accommodated and become marginalised.

7. Conflicts may originate internally, within the individual, and externally, from an individual’s interpretation of another person’s alternative viewpoint. Interpretations remain mere ‘shadows’ unless they are articulated through language. This may involve inner speech as well as exteriorised speech. Social interaction is thus centrally important.

8. Perturbations may only be accommodated if students are able to spend time in reflective abstraction. This necessitates periods of ‘stillness’ (not necessarily silence) when ‘production of answers’ gives way to ‘reflecting on alternative methods and meanings’.

9. The teacher's role is to encourage articulation of intuitive viewpoints, challenge with alternative viewpoints when these do not arise spontaneously (play 'devil's advocate'), and facilitate the reformulation of ideas by mediating learning through language which enables the student to construct his or her own new concepts. This role is proactive and contrasts strongly with the reactive roles adopted in discovery learning approaches. This requires considerable sensitivity on the part of the teacher.

10. To mediate learning, the teacher may provide 'scaffolding' — conceptual resources necessary for a higher level of cognitive functioning. Through interaction these resources may be internalised by the student as the scaffolding is progressively removed.

11. The teacher should also attempt to foster the 'institutionalisation' of the concepts and methods generated by students. The teacher must recognise and give status to students' own constructions, reveal their inadequacies, seek generalisations and set them beside socially agreed conventions.

The challenge I face in this book is considerable. How might I implement these ideas in practical contexts and engineer teaching resources and methods that can be widely applied by teachers in typical classroom environments? And, if I do so, what will be the effects? These questions underpin the rest of this book.

Sources of the extracts

Please note that references within the texts of the extracts have been omitted.

Extract 1 Views of mathematics, its learning and teaching
Improving learning in mathematics: challenges and strategies
Malcolm Swan University of Nottingham
Produced by the Department for Education and Skills Standards Unit. 2005
ISBN: 1 84478-537 X

Extract 2 Some underlying principles
Improving learning in mathematics: challenges and strategies
Malcolm Swan University of Nottingham
Produced by the Department for Education and Skills Standards Unit. 2005
ISBN: 1 84478-537 X

Extract 3 Purposes of learning mathematics and related learning activities
Collaborative learning in mathematics: a challenge to our beliefs and practices
Malcolm Swan University of Nottingham
Published by the National Research & Development Centre for Adult Literacy and Numeracy (NRDC) and the National Institute of Adult Continuing Education (NIACE) 2006
ISBN: 1 86201 311 X

Extract 4 How teachers describe their practices
Learning mathematics through reflection and discussion: the design and implementation of teaching
Malcolm Swan University of Nottingham
unpublished PhD thesis, 2005, University of Nottingham

Extract 5 How learners describe their learning strategies
Learning mathematics through reflection and discussion: the design and implementation of teaching
Malcolm Swan University of Nottingham
unpublished PhD thesis, 2005, University of Nottingham

Extract 6 How teachers describe the learning behaviours of their pupils
Learning from teachers: a personal record
Peter Lacey
Unpublished notes and records 1986 – 2007

Extract 7 Principles for the design of teaching
Collaborative learning in mathematics: a challenge to our beliefs and practices
Malcolm Swan University of Nottingham
Published by the National Research & Development Centre for Adult Literacy and Numeracy (NRDC) and the National Institute of Adult Continuing Education (NIACE) 2006
ISBN: 1 86201 311 X

Appendix (iii)

A background paper on issues related to the values and principles that underpin learning and teaching mathematics.
(sent out to conference delegates a week ahead of the conference in order to focus thinking more sharply and to assist delegates' written responses)

What constitutes the effective learning of mathematics?

A discussion paper for NCETM

Malcolm Swan[1]

Introduction

How do we interpret 'effective learning'? My dictionary defines 'effective' as 'achieving an intended or desired outcome'. This leads naturally to three questions:

- What types of learning outcomes are desired?
- How can we promote these in practice?
- How can we assess our progress towards them?

This discussion paper touches on the first two of these questions.

This paper therefore begins with a discussion of the *values* that we hold in teaching mathematics and then addresses some *principles* for achieving these that are suggested by research studies*, classroom tasks* that embody them, and *classroom cultures and discourses* that help promote more effective learning. Even if we cannot achieve consensus on all these issues, the debate itself should enable us to clarify and deepen our awareness of what is possible.

This brief paper is only intended to stimulate initial thinking. It is mostly a compilation of ideas and thoughts culled from other documents. If you disagree or think there are important gaps, then this paper will have served its purpose. This is an early first draft. We hope that at the end of the conference we can produce something better that incorporates the breadth of evidenced views heard at the conference.

There are a number of short questions scattered throughout the paper. These are just to stimulate thinking. We clearly won't have time to address all these at the conference, but would welcome written responses you may have at any time.

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[1] Any opinions expressed here are my own and not those of the NCETM. They are the starting points for the debate on what constitutes effective learning.

At the end of the paper, however, are four questions we *would* like you to think about before the conference. Please come prepared to talk about these!

1. What values underpin mathematics education?

> *Mathematics is of central importance to modern society. It provides the language and analytical tools underpinning much of our scientific and industrial research and development. Mathematical concepts, models and techniques are also key to many vital areas of the knowledge economy, including the finance and ICT industries. Mathematics is crucially important too, for the employment opportunities and achievements of individual citizens. (Smith, 2004 foreword, page v)*

As this quote from the Smith report makes clear, two purposes underpin mathematics education; to address the needs of society as a whole and the needs of its individual citizens. Individual needs are not restricted to the utilitarian (e.g. acquiring numeracy skills for survival), but also encompass other human needs:

> *Teachers in the lifelong learning sector value:*
> *AS 2 Learning, its potential to benefit people emotionally, intellectually, socially and economically, and its contribution to community sustainability.*
> *(LLUK, 2007)*

Similar aspirations are evident in the current National Curriculum for Mathematics. The first paragraph cited below identifies the need to equip students with a powerful mathematical toolkit - a cultural legacy of concepts, skills and procedures that have wide application; the second paragraph refers to about pupils creating their own mathematics and experiencing 'pleasure and wonder' at first hand:

> *Mathematics equips pupils with a uniquely powerful set of tools to understand and change the world. These tools include logical reasoning, problem-solving skills, and the ability to think in abstract ways. Mathematics is important in everyday life, many forms of employment, science and technology, medicine, the economy, the environment and development, and in public decision-making. Different cultures have contributed to the development and application of mathematics. Today, the subject transcends cultural boundaries and its importance is universally recognised.*
>
> *Mathematics is a creative discipline. It can stimulate moments of pleasure and wonder when a pupil solves a problem for the first time, discovers a more elegant solution to that problem, or suddenly sees hidden connections.*
> *(DfEE/QCA, 1999)*

There is perhaps some tension here in combining a desire to ‘transmit’ culturally valued facts and skills to students while at the same time allowing them opportunities for creativity and self-fulfilment.

- What emphasis do we place on:
  - communicating cultural knowledge and tools?
  - students creating their own mathematics?
- To what extent do these values require different styles of teaching and learning?
- What should be the balance between them?

Our National Curriculum explores the notion of self-fulfilment further by claiming that Mathematics can also encourage spiritual, moral, social, and cultural development:

> **spiritual development,** *through helping pupils obtain an insight into the infinite, and through explaining the underlying mathematical principles behind some of the beautiful natural forms and patterns in the world around us*
> **moral development***, helping pupils recognise how logical reasoning can be used to consider the consequences of particular decisions and choices and helping them learn the value of mathematical truth*
> **social development***, through helping pupils work together productively on complex mathematical tasks and helping them see that the result is often better than any of them could achieve separately*
> **cultural development***, through helping pupils appreciate that mathematical thought contributes to the development of our culture and is becoming increasingly central to our highly technological future, and through recognising that mathematicians from many cultures have contributed to the development of modern day mathematics.*
> *(DfEE/QCA, 1999)*

- Are these aspects evident in practice? How far should they be?
- What examples of 'effective practice' have we seen?

We also place value on different kinds of mathematical 'content'. The research conducted for the Cockcroft committee identified "three distinct aspects" of mathematics teaching that "require separate attention":

- **Facts and skills** - unconnected, arbitrary, information and well established routine procedures;
- **Conceptual structures** - interconnected bodies of knowledge that should underpin the performance of skills;
- **General strategies** - which guide the choice of which skills to apply, and an associated appreciation of the nature of mathematics.

(Cockcroft, 1982)

There are many other, similar taxonomies of purpose evident in curriculum and assessment specifications. Each aspect requires different methods of learning and teaching, underpinned by different learning theories. The table below expands and develops these categories in order to explore appropriate types of classroom activity that might result. Effective mathematics teaching might look very different depending on the balance of these purposes.

Different purposes of teaching may be emphasised by the same teacher within a lesson or series of lessons. Thus, within a series of lessons concerning the topics of 'fractions and decimals', an imaginative teacher may include a combination of:

- *exercises to develop fluency with multiplication or division algorithms (a skill focus);*
- *discussions concerning the meaning of place value and its links with fractional notation (a concept focus);*
- *'rich' calculator-based, investigative activities or real problems to solve (a strategy focus);*
- *discussions on the uses and abuses of fractions and decimals in the media (a social context focus); and*
- *discussions on the types and purposes of the learning activities used (for awareness of the nature and values of the educational system).*

Different purposes of learning and the types of learning activity that might result

Fluency in recalling facts and performing skills	**Interpretations for concepts and representations**	**Strategies for investigation and problem solving**	**Awareness of the nature and values of the educational system**	**Appreciation of the power of mathematics in society**
• Memorising names and notations • Practising algorithms and procedures for fluency and 'mastery'	• Discriminating between examples and non-examples of concepts • Generating representations of concepts • Constructing networks of relationships between mathematical concepts • Interpreting and translating between representations of concepts	• Formulating situations and problems for investigation • Constructing, sharing, refining, and comparing strategies for exploration and solution • Monitoring one's own progress during problem solving and investigation • Interpreting, evaluating solutions and communicating results	• Recognising different purposes of learning mathematics • Developing appropriate strategies for learning/ reviewing mathematics • Appreciating aspects of performance valued by the examination system	• Appreciating mathematics as human creativity (+ historical aspects) • Creating and critiquing 'mathematical models' of situations • Appreciating uses/abuses of mathematics in social contexts • Using mathematics to gain power over problems in one's own life

(Swan, 2006)

- Which important activities are missing in this table?
- To what extent are these types of activity compatible?
- What different kinds of teaching do they require?
- What balance of these activities would you wish to see?

The different columns in this table resonate with complementary theories/metaphors of learning:

The first column is the focus of '*behaviourists*', who emphasise the value of terminology and fluency in the performance of 'skills'. This trend is evident in learning activities that break 'mathematics' up into 'subskills' and 'key facts' that are taught until fluency is attained. Complex skills are then built by learning sequences of subskills. The process of learning is generally by clear exposition, followed by consolidation and practice.

The second and third columns reflect the focus of '*constructivists*' who recognise the value of encouraging children to construct concepts and strategies through exploration or creativity and discussion. Also reflected in these columns is the recent emphasis on metacognitive aspects in monitoring decisions in the course of problem solving.

The fourth and final columns reflect the current focus of *'social constructivists'* who emphasise that students should appreciate the way mathematics has evolved historically, is used by the world, and how they may use their mathematics to gain power over their own environment. This also includes students reflecting on their own role as a learner in an educational environment and combines elements of metacognition, in which a student develops an awareness of effective personal strategies for learning, with an awareness of the social values and discourses of education. The intention is also that the student becomes aware of the nature of the assessment system and how they may present their own abilities to their best advantage in presenting themselves to the world.

Tension results when these activities are combined inappropriately. For example, when students are given apparent freedom to 'investigate' an open situation, while the teacher wants them to 'discover' a culturally valued result or learn a particular skill, a tension arises between the divergence of the students' explorations and the convergent purposes of the teacher.

- What other tensions are evident in the purposes in our curriculum?

It is also interesting to consider the diverse range of outcomes that might result from these different types of mathematical activity. Most classrooms, I suggest produce outputs that 'exhibit technique'. We can extend this range, however, to include:

- A problem solution
- A report of a piece of research or investigation
- A mathematical model
- A plan of action
- A design
- A decision and justification
- An explanation of a concept

- What other types of product would be generated by students as a result of more 'balanced' teaching?

The rapid advance of technology continues to challenge the values we hold in teaching mathematics. When calculators were first introduced, there was much debate about the purpose of teaching the procedures of arithmetic?[2]. Similar debates continue over the role of computers. Why teach so much algebraic manipulation when computers can it all for you? New technologies redefine the way mathematics is used in the world and afford exciting new

[2] As Peter Kaner once remarked: "Why spend £1,000 per pupil giving them a skill that can be bought for £1.50?"

opportunities for education. These are actively explored in universities by educationists, but, in most teaching institutions, their use continues to be limited and unimaginative. Indeed in some schools access to computer rooms has declined with the advent of 'interactive' whiteboards (Ofsted 2004/5).

- What impact do new technologies have on the purposes of learning mathematics?

2. Teaching principles for effective learning

What makes for effective learning will depend on the purposes being served. There are, however, many research-based principles that may be drawn upon to inform practices. The list of principles below was obtained from a literature survey into effective ways of learning concepts.

1. *Students do not learn from passively 'receiving' information, but through their active participation in social practices, their reflection on these practices and through the internalisation and reorganisation of their own experience.*

2. *Students do not arrive in classrooms as 'blank slates' but as active learning participants who continually construct extensive conceptual frameworks. These pre-existing frameworks should be recognised and made explicit, not ignored. Pre-requisite knowledge must be activated before new learning can take place.*

3. *Conceptual frameworks do not develop along pre-determined linear hierarchies. Activities must be designed so as to provide opportunities for students to create their own multiple connections. This will not happen in the same way for all students.*

4. *The teacher's role is to find / deploy adidactical situations and problems that stimulate vivid 'perturbations' or 'conflicts' with students' conceptual frameworks to promote re-interpretation, reformulation and accommodation.*

5. *Students must appropriate the situations and problems, taking ownership over them, so that they can freely apply and direct their actions and thoughts. Situations must be devolved to students. While this happens, the teacher must refrain from suggesting the knowledge she wants to appear.*

6. *In order for experiences to promote vivid perturbations / conflicts, learning situations must be so designed to encourage students to recognise surprises and inconsistencies that result from using their own intuitive methods and concepts. Current methods and concepts must be brought to a state of consciousness. If students are not given this opportunity, then 'foreign' methods and concepts may fail to be accommodated and become marginalised.*

7. *Conflicts may originate internally, within the individual and externally, from an individual's interpretation of another person's alternative viewpoint. Interpretations remain mere 'shadows' unless they are articulated through language. (This may involve inner speech as well as exteriorised speech). Social interaction is thus centrally important.*

8. *Perturbations may only be accommodated if students are able to spend time in reflective abstraction. This necessitates periods of 'stillness' (not necessarily silence) when 'production of answers' gives way to 'reflecting on alternative methods and meanings'.*

9. *The teacher's role is to encourage articulation of intuitive viewpoints, challenge with alternative viewpoints when these do not arise spontaneously (play 'devils advocate'), and facilitate the reformulation of ideas by mediating learning through language which enables the student to construct his or her own new concepts. This role is pro-active and contrasts strongly with reactive roles adopted in discovery learning approaches. This requires considerable sensitivity on the part of the teacher.*

10. *To mediate learning, the teacher may provide scaffolding - conceptual resources necessary for a higher level of cognitive functioning. Through interaction these*

resources may be internalised by the learner as the scaffolding is progressively removed.

11. *The teacher should also attempt to foster the institutionalisation of the concepts and methods generated by students. The teacher must recognise and give status to students' own constructions, reveal their inadequacies, seek generalisations and set them beside socially agreed conventions.*

(Swan, 2006)

The following table below shows a set of principles used in the design of the *Improving Learning in Mathematics* resources:

Principles used in the Standards Unit material: "Improving Learning in Mathematics"

Teaching is more effective when it ...	
• **builds on the knowledge learners already have;**	This means developing formative assessment techniques and adapting our teaching to accommodate individual learning needs (Black & Wiliam, 1998).
• **exposes and discusses common misconceptions**	Learning activities should exposing current thinking, create 'tensions' by confronting learners with inconsistencies, and allow opportunities for resolution through discussion (Askew & Wiliam, 1995).
• **uses higher-order questions**	Questioning is more effective when it promotes explanation, application and synthesis rather than mere recall (Askew & Wiliam, 1995).
• **uses cooperative small group work**	Activities are more effective when they encourage critical, constructive discussion, rather than argumentation or uncritical acceptance (Mercer, 2000). Shared goals and group accountability are important (Askew & Wiliam, 1995).
• **encourages reasoning rather than 'answer getting'**	Often, students are more concerned with what they have 'done' than with what they have learned. It is better to aim for depth than for superficial 'coverage'.
• **uses rich, collaborative tasks**	The tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage 'what if' and 'what if not?' questions (Ahmed, 1987).
• **creates connections between topics**	Learners often find it difficult to generalise and transfer their learning to other topics and contexts. Related concepts (such as division, fraction and ratio) remain unconnected. Effective teachers build bridges between ideas (Askew et al., 1997).
• **uses technology in appropriate ways**	Computers and interactive whiteboards allow us to present concepts in visual dynamic and exciting ways that motivate learners.

(Swan, 2005)

- Do you agree with these principles? What additional principles would you add? Are your principles research-based? If so, what is your evidence?
- The principles articulated in this section refer mainly to the development of conceptual understanding. What principles would apply to some of the other purposes outlined above?

3. Choosing appropriate mathematical tasks

The Professional Standards for Teaching Mathematics in the US, offers a set of 'standards' that are central to a teacher's effectiveness (National Council of Teachers of Mathematics (NCTM),1991). The first of these is the choice of tasks for students:

> *The teacher should pose tasks that are based on: sound and significant mathematics; knowledge of students' understandings, interests and experiences and of the range of ways that diverse students learn mathematics.*
>
> *In addition, tasks should:*
> - *engage students' intellect;*
> - *develop students' mathematical understandings and skills;*
> - *stimulate students to make connections and develop a coherent framework for mathematical ideas;*
> - *call for problem formulation, problem solving and mathematical reasoning;*
> - *promote communication about mathematics;*
> - *represent mathematics as an ongoing human activity;*
> - *draw on students' diverse background experiences and dispositions*
> - *promote the development of all students' dispositions to do mathematics*
>
> *(NCTM, 1991)*

This is perhaps just 'peace, motherhood and apple pie'. Who could disagree with it? The difficulty for teachers is in recognising the characteristics of 'worthwhile' mathematical tasks and the different purposes they serve.

"Better Mathematics" (Ahmed, 1987) developed criteria and exemplification to support the identification of 'rich' mathematical tasks. By 'rich', the authors refer to tasks which are accessible, yet admit further challenges; tasks which invite pupils to make decisions; which involve pupils in speculating, hypothesising, explaining, proving, reflecting and interpreting; which promote discussion and questioning; which encourage originality and invention; and which have an element of surprise and are enjoyable.

The original Non-Statutory guidance for the National Curriculum (NCC, 1989) tried to give a flavour of the types of activity that students would engage in a balanced scheme of work:

> *The design and balance of a scheme of work should take into account the following guidelines:*
> - *Activities should bring together different areas of mathematics;*
> - *The order of activities should be flexible;*
> - *Activities should be balanced between tasks which develop knowledge, skills and understanding and those which develop the ability to tackle practical problems;*
> - *Activities should be balanced between the applications of mathematics and ideas which are purely mathematical;*
> - *Activities should, where appropriate, use pupils' own interests or questions as starting points or as further lines of development;*
> - *Activities should, where appropriate, involve both independent and cooperative work;*
> - *Tasks should be both of the kind which have an exact result or answer and those which have many possible outcomes;*

- *Activities should be balanced between different modes of learning: doing, observing, talking and listening, discussing with other pupils, reflecting, drafting, reading and writing, etc.*
- *Activities should encourage pupils to use mental arithmetic and to become confident in the use of a range of mathematical tools;*
- *Activities should enable pupils to communicate their mathematics;*
- *Activities should enable pupils to develop their personal qualities;*
- *Activities should enable pupils to develop a positive attitude to mathematics.*

(NCC, 1989)

(Source: Mathematics non-statutory guidance NCC, 1989)

All things are permitted in 'balance', but what should this balance look like? (Is it a 'rabbit-and-elephant-pie' type of balance - one rabbit and one elephant?).

The following 'activity analysis sheet' was designed to enable teachers to come to grips with some of the possible dimensions of variation possible in choosing tasks for students so that they could consider whether or not a particular scheme of work was balanced.

Activity analysis sheet from "Mathematics Programmes of Study INSET materials".

Classroom Activity Analysis Sheet

Description of the activity

Length
Short (< 20 minutes)
Long (20mins - 2 hours)
Extended (2 hours - 10 hours)

Nature of component tasks within the activity
Is the activity fragmented into many short subtasks, does it consist of longer, more substantial subtasks, or is it one holistic task ?

Main content objectives
Note the range of levels that pupils may work at under the appropriate NATs.

Using and applying	Number	Algebra	Space and shape	Handling data

Main process objectives
Tick the aspects of NAT 1 that are the centre of attention in this activity.

Designing / devising tasks
Planning / organising
Selecting maths /materials
Completing / checking
Reflecting / reviewing

Interpreting / asking
Discussing / explaining
Presenting / recording
Generalising / testing
Reasoning / justifying

Context and purpose
Is the *primary* purpose to tackle
a real - life situation,
a mathematical idea illustrated by applications,
a situation in pure mathematics ?

Intended outcomes of the activity
Has the activity,
a single intended outcome (closed),
or many possible outcomes (open) ?

Presentation of the activity
Tick the modes in which the activity will be introduced, adding your own if necessary.

Oral
Book(let) / Worksheet
Computer / calculator

Video
Practical equipment
____

Ways of working on the activity
Tick the ways in which pupils will work

Listening to exposition
Discussing in small groups
Using practical methods
Consolidating and practising
Solving problems
Investigating
Linking ideas in mathematics
Using computers / calculators.

Grouping
Show approx. proportion of time spent working in each grouping.

Whole class	%
Small group	%
Individual / pairs	%

Opportunities for independence and flexibility

Are pupils nearer to being..
Imitative
Autonomous

Are activities nearer to being....
Familiar / routine
Non - routine

- Is it adequate to list different kinds of activity/task and say that we need "balance"?
- Can we come up with something more helpful? What would it look like?

Perhaps a more helpful approach would to begin to articulate and illustrate how different types of task address each different type of purpose. In the past we may have been far too unimaginative in this regard.

Cockcroft's famous paragraphs 243 - 252 declared that all mathematics teaching should include opportunities for *Exposition, Discussion, Practical work, Practice, Problem solving,* and *Investigational work.* The formulaic, narrow interpretations that have subsequently been given to these terms serves us an object lesson as to how general advice can be mutated in practice. Consider how the term 'investigation' has mutated away from the spontaneous exploration of pupils' 'what if..?' questions so advocated in Cockcroft. Exemplification is crucial if purposes are not to become corrupted in practice. Below I offer just a few notes on specific purposes.

Concepts and strategies. Sample activities designed to foster the development of concepts and strategies may be found in the recent DfES materials. These illustrate a number of task 'types' that facilitate (mainly) conceptual development and different forms of 'mathematical thinking' that may be applied to all areas of mathematical content.

> ***Classifying mathematical objects***
> *Learners devise their own classifications for mathematical objects, and apply classifications devised by others. They learn to discriminate carefully and recognise the properties of objects. They also develop mathematical language and definitions.*
>
> ***Interpreting multiple representations***
> *Learners match cards showing different representations of the same mathematical idea. They draw links between different representations and develop new mental images for concepts.*
>
> ***Evaluating mathematical statements***
> *Learners decide whether given statements are always, sometimes or never true. They are encouraged to develop rigorous mathematical arguments and justifications, and to devise examples and counterexamples to defend their reasoning.*
>
> ***Creating problems***
> *Learners devise their own problems or problem variants for other learners to solve. This offers them the opportunity to be creative and 'own' problems. While others attempt to solve them, they take on the role of teacher and explainer. The 'doing' and 'undoing' processes of mathematics are vividly exemplified.*
>
> ***Analysing reasoning and solutions***
> *Learners compare different methods for doing a problem, organise solutions and/ or diagnose the causes of errors in solutions. They begin to recognise that there are alternative pathways through a problem, and develop their own chains of reasoning.*
>
> *(Swan, 2005)*

Such task-types are more specific than the general advice given in earlier documents and so provide more useable models for further development. In most teaching resources it is assumed that processes such as classifying, interpreting, evaluating, creating and analysing, will occur naturally in the course of tackling problems. Unfortunately, this often does not happen. These task-types address this issue by making the whole *focus* of each activity a particular mathematical process. The evidence would suggest that these communicate more effectively.

Problem-solving. It seems self-evident that tasks suitable for problem solving should be *non-routine* to the student. The student must make strategic choices as to which methods to apply. When the teacher describes the method to be used in advance then the problem simply becomes an exercise. Problems should illustrate the power of mathematics in a range of generic real-life tasks such as: when planning, organising, designing, evaluating and so on. They should involve a *formulation* phase in which the student is expected to define the problem from the context, the important variables and the relationships between them, a *solving* phase and a *verification* phase in which the solution is tested in practice. At present most of the curriculum is imitative and does not develop these 'modelling' skills.

Awareness-raising. Tasks that are suitable for raising pupils' awareness of concepts and connections, of their own learning and ways of learning, and of the values of the educational system may include activities that involve them adopting novel classroom roles. For example, they may become ‘teachers’ and construct their own worksheets for use with another class or they may become ‘assessors’, and devise tests and mark schemes. Such role shifts have raised the general level of reflective activity in a number of classrooms (Bell et al., 1993b).

During a lesson, students may

- *discuss key conceptual obstacles and common errors*
- *assess, correct and explain errors in ‘typical’ work*
- *make up problems that satisfy given constraints*
- *keep personal ‘dictionaries’ which explain important concepts and strategies*
- *orally review the purpose of each lesson*

After a sequence of lessons, students may

- *prepare worksheets and review materials for other students to use*
- *conduct student-student interviews on what was learned*
- *construct tests for other students to try (and mark schemes)*

Occasionally, students may

- *plan how they would teach a topic to other students (then do it)*
- *plan an outline for a new textbook, deciding which are important concepts and how these link together*
- *observe other students working and decide how their problem solving approaches could be improved*
- *conduct ‘mini debates’ on general learning issues such as: "Do we learn more from working on a few hard problems or from working on a lot of short exercises ?"*
- *assess their own progress against given criteria.*

(Bell et al., 1993b)

- What range of task types, based on 'sound and significant' mathematics, would you suggest should be included in the mathematics curriculum?
- How do you relate these types to the purposes elicited earlier?

4. Creating an effective learning climate

Effective learning requires more than carefully articulated purposes, principles and well-designed tasks. Also needed is an appropriate climate (or culture) for learning. Here is not the place to discuss general pedagogical issues, but perhaps there is a need to try to articulate what effective mathematics-specific pedagogies might look like. Again these will depend on the values and purposes under consideration. Different purposes will require different teaching strategies. There is no 'definitive style' for teaching mathematics:

> *We are aware that there are some teachers who would wish us to indicate a definitive style for the teaching of mathematics, but we do not believe that this is either desirable or possible. (Cockcroft, 1982 para 242)*

The teacher's vital role in constructing an appropriate classroom climate is well articulated in the NCTM[3] professional standards. I have chosen to base the following analysis on these US standards as we do not have anything similar specifically written for mathematics teachers. (The 'New overarching professional standards for teachers, tutors and trainers' by LLUK, and the 'Draft Revised Professional Standards for Teachers in England' from the TDA, while similar in many ways, are more general in nature).

This NCTM standards consider the role of the teacher in the:

- ***orchestration of discourse***
  (including the promotion of student-student discourse);
- ***selection of appropriate tools to enhance this discourse***
  (e.g. computers, concrete materials, stories and metaphors, dramatisations);
- ***creation of an appropriate learning environment***
  (e.g. allowing time to grapple with significant mathematics, a 'safe' place for taking intellectual risks)
- ***ongoing reflective analysis of teaching and learning***
  (e.g. formative assessment).

The first of these shows clearly the teacher's proactive role in making minute-by-minute decisions and the promotion and management of appropriate learner activity:

> *The teachers should orchestrate discourse by:*
> - *posing questions and tasks that elicit, engage and challenge each student's thinking;*
> - *listening carefully to students' ideas;*
> - *asking students to clarify and justify their ideas orally and in writing;*
> - *deciding what to pursue in depth from among the ideas that students bring up during a discussion;*
> - *deciding when and how to attach mathematical notation and language to students' ideas;*
> - *deciding when to provide information, when to clarify an issue, when to model, when to lead and when to let a student struggle with a difficulty;*

- *monitoring students' participation in discussions and deciding when and how to encourage each student to participate.*

The teacher should promote discourse in which students:
- *listen and respond to, and question the teacher and one another;*
- *use a variety of tools to reason, make connections, solve problems and communicate;*
- *initiate problems and questions*
- *make conjectures and present solutions;*
- *explore examples and counterexamples;*
- *try to convince themselves and one another of the validity of representations, solutions, conjectures, answers;*
- *rely on mathematical evidence and argument to determine validity. (NCTM, 1991)*

It is interesting to note the overlap between the desired activities listed here and the types of task that might foster them, as described above. This overlap is considerable. This leads to further questions:

- Is it the teachers' job to *design* tasks that encourage these types of activity?
- If it is, then how can this expertise accumulate within a community of teachers?
- If it is not, then who should do it?

After discussing the role of the teacher in encouraging the wide range of appropriate tools that will enhance this discourse, the NCTM standards describe aspects of the learning environment that are also under the control of the teacher:

The teacher should foster the development of each student's mathematical power by:
- *providing and structuring the time necessary to explore sound mathematics and grapple with significant ideas and problems;*
- *using the physical space and materials in ways that facilitate students' learning of mathematics;*
- *providing a context that encourages the development of mathematical skill and proficiency;*
- *respecting and valuing students' ideas, ways of thinking and mathematical dispositions;*

and by consistently expecting and encouraging students to:
- *work independently or collaboratively to make sense of mathematics;*
- *take intellectual risks by raising questions and formulating conjectures;*
- *display a sense of mathematical competence by validating and supporting ideas with mathematical argument. (NCTM, 1991)*

Of particular interest here is the issue of allowing sufficient time to explore mathematical ideas in depth and encouraging the serious consideration of learners' own ideas and contributions.

- To what extent do current values and practices encourage 'rapid coverage' at the expense of these aspects?

Finally, the US NCTM consider the role played by ongoing formative assessment:

The teacher should engage in ongoing analysis of teaching and learning by:
- *observing, listening to, and gathering other information about students to assess what they are learning;*

- *examining effects of the tasks, discourse, and learning environment on students' mathematical knowledge, skills and dispositions;*

in order to:

- *ensure that every student is learning sound and significant mathematics and is developing a positive disposition toward mathematics;*
- *challenge and extend students' ideas;*
- *adapt or change activities while teaching;*
- *make plans, both short and long range;*
- *describe and comment on each student's learning to parents and administrators, as well as to the students themselves. (NCTM, 1991)*

> - Do we need exemplified standards of this kind for this country?
> - What form might this exemplification take?
> - What else might be helpful?

A contention of this paper is that guidelines have often been too one-dimensional in the past. We cannot hope to define a single teaching strategy that will equally apply to all purposes. For me, the key question is 'what teaching styles are most appropriate/effective for each given purpose?' Research studies find it difficult to come to firm conclusions about this. The measures used to evaluate progress are often based on the assessment of procedural performances and so cannot enable evaluation of effective practice in terms of other purposes.

> - Do we know which methods are more/less effective for different purposes?
> - Can we articulate more effective/less effective modes of working for different purposes?

As one example of a response to the above box, the following list was obtained from close observation of two teachers using the same teaching resources. Both teachers were trying to develop a deep understanding of algebraic concepts. The research judged teachers who operated in the style exemplified by teacher A to be more effective.

Teacher A: strong 'connectionist' beliefs	Teacher B: strong 'transmission' beliefs
Lesson organisation • *Flexible*– schedule and pace determined partly by student responses. Some tasks spread over two lessons. • *Extended whole-class discussions* held at the beginning and end of every lesson and within lessons '*when the need arose'*. • *Students worked in groups for shorter periods* without teacher intervention.	• *Predetermined*– schedule and pace timetabled at outset of the project. Spent one lesson on each activity. • *Short whole-class discussions* usually at the beginning and end of each lesson. These were carefully planned beforehand. • *Students worked in groups for lengthy periods* without teacher intervention.
During whole class and small group discussion • *Offered challenge before help*. Allowed students to struggle before offering advice and help.	• *Offered help before challenge*. Offered 'easy' methods for doing tasks before students tackled them.

• *Questioning encouraged reflection* Asked more questions eliciting description of methods. Longer pauses between questions. Emerging issues often left unresolved. • *Students led whole-class discussions*. - teacher sat at back of room - students reported on the work of groups - students explained to other students. • *Listened before intervening with small groups* Listened before making suggestions. Agenda emerged from observing students. Left discussions unresolved. • *Encouraged students to explain and resolve their own difficulties.* When student were stuck, the teacher asked other students to contribute, or asked them to explain part of something.	• *Questioning encouraged recall* Asked more questions eliciting factual knowledge. Shorter pauses between questions. Emerging issues were mostly resolved by TT. • *Teacher led whole-class discussions.* - teacher remained at front of room - teacher controlled interactions - little student-student interaction. • *Intervened before listening to small groups.* Intervened whenever something incorrect noticed. Predetermined agenda apparent. Ensured discussions resolved. • *Attempted to resolved students' difficulties through repeated explanation.* When student were stuck, the teacher appropriated the role of explainer.

(Swan, 2006)

Similar contrasts may be drawn from other detailed studies of 'effective practice' (Askew et al., 1997).

When more global, observable criteria are used, however, (e.g. uses worksheets, students work in pairs etc) it is often hard to find any correlation between effective practice and teaching styles. These do not address the content or cognitive demand of the activities.

Conclusions

This paper contends that any communication of what constitutes the effective learning of mathematics must address four aspects:

1. The **values/purposes** we have in teaching mathematics
2. The **teaching principles** that emerge from each of these purposes
3. The **mathematical tasks** that are appropriate for each purpose
4. The **learning culture** that needs to be created.

It may be possible for the NCETM to create some form of resource that would begin to exemplify these aspects in a way that would greatly enlarge the vision of what is possible. There is unlikely to be total agreement, but we hope to be able to articulate something of the debate and to tease out any common principles.

The following questions are some that may be addressed at the conference itself. Please have a think about these beforehand and come prepared to discuss them.

1.	**What values/purposes and principles do *you* think should underpin mathematics education?** Do you agree with the ones outlined in this paper? Which new ones would you add? Which would you delete? Which ones are currently given most prominence?
2.	**Can you think of any vivid examples of effective mathematics learning?** What types of task were used? What kind of learning culture was created? How can we 'capture' these vignettes and communicate them to others?
3.	**What factors inhibit or modify practice so that it becomes less than effective?** What are the major obstacles to progress? What practical steps can we take to inform/equip others to help overcome these obstacles?

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Appendix (iv)

A paper tabled at the conference for delegates to express their views on each of the issues raised in the background paper, prompted by a fifteen-minute introduction.

What constitutes the effective learning of mathematics?

Name: ....

Organisation: ....

DISCUSSION 1

Before sharing your thoughts, take time to collect them and write them down.

1. What values/purposes do *you* think *should* underpin mathematics education? (e.g. the values on page 2)
2. What principles for teaching and learning do *you* think *should* underpin mathematics education? (e.g. the principles on page 3)

Continue your notes on the back if necessary.

2. Ideal and implemented values

Write "**A**" in the appropriate box on each row to show your vision for an **ideal** mathematics curriculum.
Write "**B**" in the appropriate box on each row to show the values implied by the curriculum that is **currently implemented** in most schools and other institutions.

4 = Most important: almost all mathematics lessons should contain this aspect
3 = most mathematics lessons should contain this aspect
2 = less than half of mathematics lessons should contain this aspect
1 = Least important: few mathematics lessons should contain this aspect

	Most important ←→			Least important
Types of outcome and types of activity	**4**	**3**	**2**	**1**
Fluency in recalling facts and performing skills E.g. Memorising names and notations Practising routine procedures				
Interpretations for concepts and representations E.g. Discriminating between examples/non-examples Generating representations Constructing relationships Translating between representations				
Strategies for investigation and problem solving E.g. Formulating questions / problems Developing / comparing strategies for solution Monitoring progress Interpreting / evaluating solutions Communicating results				
Awareness of the nature and values of the educational system E.g. Recognising the purposes of learning maths Developing learning/ reviewing strategies Knowing what others value.				
Appreciation of the power of mathematics in society E.g. Appreciate history / cultural foundations Creating / critiquing models of real situations Recognising uses / abuses of maths in society Gaining power over problems in ones own life.				

Please write comments and notes on the back of this sheet.

3. Principles for teaching and learning

Here are some principles that have been emphasised in the recent "Improving Learning in Mathematics" materials.

Principles used in the Standards Unit material: "Improving Learning in Mathematics"

Teaching is more effective when it ...	
• **builds on the knowledge learners already have;**	This means developing formative assessment techniques and adapting our teaching to accommodate individual learning needs (Black & Wiliam, 1998).
• **exposes and discusses common misconceptions**	Learning activities should exposing current thinking, create 'tensions' by confronting learners with inconsistencies, and allow opportunities for resolution through discussion (Askew & Wiliam, 1995).
• **uses higher-order questions**	Questioning is more effective when it promotes explanation, application and synthesis rather than mere recall (Askew & Wiliam, 1995).
• **uses cooperative small group work**	Activities are more effective when they encourage critical, constructive discussion, rather than argumentation or uncritical acceptance (Mercer, 2000). Shared goals and group accountability are important (Askew & Wiliam, 1995).
• **encourages reasoning rather than 'answer getting'**	Often, students are more concerned with what they have 'done' than with what they have learned. It is better to aim for depth than for superficial 'coverage'.
• **uses rich, collaborative tasks**	The tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage 'what if' and 'what if not?' questions (Ahmed, 1987).
• **creates connections between topics**	Learners often find it difficult to generalise and transfer their learning to other topics and contexts. Related concepts (such as division, fraction and ratio) remain unconnected. Effective teachers build bridges between ideas (Askew et al., 1997).
• **uses technology in appropriate ways**	Computers and interactive whiteboards allow us to present concepts in visual dynamic and exciting ways that motivate learners.

How far do you agree with these?
What principles would you add?
Are there principles, commonly believed by teachers, that are positively *unhelpful* for learning mathematics? If so, then what are they?

Now share and discuss your ideas and their implications.

DISCUSSION 2

Describe a vivid example of effective mathematics learning.
(It may, for example, be a lesson or lesson segment.)

(a) What was the mathematical task(s)?
(b) What learning culture was created? How was this achieved?

(c) What purposes and principles were being promoted? (refer to the tables on pages 2 and 3)	
(d) How could you tell that the task(s) achieved the intended purposes? Do you have any evidence?	
(e) Is this example available to see/ read about? (Give reference)	
(f) Can you say ***why*** you chose this example? What criteria were in your mind?	

Share your examples and discuss their implications

DISCUSSION 3

What are the main factors that inhibit or modify practice so that it becomes less than effective?

(a) What are the major obstacles to progress? How do these obstacles function?
(b) What practical steps can we take to help ourselves and others to overcome these obstacles?

Additional comments

Use this space to add your own further advice or comments that would help us to further the debate into effective mathematics learning.

Important note:

The purpose of today is to collect and share the views of informed professional mathematics educators. We therefore seek your permission to include attributed comments in papers and reports that will arise out of the conference**. You will be sent draft copies of these before they are widely circulated and offered the chance to amend or withdraw your contribution**.

PLEASE TICK ONE BOX

I am happy for my comments to be quoted and attributed. ☐

I am happy for my comments to be quoted but not attributed. ☐

I am not happy for my comments to be quoted. ☐

Signed: ....

Email address: ....